XXVI$^{\text{th}}$ INTERNATIONAL CONFERENCE

GEOMETRY, INTEGRABILITY AND QUANTIZATION

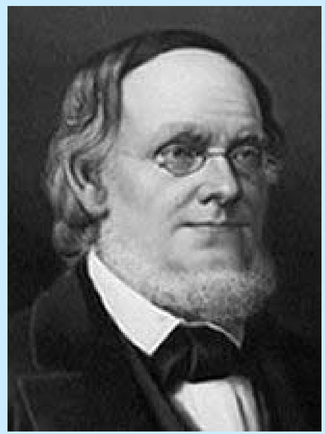
June 4$^{\text{th}}$-11$^{\text{th}}$ 2026, Varna, Bulgaria

APPLICATIONS OF CLIFFORD ALGEBRA TO EUCLIDEAN AND PROJECTIVE GEOMETRIES

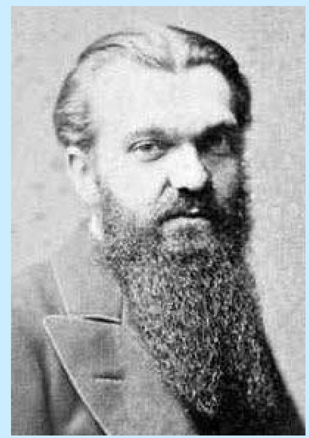
Ramon González Calvet

Leibniz

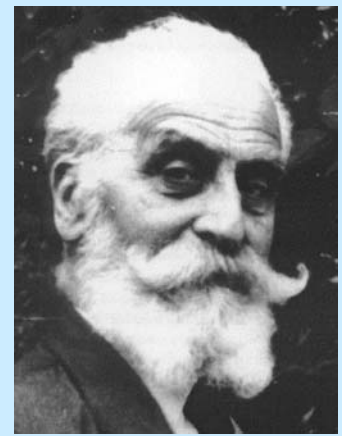
Möbius



Grassmann



Clifford



Peano

ANTECEDENTS

- In 1679-1680 Leibniz wrote to Huygens some letters in French on the *Analisis situs* or the *Characteristica geometrica*.
- In 1827, Möbius published *Der barycentrische Calculus* (*The barycentric calculus*).
- In October 16th 1843, Hamilton discovered the *Quaternions*, which he defined as quotients of vectors.
- In 1844 Grassmann published *Die Ausdehnungslehre* (*Extension Theory*) and the second edition in 1862.
- In 1878 Clifford published the paper *Applications of Grassmann's Extensive Algebra*.
- In 1888 Peano publishes *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann preceduto dalle operazione della logica dedutiva* (*Geometric Calculus according to H. Grassmann's Ausdehnungslehre preceded by the operations of deductive logic*).
- In 1901 Wilson publishes Gibbs' lessons in *Vector Analysis*.

QUATERNIONS AS GEOMETRIC QUOTIENTS OF VECTORS

- Hamilton defined a quaternion as a geometric quotient of two vectors.
- A quaternion is defined by four quantities: the ratio of norms of both vectors, the oriented angle they form, and two angles of orientation of the plane where the vectors lie.

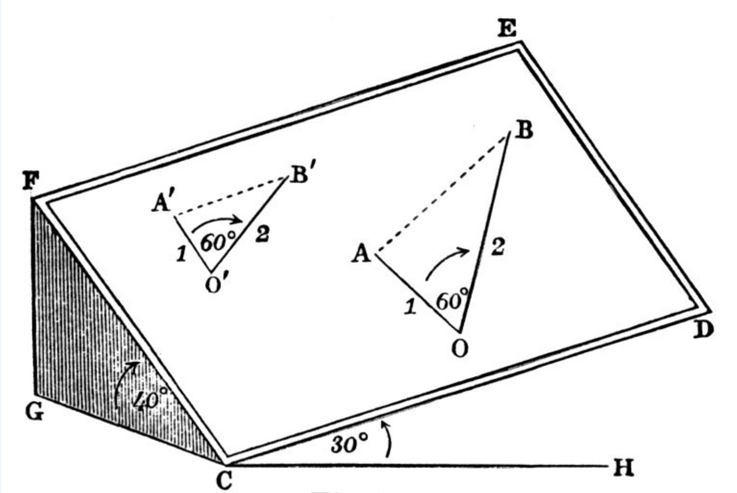


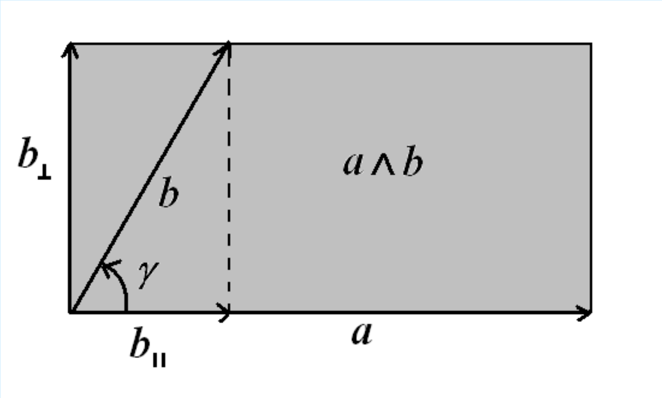
Fig. 34.

[W. R. Hamilton, *Elements of Quaternions*, p. 113]

GEOMETRIC PRODUCT OF TWO VECTORS

- The geometric product of two vectors in the Euclidean space is also a quaternion.

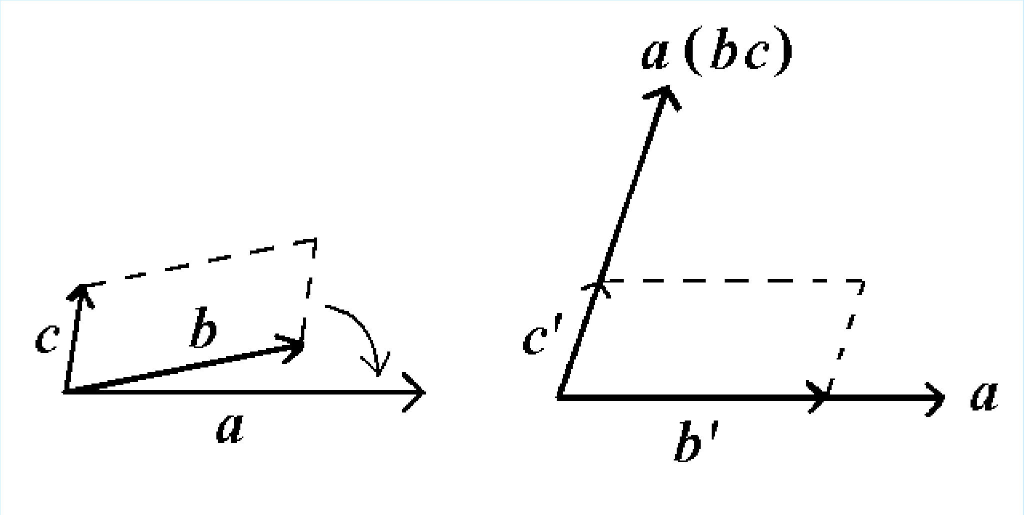
$$a\, b = a\,(b_{\|} + b_{\perp}) = a\, b_{\|} + a\, b_{\perp} = \| a \|\, \| b \|\,(\cos\gamma + u \sin\gamma)$$



- The geometric and Clifford products are identical. In the Clifford algebra $Cl_{3,0}$, the correspondence between quaternion unities and Clifford unities is $i = -e_{23}$, $j = -e_{31}$, $k = -e_{12}$.

GEOMETRIC PRODUCT OF THREE VECTORS

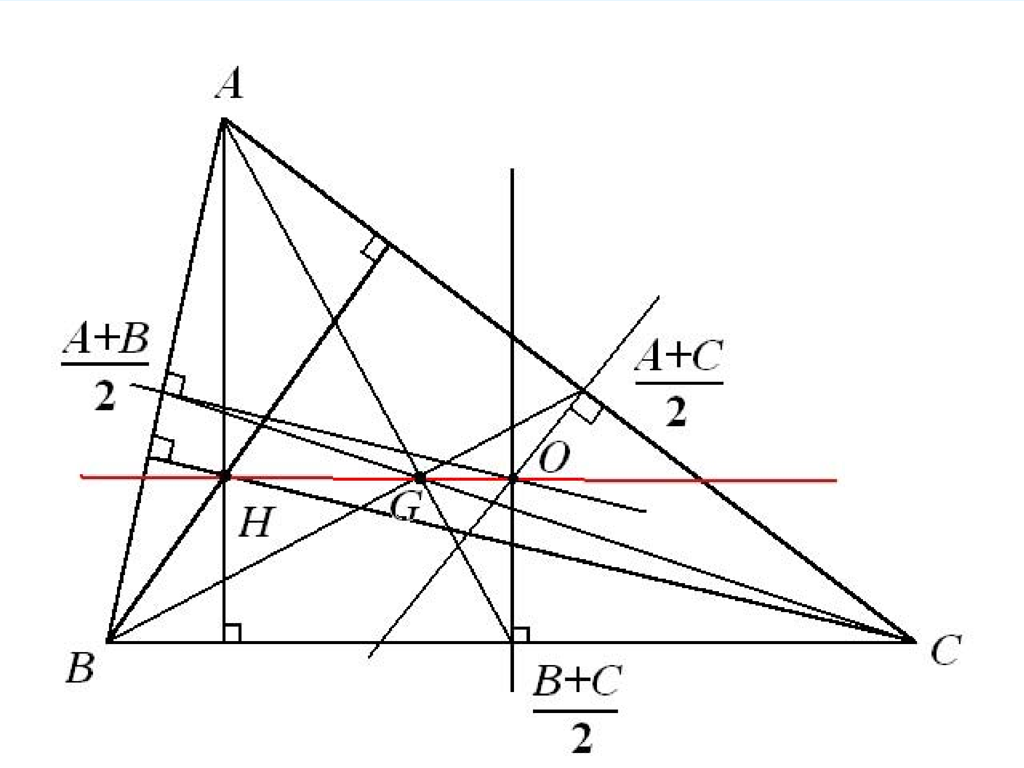
- The geometric product is assumed to be associative and distributive.
- The geometric product is not commutative but it satisfies the permutative property
  - In the plane $a\,b\,c = c\,b\,a$ or $a\,z = z^*\,a$ $z \in \mathbb{C}$
  - In the space $a \wedge b \wedge \mathrm{c} = (\,a\,b\,c - c\,b\,a\,)\,/\,2$
- The figure shows the geometric product of three vectors in the plane.
- In the space, the product of two vectors by the perpendicular component generates a volume and a perpendicular vector.



THE EULER LINE OF A TRIANGLE

- The vector from the circumcenter O to the ortocenter H is obtained from the sides of a triangle ABC through the formula

  $OH = -(AB\ BC\ CA + BC\ CA\ AB + CA\ AB\ BC)\ (2\ AB \wedge BC)^{-1}$



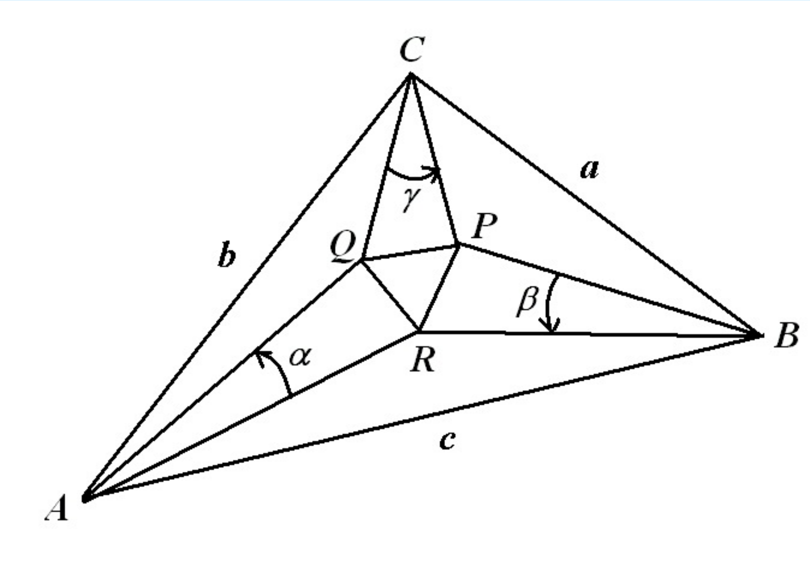
THE MORLEY THEOREM

- The intersections of the trisectors of the interior angles of a triangle form an equilateral triangle.

$$PQ = 8\, r \sin\alpha \sin\beta \sin\gamma\, u_{BC} \exp(\mathrm{i}(\alpha + 2\beta))$$
$$QR = 8\, r \sin\alpha \sin\beta \sin\gamma\, u_{CA} \exp(\mathrm{i}(\beta + 2\gamma))$$
$$RP = 8\, r \sin\alpha \sin\beta \sin\gamma\, u_{AB} \exp(\mathrm{i}(\gamma + 2\alpha))$$

- The geometric quotient of two sides of the equilateral triangle yields a rotation of 120º.

$$PQ^{-1}\, QR = \exp(-\mathrm{i}(\alpha+2\beta))\, u_{BC}\, u_{CA} \exp(\mathrm{i}(\beta + 2\gamma))$$
$$= \exp(-\mathrm{i}(\alpha+2\beta)) \exp(\mathrm{i}(\pi-3\gamma)) \exp(\mathrm{i}(\beta + 2\gamma)) = \exp(\mathrm{i}\ 2\pi/3)$$



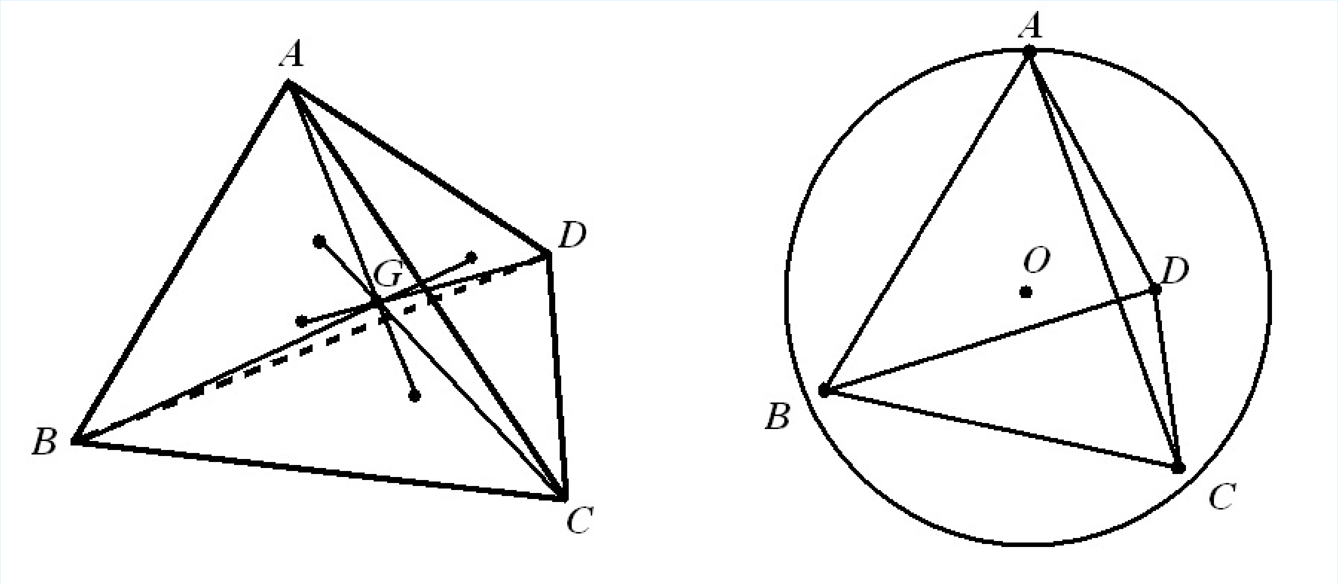
THE EULER LINE OF A TETRAHEDRON

- Let $ABCD$ be a tetrahedron. Then the director vector of the Euler line is given by

$$OM = (AB\ BC\ CD\ DA - BC\ CD\ DA\ AB + CD\ DA\ AB\ BC$$
$$- DA\ AB\ BC\ CD)\ (4\ AB \wedge BC \wedge CD)^{-1}$$

where M is Monge's point, the intersection of the six planes that pass through the midpoint of each edge and are perpendicular to the opposite edge.

- The centroid G satisfies $G = (O + M\,)\ /\ 2$

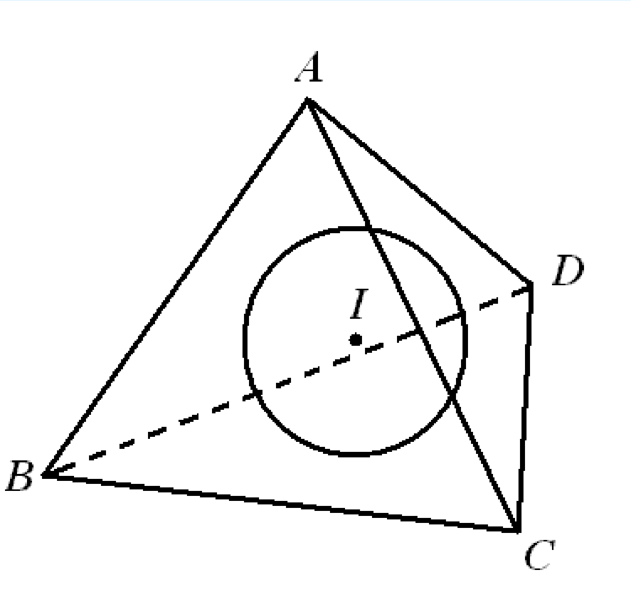


THE INCENTER OF A TETRAHEDRON

- The incenter of a triangle ABC is given by
  $I = (A \,\|BC\| + B \,\|CA\| + C \,\|AB\|)\,(\|AB\| + \|BC\| + \|CA\|)^{-1}$

- The incenter of a tetrahedron $ABCD$ is given by

$$I = (A \,\|BC \wedge CD\| + B \,\|CD \wedge DA\| + C \,\|DA \wedge AB\| + D \,\|AB \wedge BC\|)$$
$$(\|BC \wedge CD\| + \|CD \wedge DA\| + \|DA \wedge AB\| + \|AB \wedge BC\|)^{-1}$$



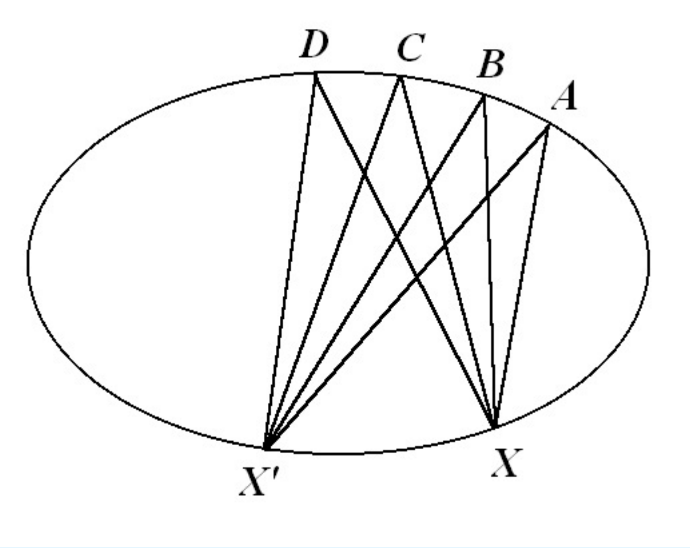
PROJECTIVE GEOMETRY: THE CHASLES THEOREM

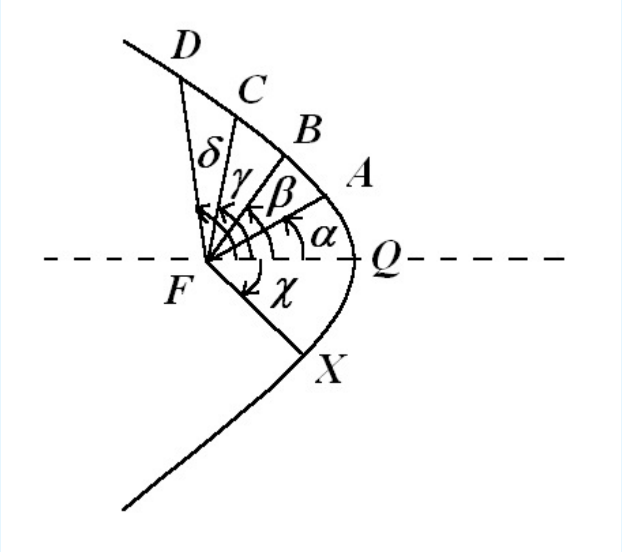
- The cross ratio of a conic is given by

$$(X, ABCD) = (XA \wedge XC)\,(XB \wedge XD)\,(XA \wedge XD)^{-1}(XB \wedge XC)^{-1}$$

- The constancy of the cross ratio is proven as

$$(X, ABCD) = \sin((\gamma - \alpha)/2)\,\sin((\delta - \beta)/2)\;/\;(\sin(\delta - \alpha)/2)\,\sin(\gamma - \beta)/2)$$





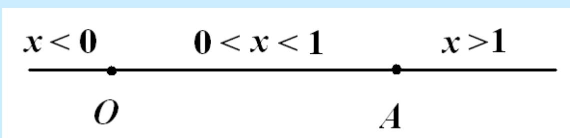
A BASIS OF POINTS IN THE SPACE

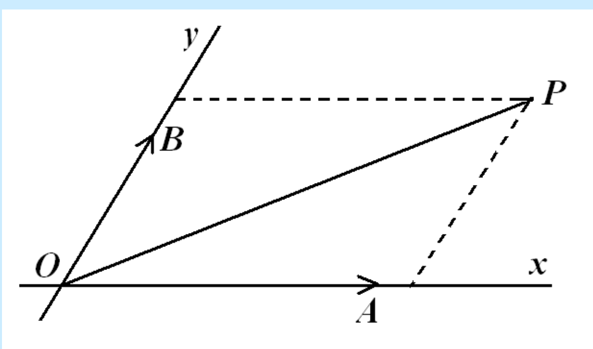
- According to Möbius, every point on a line, in a plane or in the space can be written as a linear combination of two points, three non-collinear points, or four non-coplanar points respectively.

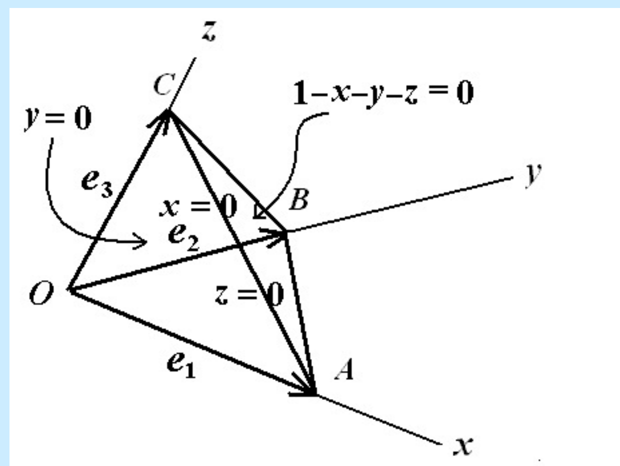
  $P = O + x\, OA = (1 - x)\, O + x\, A$ on a line,

  $P = O + x\, OA + y\, OB = (1 - x - y - z\,)\, O + x\, A + y\, B$ in the plane,

  $P = O + x\, OA + y\, OB + z\, OC = (1 - x - y - z\,)\, O + x\, A + y\, B + z\, C$ in the space.
- $(1 - x - y - z,\ x, y, z\,)$ are the *barycentric coordinates.*







THE DESARGUES THEOREM

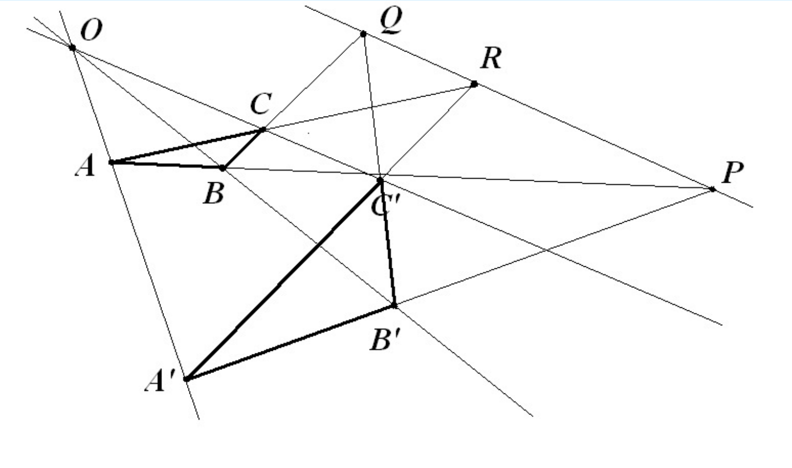
- **Theorem**. Let ΔABC and $\Delta A'B'C'$ be two triangles. Let P be the intersection of the prolongations of sides AB and $A'B'$, Q the intersection of the prolongations of BC and $B'C'$,and R the intersection of the prolongations of CA and $C'A'$. Points P, Q and R are collinear if and only if the lines AA', BB' and CC' meet at the same point O.
- **Proof**

$O = a\,A + (1-a)\,A' = b\,B + (1-b)B' = c\,C + (1-c)\,C' \qquad a, b, c \in \mathbb{R}$

$\Rightarrow \quad a\,A - b\,B = -(1-a)\,A' + (1-b)\,B'$

$\Rightarrow \quad P = (a\,A - b\,B) / (a-b) = (-(1-a)\,A' + (1-b)\,B') / (a-b)$

In the same way, $Q = (b\,B - c\,C) / (b-c)$ and $R = (c\,C - a\,A) / (c-a)$



Then

$$\det(PQR) \propto \begin{vmatrix} a & -b & 0 \\ 0 & b & -c \\ -a & 0 & c \end{vmatrix} = 0$$

and P, Q, and R are collinear.

FROM BARYCENTRIC TO PROJECTIVE COORDINATES

- The Pappus and Pascal hexagram theorems are proven in the same way as the Desargues theorem.
- Let us denote the fourth barycentric coordinate as $t = 1 - x - y - z$. The equation of a plane is written in homogeneous form

$$\begin{bmatrix} d & e & f & g \end{bmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = d\,t + e\,x + f\,y + g\,z = 0$$

- The equation of a line is written as $\begin{bmatrix} d_1 & e_1 & f_1 & g_1 \\ d_2 & e_2 & f_2 & g_2 \end{bmatrix} \begin{pmatrix} t_1 & t_2 \\ x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{pmatrix} = 0$

- We can take homogeneous coordinates. It is not necessary that $t + x + y + z = 1$. Even, this sum can be zero (points at infinity=vectors).

QUADRICS

- Every quadric can be written with projective coordinates as a homogeneous equation

$$\begin{pmatrix} t & x & y & z \end{pmatrix} \mathbf{M}_{4\times 4} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = 0$$

where $(t \; x \, y \, z)$ are point coordinates

- Its tangential quadric is

$$\begin{bmatrix} d & e & f & g \end{bmatrix} \mathbf{M}_{4\times 4}^{-1} \begin{bmatrix} d \\ e \\ f \\ g \end{bmatrix} = 0$$

where $[d \; e \; f \; g]$ are dual coordinates defining a plane.

EXTERIOR ALGEBRA OF POINTS

- The product of a point by itself is null.
- The product of two points is the line passing through both points.
- The product of three non-collinear points is the plane that contains them. If they are collinear, the product is zero.
- The product of four non-coplanar points is the space where they live. If they are coplanar, the product is zero.
- Let us consider three points expressed in the frame $(P_1\ P_2\ P_3\ P_4)$

$$A = a_1 P_1 + a_2 P_2 + a_3 P_3 + a_4 P_4 \qquad B = b_1 P_1 + b_2 P_2 + b_3 P_3 + b_4 P_4$$
$$C = c_1 P_1 + c_2 P_2 + c_3 P_3 + c_4 P$$

- Then the line AB is

$$AB = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} P_1 P_2 + \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} P_1 P_3 + \begin{vmatrix} a_1 & b_1 \\ a_4 & b_4 \end{vmatrix} P_1 P_4$$
$$+ \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} P_2 P_3 + \begin{vmatrix} a_2 & b_2 \\ a_4 & b_4 \end{vmatrix} P_2 P_4 + \begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix} P_3 P$$

- The plane ABC is given by

$$ABC = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} P_1P_2P_3 + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4 \end{vmatrix} P_1P_2P_4$$

$$+ \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} P_1P_3P_4 + \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} P_2P_3P_4$$

- The basis of lines has six elements: the six lines containing the edges of the fundamental tetrahedron. The basis of planes has four elements: the planes containing the faces of the fundamental tetrahedron.
- In the same way, the exterior product of the dual coordinates of two planes yields the line that is intersection of both planes, and the exterior product of the dual coordinates of three planes yields their intersection point.

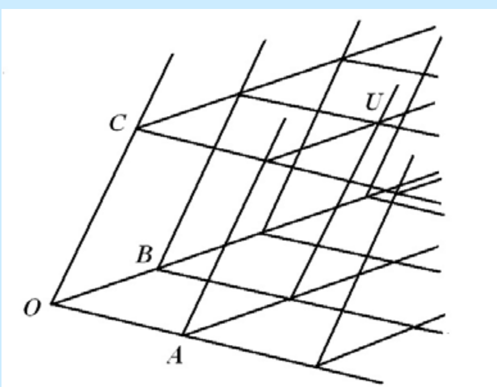
PROJECTIVITIES

- A *projectivity* is a homogeneous linear transformation of projective coordinates (passive definition) in the same frame.

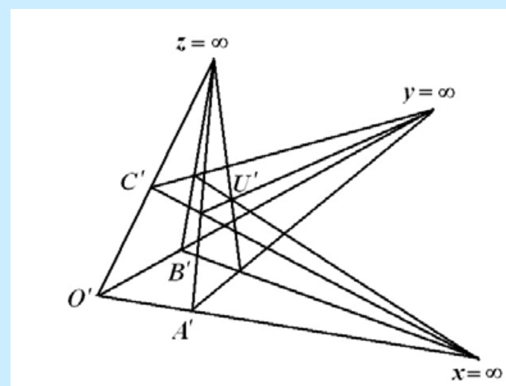
$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = k \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

- $k \neq 0$ is undefined because projective coordinates are homogeneous.
- A *projectivity* is a mapping between two projective frames (active definition) that keeps projective coordinates. A projective frame is given by one point more than an affine frame.

$U = o\ O + a\ A + b\ B + c\ C + d\ D \quad \rightarrow \quad U' = o\ O' + a\ A' + b\ B' + c\ C' + d\ D'$



$\rightarrow$

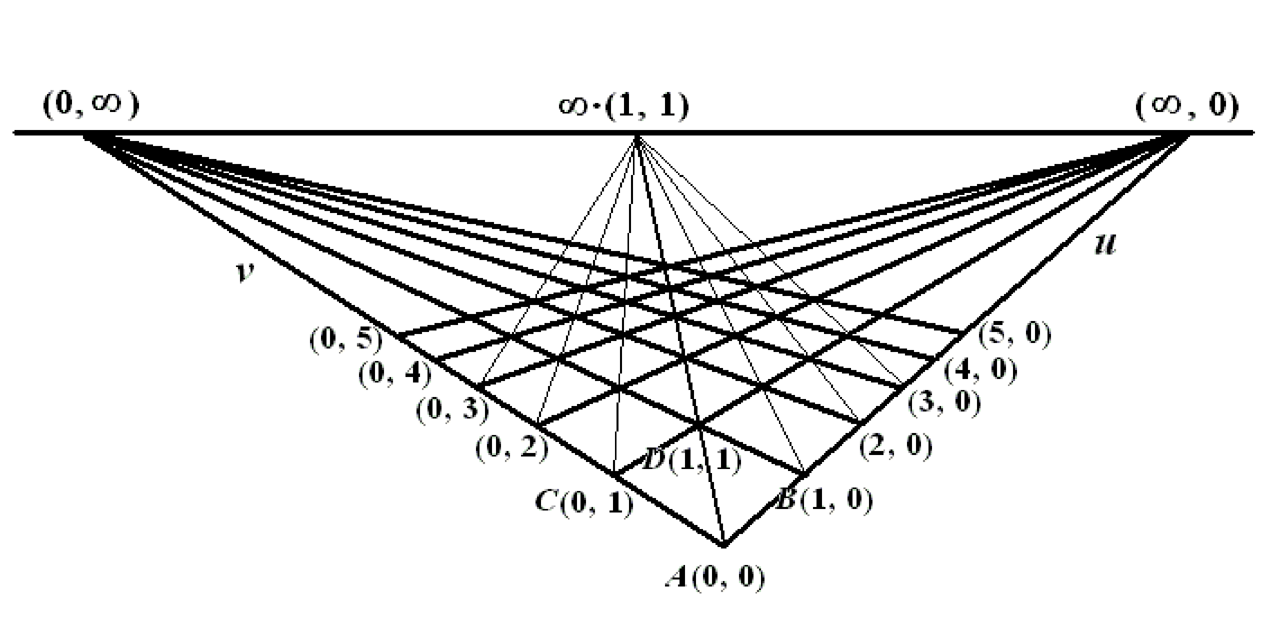


PROJECTIVE FRAMES

- Let us see an example of projective frame *ABCD* in the plane. The projective coordinates are defined by these four points. There are three projective coordinates, which only the last two are shown in the figure. For example,

  $D = -A + B + C = (-1, 1, 1) = (1, 1)$

- The line at infinity is the union of the points at infinity.
- The distances between points are measured with the cross ratio. For example $(ABU_2U_\infty)=AU_2\, BU_\infty / (AU_\infty\, BU_2) = 2$



REFERENCES

- H. S. M. Coxeter, *Fundamentos de Geometría*, Limusa-Wiley, p. 251 and p. 271 (México D. F., 1971).
- H. S. M. Coxeter, “Coordinates”. In: *Projective Geometry*, 2nd Ed., pp. 111-125, Springer (New York 2003).
- R. González Calvet, “Conics”. In: *Treatise of Plane Geometry through Geometric Algebra* pp. 117-138 (Barcelona 2007).
- R. González Calvet, “Geometría del tetraedro” (2016). In: *El álgebra geométrica del espacio y tiempo*, pp. 87-94. https://www.rgonzalezcalvet.cat/espacio.htm.
- H. Grassmann, “Combinatorial multiplication of points”. In: *Extension Theory* (translation of the second edition of *Die Ausdehnungslehre* [1862] by Lloyd C. Kannenberg) p. 138, American Mathematical Society & London Mathematical Society (Providence 2000).
- W. R. Hamilton, “Fundamentals Principles Respecting Quotients of Vectors”. In: *Elements of Quaternions* [1869], p. 107. Ed. By Charles Jasper Joly, 3rd edition, Chelsea (New York 1969).
- A. F. Möbius, *Der Barycentrische Calculus* [Leipzig 1827]. Facsimile edition of Georg Olms Verlag (Hildesheim 1976).
- G. Peano, “Saggio di calcolo geometrico” [1896]. In: *Opere Scelte*, vol. III, pp. 167-186. Edizioni Cremonese (Roma 1957-1959).
- A. A. Ungar, *Barycentric Calculus in Euclidean and Hyperbolic Geometry. A Comparative Introduction*. World Scientific (Singapore 2010).

THANK YOU VERY MUCH FOR YOUR ATTENTION