Integrable Curves and Surfaces

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Summary

The connection of curves and surfaces in $\mathbb{R}^3$ to some nonlinear partial differential equations is very well known in differential geometry [1], [2]. Motion of curves on two dimensional surfaces in differential geometry lead to some integrable nonlinear differential equations such as nonlinear Schrödinger (NLS) equation [3], Korteweg de Vries (KdV) and modified Korteweg de Vries (mKdV) equations [4],[5].

Surface theory in three dimensional Euclidean space ($\mathbb{R}^3$) is widely used in different branches of science, particularly mathematics (differential geometry, topology, Partial Differential Equations (PDEs)), theoretical physics (string theory, general theory of relativity), and biology [6]-[11]. There are some special subclasses of 2-surfaces which arise in the branches of science aforementioned. For the classification of surfaces in $\mathbb{R}^3$, particular conditions are imposed on the Gaussian and mean curvatures. These conditions are sometimes given as algebraic relations between curvatures and sometimes given as differential equations for these two curvatures. Here are some examples of some subclasses of 2-surfaces:

i) Minimal surfaces: $H = 0$,

ii) Surfaces with constant mean curvature : $H = constant$,

iii) Surfaces with constant positive Gaussian curvature: $K = constant > 0$,

iv) Surfaces with constant negative Gaussian curvature: $K = constant < 0$,

v) Surfaces with harmonic inverse mean curvature: $\nabla^2(1/H) = 0$,

vi) Bianchi surfaces: $\nabla^2(1/\sqrt{K}) = 0$ and $\nabla^2(1/\sqrt{-K}) = 0$, for positive Gaussian curvature and negative Gaussian curvature, respectively,

vii) Weingarten surfaces: $f(H, K) = 0$. For example; linear Weingarten surfaces, $c_1\, H + c_2\, K = c_3$, and quadratic Weingarten surfaces, $c_4\, H^2 + c_5\, H\, K + c_6\, K^2 + c_7\, H + c_8\, K = c_9$, where c_j are constants, $j = 1, 2, ..., 9$,

viii) Willmore surfaces: $\nabla^2 H + 2\, H(H^2 - K) = 0$,

ix) Surfaces that solve the shape equation of lipid membrane:

$$p - 2\,\omega H + k_c \nabla^2 (2H) + k_c (2H + c_0)(2H^2 - c_0 H - 2K) = 0,$$

where p, ω, k_c, and c_0 are constants.

Here, H and K are mean and Gaussian curvatures of the surface, respectively. On the other hand soliton equations play a crucial role for the construction of surfaces. The theory of nonlinear soliton equations was developed in 1960s. Lax representation of integrable equations should exist in order to apply inverse scattering method for finding solutions of these integrable equations. For details of integrable equations one may look [12], [13], and the references therein. Lax representation of nonlinear PDEs consists of two linear equations which are called Lax equations

$$\Phi_x = U\,\Phi, \quad \Phi_t = V\,\Phi, \tag{1}$$

and their compatibility condition

$$U_t - V_x + [U, V] = 0, \tag{2}$$

where x and t are independent variables. Here U and V are called Lax pairs. They depend on independent variables x and t, and a spectral parameter λ. Hereafter, subscript x and t denote the partial derivatives of the object with respect to x and t, respectively. For our cases, U and V are 2×2 matrices and they are in a given Lie algebra $\mathfrak{g}$. Eq. (2) is also called the zero curvature condition. Integrable equations arise as the compatibility conditions, Eq. (2), of the Lax equations [Eq. (1)]. Since Gauss-Mainardi-Coddazi (GMC) equations are compatibility conditions of Gauss-Weingarten (GW) equations, there is a close relationship between surfaces and Lax equations. GW equations and Lax equations play similar roles but they are not exactly the same. While Lax equations depend on spectral parameters, GW equations do not. Moreover GW equations are written in terms of 3×3 matrices whereas Lax pairs are 2×2 matrices. The former problem can be solved easily by inserting spectral parameters in GW equations using the one dimensional symmetry group of GW equations. The latter problem was solved by Sym [16]. By making use of the isomorphism $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$, he rewrote the GW equations in terms of 2×2 matrices. So for integrable surfaces, GW equations can be written in terms of 2×2 matrices using the conformal parametrization.

2-surfaces and integrable equations can be related by the analogy between GW equations and Lax equations. Such a relation is established by the use of Lie groups and Lie algebras. Using this relation, soliton surface theory was first developed by Sym [14]-[16]. He studied the surface theory in both directions: from geometry to solitons and from solitons to geometry. In the first direction, he obtained some well known soliton equations as a consequence of GMC equations. In the second direction, he obtained the following formula using the deformation of Lax equations for integrable equations

$$F = \Phi^{-1} \frac{\partial \Phi}{\partial \lambda}, \tag{3}$$

which gives a relation between a family of immersions (F) into the Lie algebra and the Lax equations for given Lax pairs. Fokas and Gelfand [17] generalized Sym's formula as

$$\begin{aligned} F &= \alpha_1 \Phi^{-1} U\,\Phi + \alpha_2 \Phi^{-1} V\,\Phi + \alpha_3 \Phi^{-1} \frac{\partial \Phi}{\partial \lambda} + \alpha_4\, x\, \Phi^{-1} U\,\Phi \\ &\quad + \alpha_5\, t\, \Phi^{-1} V\,\Phi + \Phi^{-1} M\,\Phi, \end{aligned} \tag{4}$$

where α_i, $i = 1, 2, 3, 4, 5$ and $M \in \mathfrak{g}$ are constants. So by this technique, which is called *the soliton surface technique*, using the symmetries of the integrable equations and their Lax equations we can find a large class of soliton surfaces for given Lax pairs. One may find 2-surfaces developed by soliton surface technique, which belong to subclasses of the surfaces, mentioned in **(i)**-**(ix)** on page 2, in the references [28], [9], [14]-[32].

On the other hand, there are some surfaces that arise from a variational principle for a given Lagrange function, which is a polynomial of degree less than or equal to two in the mean curvature of the surfaces. Examples of this type are minimal surfaces, constant mean curvature surfaces, linear Weingarten surfaces, Willmore surfaces, and surfaces solving the shape equation for the Lagrange functions. Taking more general Lagrange function of the mean and Gaussian curvatures of the surface, we may find more general surfaces that solve the generalized shape equation (see [34]-[40]) . Examples for this type of surfaces can be found in [20] - [25].

Examples of some of these surfaces like Bianchi surfaces, surfaces where the inverse of the mean curvature is harmonic [9], and the Willmore surfaces [10], [11] are very rare. The main reason is the difficulty of solving corresponding differential equations. For this purpose, some indirect methods [14]-[32] have been developed for the construction of two surfaces in $\mathbb{R}^3$ and in three dimensional Minkowskian geometries (M_3). Among these methods, soliton surface technique is very effective. In this method, one mainly uses the deformations of the Lax equations of the integrable equations. This way, it is possible to construct families of surfaces corresponding to some integrable equations such as sine Gordon (SG), Korteweg de Vries (KdV) equation, modified Korteweg de Vries (mKdV) equation and Nonlinear Schrödinger (NLS) equation [14]-[25], belonging to the afore mentioned subclasses of 2-surfaces in a three dimensional flat geometry. In particular, using the symmetries of the integrable equations and their Lax equation, we arrive at classes of 2-surfaces. There are many attempts in this direction and examples of new two surfaces.

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10) Conclusion

References

[1] L.P. Eisenhart, *A Treatise on the Differential Geometry of Curves and Surfaces*, Dover Pub., Inc., New York, (1909).

[2] L.P. Eisenhart, *Riemannian Geometry*, Princeton Univ. Press, (1964).

[3] H. Hasimoto, *A soliton on a vortex filament*, J. Fluid. Mech. **51**,477 (1972).

[4] U. Pinkall, *Hamiltonian flows on the space of star-shaped curves*, Results Math. **27**, 328-332 (1995).

[5] M. Gurses, *Motion of Curves on two dimensional curves surfaces and soliton equations*, Phys. Lett. **241 A**, 329-332 (1998).

[6] M. P. Do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Inc., Englewood Cliffs, NJ, (1976).

[7] R. Parthasarthy and K. S. Viswanathan, *Geometric properties of QCD string from Willmore functional*, J. Geom. Phys. **38**, 207-216 (2001) .

[8] Z. C. Ou-Yang, J. Liu and Y. Xie, *Geometric Methods in the Elastic Theory of Membranes in Liquid Crystal Phases*, World Scientific, Singapore, (1999).

[9] A. I. Bobenko, *Integrable Surfaces*, Funct. Anal. Prilozh. **24**, 68-69 (1990).

[10] T.J. Willmore, *Total Curvature in Riemannian Geometry*, John Willey and Sons, New York, (1982).

[11] T.J. Willmore, *Surfaces in Conformal Geometry*, Annals of Global Analysis and Geometry **18**, 255-264 (2000).

[12] P. G. Drazin, *Solitons: An Introduction*, Cambridge Univ. Press, New York (1989).

[13] M. J. Ablowitz and H. Segur, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, London Math. Soc. Lecture Notes Series **149** (1991).

[14] A. Sym, *Soliton Surfaces*, Lett. Nuovo Cimento **33**, 394-400 (1982).

[15] A. Sym, *Soliton Surfaces II*, Lett. Nuovo Cimento **36**, 307-312 (1983).

[16] Sym A, *Soliton Surfaces and their Applications* in Geometrical Aspects of the Einstein Equations and Integrable Systems, Lecture Notes in Physics Vol. **239**, ed. Martini R, Springer- Berlin, 154-231 (1985).

[17] A.S. Fokas and I.M. Gelfand, *Surfaces on Lie Groups, on Lie Algebras, and Their Integrability*, Commun. Math. Phys. **177**, 203-220 (1996).

[18] A.S. Fokas, I.M. Gelfand, F. Finkel and Q.M. Liu , *A Formula for Constructing Infinitely Many Surfaces on Lie Algebras and Integrable Equations*, Selecta Math., New Ser. **6**, 347-375 (2000).

[19] Ö. Ceyhan, A.S. Fokas, and M. Gürses, *Deformations of Surfaces Associated with Integrable Gauss-Minardi-Codazzi Equations* , J. Math. Phys. **41**, 2251-2270 (2000).

[20] M. Gürses, *Some Special Integrable Surfaces*, J. Nonlinear Math. Phys. **9**, 59-66 (2002).

[21] S. Tek, *Modified Kortewegde Vries surfaces*, J. Math. Phys. **48**, 013505 (2007).

[22] S. Tek, *Soliton Surfaces and Surfaces from a Variational Principle*, Ph.D. Thesis. Bilkent University, 2007.

[23] S. Tek, *Some Classes of Surfaces in R^3 and M_3 arising from Soliton Theory and a Variational Principle*, Discrte and Continuous Dynamical System Supplement, 761-770, 2009.

[24] S. Tek, *Using Nonlinear Schrödinger Equation to Obtain Some New Surfaces in* $\mathbb{R}^3$, Journal of Nonlinear Mathematical Physics, *submitted, 2014.*

[25] M. Gurses and S. Tek, *Korteweg de Vries Surfaces*, Nonlinear Analaysis: Theory, Method and Applications, **95**, 11-22 (2014).

[26] B.G. Konopelchenko, *Nets in* $\mathbb{R}^3$*, their integrable evolutions and the DS hierarchy*, Phys. Lett. A **183**, 153-159 (1993).

[27] B.G. Konopelchenko and I. A. Taimanov, *Generalized Weierstrass formulae, soliton equations and Willmore surfaces. I. Tori of revolution and the mKdV equation*, Report No. dg-ga/9506011.

[28] A. I. Bobenko, *Surfaces in Terms of 2 by 2 Matrices, Old and New Integrable Cases* in Harmonic Maps and Intgerable Systems, eds. Fordy A P and Wood J C, Aspects of Mathematics, E23, Friedr. Vieweg & Sohn-Braunscweig, 83-127 (1994).

[29] M. Melko and I. Sterling, *Integrable systems, harmonic maps and the classical theory of surfaces* in Harmonic maps and integrable systems, eds. A. P. Fordy and J. C. Wood , Aspects of Mathematics, E23, Friedr. Vieweg & Sohn-Braunscweig, 129-144 (1994).

[30] M. Gurses and Y. Nutku, *New Nonlinear Evolution Equations From Surface Theory* , J. Math. Phys. **22**,1393 (1981).

[31] J. Cieslinski, P. Goldstein, and A. Sym, *Isothermic surfaces in* E^3 *as soliton surfaces*, Phys. Lett. A **205**, 37-43 (1995).

[32] J. Cieslinski, *A Generalized Formula for Integrable Classes of Surfaces in Lie Algebraas*, J. Math. Phys. **38**, 4255-4272 (1997).

[33] R. Osserman, *A Survey of Minimal Surfaces*, Dover Pub., Inc., New York, (1986).

[34] Z.C. Tu and Z.C. Ou-Yang, *A Geometric Theory on the Elasticity of Bio-membranes*, J. Phys. A: Math. Gen. **37**, 11407-11429 (2004).

[35] Z.C. Tu and Z.C. Ou-Yang, *Lipid Membranes with Free Edges*, Phys. Rev. E **68**, 061915 (2003).

[36] Z. C. Tu and Z. C. Ou-Yang, Proceeding of the Seventh International Conference on Geometry, Integrability and Quantization, Varna, Bulgaria, 2005, edited by I. M. Mladenov and M. de León, SOFTEX, Sofia, 237 (2005), Report No. math-ph/0506055.

[37] Z. C. Ou-Yang and W. Helfrich, *Instability and Deformation of a Spherical Vesicle by Pressure*, Phys. Rev. Lett. **59**, 2486-2488 (1987).

[38] Z. C. Ou-Yang and W. Helfrich, *Bending Energy of Vesicle Membranes: General Expansion for the First, Second, and Third Variation of the Shape Energy and Applications to Sphere and Cylinders*, Phys. Rev. A **39**, 5280-5288 (1989).

[39] I. M. Mladenov, *New Solutions of the Shape Equation*, Eur. Phys. J. B **29**, 327-330 (2002).

[40] Z.C. Tu, *Elastic Theory of Biomembranes*, Thin Solid Films **393**, 19-23 (2001).