Symmetries of Hamiltonian Dynamical Systems, Momentum Maps and Reduction

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Abstract

Hamiltonian systems on symplectic or Poisson manifolds very often appear in Mechanics and in Mathematical Physics. Their symmetry properties can be described by the use of Lie groups actions on these manifolds. I will first recall the basic notions about symplectic and Poisson manifolds, and about Hamiltonian systems on these manifolds. I will discuss the action of a Lie group on a symplectic manifold; that action is called *symplectic* when it preserves the symplectic form, *Hamiltonian* when the infinitesimal generators of the action are Hamiltonian vector fields (such an action admits a momentum map, defined up to addition of an arbitrary constant), and *strongly Hamiltonian* when the momentum map can be chosen equivariant with respect to the coadjoint action of the group on the dual of its Lie algebra. Any momentum map of a Hamiltonian, but not strongly Hamiltonian action is equivariant with respect to an affine action of the group on the dual of its Lie algebra whose linear part is the coadjoint action, the additional term being a symplectic cocycle of the Lie group. Several examples will be presented. Poisson actions of a Lie group on a Poisson manifold will be described.

When a Lie group acts on the phase space of a Hamiltonian system by a Hamiltonian action which leaves the Hamiltonian unchanged, the momentum map is a first integral of the system (Noether's first theorem in Hamiltonian formalism). The reduction procedure of Marsden and Weinstein, which uses that property, will be discussed and several examples will be presented.

Momentum maps have remarkable convexity properties, and can be generalized in various ways. I will present the convexity theorems of Atiyah, Guillemin and Sternberg and their generalizatrion by Frances Kirwan.

Poisson-Lie groups were introduced Vladimir Drinfeld. A Poisson-Lie group is a Lie group G with a Poisson structure for which the multiplication is a Poisson map. It is a beautiful mathematical object with remarkable properties: the dual of its Lie algebra has a natural structure of Lie algebra. One can define a notion of Poisson action of a Poisson-Lie group on a Poisson manifold. The previously defined action of a Lie group on a Poisson manifold appears as the special case in which the Poisson-Lie group is equipped with the zero Poisson structure. For such an action Jiang Hua Lu has defined a generalized momentum map which takes its value in the dual group of the Poisson-Lie group.

If I have enough time left, I will define symplectic groupoids and explain how the source and target maps appear as natural generalizations of the momentum maps of the right and left actions of a Lie group on its cotangent bundle.

Contents

1. Hamiltonian systems on symplectic and Poisson manifolds

Symplectic manifolds. Their importance in Mechanics and in Physics: the phase space and the space of motions of a conservative classical mechanical system are symplectic manifolds; the space of oriented straight lines in a 3-dimensional Euclidean space is a symplectic manifold, whose structure naturally appears in Geometric Optics. Isotropic, coisotropic, Lagrangian and symplectic submanifolds of a symplectic manifold. Examples in Mechanics and in Geometric Optics.

Poisson manifolds. They often appear as quotients of a symplectic manifold by an appropriate equivalence relation, in particular as the space of leaves of an appropriate foliation; the symplectic orthogonal of the vector sub-bundle tangent to the leaves is integrable and determines another foliation of the symplectic manifold; therefore the quotient Poisson manifolds come in pairs. Example: the cotangent bundle T^*G of a Lie group G has a natural symplectic structure, and admits two symplectically orthogonal foliations, by the images of right-invariant or left-invariant differential 1-forms on G. The sets of leaves of these two foliations can both be identified with the dual $\mathcal{G}^*$ of the Lie algebra $\mathcal{G}$ of the group, with two opposite (but isomorphic) Poisson structures: the plus and minus Kirillov-Kostant-Souriau Poisson structure.

Hamiltonian systems on symplectic or Poisson manifolds. Examples: The Kepler problem for the motion of a planet around the Sun; light rays in a transparent non-homogeneous medium; motion of a rigid body around a fixed point; motion of an incompressible ideal fluid in a compact Riemannian manifold with boundary.

2. Action of a Lie group on a symplectic or Poisson manifold

Symplectic, Hamiltonian and strongly Hamiltonian actions of a Lie group on a symplectic manifold. Poisson and Hamiltonian actions of a Lie group on a Poisson manifold. Momentum maps. Affine action of the group on the dual of its Lie algebra for which the momentum map is equivariant. Symplectic cocycles of Lie groups and Lie algebras. For the Galilean group, interpretation of the cohomology class of the cocycle as the total mass of the system.

Momentum maps and first integrals of a Hamiltonian system with symmetry, Noether's first theorem. Examples: system of gravitationally interacting point particles, total linear and angular momentum; motion of a rigid body around a fixed point, Euler-Poinsot, Euler-Lagrange and Kovalevskaya special cases; the Kepler problem, angular momentum and the eccentricity vector (also calles the Laplace-Runge-Lenz vector).

3. Reduction methods of Hamiltonian systems with symmetries

Reduction of a symplectic manifold with symmetry: the Marsden-Weinstein reduced symplectic manifold. Examples in Mechanics and Geometric Optics of axially symmetric instruments.

Euler equations for the motion of a rigid body with a fixed point and for the motion of an ideal incompressible fluid. The vorticity field of the fluid as an element of the dual space of the infinite-dimensional Lie algebra of volume preserving vector fields.

The Marsden-Weinstein and Euler-Poincaré reduction procedures for mechanical systems whose configuration space is a Lie group.

4. Properties of the momentum map and generalizations

Convexity of the image of the momentum map for the Hamiltonian action of a torus on a compact symplectic manifold, theorems of Atiyah, Guillemin and Sternberg. Generalization for the momentum map of the action of a compact Lie group by Kirwan.

Poisson-Lie groups. Definition, properties, and Poisson actions of a Poisson-Lie group on a Poisson manifold. Group-valued generalized momentum maps.

5. Symplectic groupoids

Lie groupoids appear as natural mathematical objects: the contangent bundle of a Lie group is not a Lie group: it is a Lie groupoid. Definition and properties of Lie groupoids and Lie algebroids, source and target maps.

Use of a symplectic groupoid for the symplectization of Poisson manifolds.

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