Lectures on Geometric Quantization

Jędrzej Śniatycki
Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta, Canada

XVII International Conference
Geometry, Integrability and Quantization
June 5-10, 2015, Varna, Bulgaria

Abstract

Quantum mechanics is considered by physicists to be the primary theory describing the behaviour of matter at microscopic level. For states of a quantum system, in which its quantum numbers are very large comparing to Planck's constant $\hbar$, the quantum theory is approximated by the classical theory of the system. This relation between the quantum and the classical mechanics is usually subsumed by the statement that the classical mechanics is the limit of the quantum mechanics when $\hbar$ tends to zero.

Quantization is an attempt to determine the quantum theory of a physical system from the knowledge of its classical theory. Since different quantum theories may have the same limit as $\hbar \to 0$, different quantizations of a system may lead to inequivalent quantum theories.

In geometric quantization applied to physical systems, we describe the choices made in the process of quantization in terms of the geometry of the phase space of the classical system under consideration. This helps us with understanding the degree of arbitrariness of the resulting quantum theory. On the other hand, geometric quantization applied to a Hamiltonian action of a connected Lie group G on a symplectic manifold gives rise to a unitary representation of G. In particular, geometric quantization of co-adjoint orbits of connected Lie groups gives rise to irreducible unitary representations of these groups. This dual role of geometric quantization enables us to use the representation theory to test hypotheses in quantum mechanics and vice versa.

Outline

Lecture 1. Discovery of Quantum Mechanics
1.1. Black body radiation (M. Planck).
1.2. Photoelectric effect (A. Einstein).
1.3. Energy spectrum of the harmonic oscillator (M. Born).
1.4. Radiation of the hydrogen atom (A. Sommerfeld).
1.5. Transition amplitudes (W. Heisenberg).
1.6. Schrödinger equation.
1.7. Foundations of quantum mechanics (P.A.M. Dirac).
Lecture 2. Prequantization
2.1. Symplectic manifolds.
2.2. Symplectic structure in mechanis.
2.3. Coadjoint orbits.
2.4. Momentum map.
2.5. Prequantization of a symplectic manifold.
2.6. Prequantization representation of a compact Lie group.
2.7. Prequantization representation of the Poisson algebra.
Lecture 3: Polarization
3.1. Complete set of commuting observables.
3.2. Definition of a polarization.
3.3. Quantization corresponding to a given polarization:
  3.3.1. Representation space;
  3.3.2. Operators.
3.4. Quantization of the harmonic oscillator:
  3.4.1. Schrödinger;
  3.4.2. Bargmann;
  3.4.5. Bohr.
3.5. Quantization representation of the rotation group.
  3.5.1. Complex polarization;
  3.5.2. Real polarization.

Lecture 4. Metaplectic corrections
4.1. Quasiclassical approximation to the Schrödinger equation.
4.2. Correction $\frac{1}{2}\hbar$ to Bohr-Sommerfeld conditons indicates the presence of a double covering.
4.3. Metaplectic frame bundle.
4.4. A half-form bundle.
4.5. Modification of the representation space.
4.6. Blattner-Kostant-Sternberg kernels.
4.7. Pairing of Bohr and Bargmann quantization of the harmonic oscillator.
4.8. Quantization of energy in the position representation.

References

L. Auslander and B. Kostant, "Polarizations and unitary representations of solvable Lie groups", *Invent. Math.*, **14** (1971) 255-354.

R.J. Blattner, "Intertwining operators and the half-density pairings", *Lecture Notes in Mathematics*, 466 (1975) 1-12, Springer, New York.

R.J. Blattner, "Pairing of half-form spaces" in Géométrie symplectique et physique mathématique, Coll. Int. C.N.R.S. No. 237 (1975) 175-186, C.N.R.S., Paris.

R.J. Blattner, "The metalinear geometry of non-real polarizations", Lecture Notes in Mathematics, 570 (1977) 11-45, Springer, New York.

V. Guillemin and S. Sternberg, "Geometric quantization and multiplicities of group representations". *Invent. Math.* **67** (1982) 515-538.

B.C. Hall, *Quantum Theory for Mathematicians*, Springer, 2013.

A.A. Kirillov, *Lectures on the Orbit Method,* American Mathematical Society, Providence, R.I., 2004.

B. Kostant, "Orbits, symplectic structures, and representation theory". *Proc. US-Japan Seminar in Differential Geometry, Kyoto (1965)*, Nippon Hyoronsha, Tokyo, 1966.

B. Kostant, "Quantization and unitary representations". In *Modern Analysis and Applications*, Lecture Notes in Mathematics, vol. 170, pp. 87-207, Springer, Berlin-Heidelberg-New York, 1970.

J. Śniatycki, *Geometric Quantization and Quantum Mechanics,* Applied Mathematical Science, Vol. **30**, Springer-Verlag, New York, 1980.

J. Śniatycki, *Differential Geometry of Singular Spaces and Reduction of Symmetry,* Cambridge University Press, Cambridge, 2013.

S. Sternberg, "Some recent results on the metaplectic representation", Lecture Notes in Physics, 79 (1978) 117-143.

N.M.J. Woodhouse, *Geometric Quantization*, second edition, Clarendon Press, Oxford, 1992.