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Applications of Lusternik-Schnirelmann Category and its Generalizations

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Lectures

These lectures are meant for a general audience of mathematicians, so background in algebraic topology and differential geometry is not a pre-requisite.

Lecture 1: Introduction to LS Category. In the early 1930's Lusternik and Schnirelmann described a new invariant of a manifold, called *category*, which provides a lower bound on the number of critical points for any smooth function on a compact manifold. In this sense, category is a cousin of Morse theory and in situations where it is not allowed to invoke genericity to obtain a Morse function, it is Lusternik-Schnirelmann theory that must be relied on. Once reformulated by Ralph Fox, category (or LS-category as it became known) found its place as a useful homotopy invariant — but one whose explicit computation is a very difficult task. In this lecture, we will describe basic properties and re-formulations of category as well as classical applications of category to, for instance, critical point theory and the Borsuk-Ulam theorem. References are [CLOT, Fo, EG, CP, OS, OS2].

Lecture 2: LS Category in Symplectic Geometry. In applying LS category to geometry, it is often the case that hard analytic results open the door to a calculation involving category which solves a problem. This lecture will focus on one such instance — the (strong) Arnold Conjecture on fixed points of Hamiltonian diffeomorphisms on symplectically aspherical manifolds. In this case, hard analysis produces a particular type of map arising from dynamics whose properties are especially well-suited to an LS category-theoretic interpretation. This then leads to a proof of the conjecture. References are [MS, Rud, RO, TO].

Lecture 3: LS Category and Non-Negative Ricci Curvature. Another very hard theorem with an interesting category interpretation is the Cheeger-Gromoll Splitting Theorem for manifolds with non-negative Ricci curvature. In this lecture, we will see how the Cheeger-Gromoll theorem leads to a powerful LS category-type refinement of Bochner's classical result that the first Betti number of a non-negative Ricci-curved manifold is bounded above by the manifold's dimension. We shall also consider some similar applications to manifolds of almost non-negative sectional curvature. References are [CG, YB, KPT, Op, OS3].

Lecture 4: New LS Categorical Ideas in Applied Mathematics. The *motion planning problem* asks for an *algorithm* that provides a path (in configuration space) for a system to take from one configuration to another. Perhaps surprisingly, the existence of such a path for any initial and terminal values is a question in topology involving a variant of category called *sectional category.* This category-type notion was used by Michael Farber to define the *topological complexity* (TC) of a space (e.g. a configuration manifold of a system) and to apply TC to the motion planning problem. In this lecture, we will describe TC, give some straightforward examples and mention some recent work on the TC of aspherical spaces. References are [Fa1, Fa2, GLO1, GLO2].

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