Neutrosophic Almost convergence

Bridging Functional Analysis and Neutrosophic Theory

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The Evolution of Uncertainty Modeling

Classical Approaches

- **Fuzzy Sets (Zadeh):** Introduced the concept of partial truth. Membership is defined by a single degree $T \in [0, 1]$.
- **Intuitionistic Fuzzy Sets (Atanassov):** Added a degree of hesitation or non-membership $F \in [0, 1]$.

The Structural Flaw

While conceptually sound for many applications, these classical approaches possess a critical limitation:

- They cannot mathematically handle **strictly independent** indeterminate and inconsistent information.
- In intuitionistic fuzzy sets, the degrees are dependent (e.g., $T + F \leq 1$). If data is noisy or contradictory, this restriction fails.

The Neutrosophic Paradigm

Smarandache's Solution
To overcome the limitations of classical fuzzy logic, Florentin Smarandache proposed **Neutrosophic Theory**. It is a new branch of philosophy and mathematics that studies the origin, nature, and scope of neutralities.

The Three Independent Dimensions
Neutrosophic logic evaluates every entity or statement across three entirely *independent* coordinates:

- **T:** Degree of Truth
- **I:** Degree of Indeterminacy (Neutrality)
- **F:** Degree of Falsity

Because they are independent, the sum $T + I + F$ is not restricted to 1, allowing it to model incomplete (< 1), exact $(= 1)$, and inconsistent/paraconsistent (> 1) information.

Structural Differences: Intuitionistic vs. Neutrosophic

Feature	Intuitionistic Fuzzy Sets	Neutrosophic Sets
Components	Truth (T), Falsity (F)	Truth (T), Indeterminacy (I), Falsity (F)
Indeterminacy	Dependent: $I = 1 - (T + F)$	Completely Independent
Constraint	$T + F \leq 1$	$0 \leq T + I + F \leq 3$
Best Used For	Incomplete information	Contradictory, noisy, and indeterminate data

Why it matters

In real-world complex phenomena, we often have independent sources of evidence. Neutrosophic theory allows us to process a statement that is simultaneously highly true, highly false, and highly uncertain without breaking the mathematical model.

Application to Functional Analysis

Bridging the Gap
The intersection of functional analysis and neutrosophic theory has recently emerged as a highly active research area.

Why apply it to Sequence Spaces?

- Classical functional analysis assumes exact measurements of distances (norms).
- **Neutrosophic Normed Spaces (NNS)** allow us to evaluate limits and convergence when the sequence terms themselves, or the functional distances between them, are subject to deep indeterminacy.
- It provides a rigorous topological framework to answer: *"What happens to summability when the data streams are inconsistent?"*

Neutrosophic Almost Convergence: An Overview i

- **Theoretical Foundation**
  - *Classical Almost Convergence (Lorentz):* Evaluates non-classical limits uniformly using Banach limits ($\sup_m \left|\frac{1}{n}\sum_{k=1}^{n} x_{k+m} - L\right| \to 0$ as $n \to \infty$).
  - *Neutrosophic Theory (Smarandache):* Rigorously models complex uncertainty via independent degrees of truth, indeterminacy, and falsity.

- **The Research Gap**
  - While Neutrosophic Normed Spaces (NNS) and various statistical convergences are well-studied, the structural extension of *almost convergence* is missing in current literature.

- **Core Innovations & Key Results**
  - Introduces a novel neutrosophic norm to handle deep indeterminacy.
  - The sequence spaces form a vector space over $\mathbb{R}$ or $\mathbb{C}$ (but not an algebra).

Neutrosophic Almost Convergence: An Overview ii

- Neutrosophic almost convergence strictly contains neutrosophic convergence and implies neutrosophic boundedness.
- Establishes necessary inclusion criteria linking classical almost convergence to its neutrosophic counterpart.

Almost Convergence (Lorentz)
A bounded sequence (x_k) is said to be **almost convergent** to a limit L if all its Banach limits equal L. Equivalently, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that:

$$\sup_{m \geq 0} \left| \frac{1}{n} \sum_{k=1}^{n} x_{k+m} - L \right| < \epsilon \quad \forall n \geq N.$$

Preliminaries: Core Definitions ii

Continuous t-norm (^CTN) & t-conorm (^CTCN)
Let $*, \diamond : [0,1] \times [0,1] \to [0,1]$ be commutative, associative, and continuous binary operations satisfying monotonicity ($a \leq c, b \leq d \implies a * b \leq c * d$ and $a \diamond b \leq c \diamond d$).

- The operation $*$ is a ^CTN if it acts with identity 1:

$$a * 1 = a \quad \forall a \in [0,1].$$

- The operation $\diamond$ is a ^CTCN if it acts with identity 0:

$$a \diamond 0 = a \quad \forall a \in [0,1].$$

Axioms of an NNLS (1/2)

Let X be a linear space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), $*$ be a continuous t-norm, and $\diamond$ be a continuous t-conorm. Then $(X, (T, I, F), *, \diamond)$ is called a NNLS if $\forall \alpha \in \mathbb{K}$, $x, y \in X$ and $t, s > 0$ the following axioms hold:

(NN1) $0 \leq T(x,t),\ I(x,t),\ F(x,t) \leq 1$

(NN2) $T(x,t) = 1 \iff x = 0,\ \ I(x,t) = 0 \iff x = 0,$ and $F(x,t) = 0 \iff x = 0$

(NN3) If $x \neq 0$, then $T(x,t) < 1,\ \ I(x,t) > 0,$ and $F(x,t) > 0\ \ \forall t > 0$

(NN4) For each fixed $x \in X$, $T(x,t)$ is continuous and non-decreasing in t, while $I(x,t)$ and $F(x,t)$ are continuous and non-increasing in t

(NN5) For $x \neq 0$:

$$\lim_{t\to\infty} T(x,t) = 1, \quad \lim_{t\to\infty} I(x,t) = 0, \quad \lim_{t\to\infty} F(x,t) = 0$$

Axioms of an NNLS (2/2)

(NN6) $T(-x,t) = T(x,t)$, $I(-x,t) = I(x,t)$, and $F(-x,t) = F(x,t)$

(NN7) For $\alpha \neq 0$:

$$T(\alpha x, t) = T\left(x, \frac{t}{|\alpha|}\right), \quad I(\alpha x, t) = I\left(x, \frac{t}{|\alpha|}\right), \quad F(\alpha x, t) = F\left(x, \frac{t}{|\alpha|}\right)$$

(NN8) $T(x+y, t+s) \geq T(x,t) * T(y,s)$

(NN9) $I(x+y, t+s) \leq I(x,t) \diamond I(y,s)$

(NN10) $F(x+y, t+s) \leq F(x,t) \diamond F(y,s)$

(NN11) For all $t \leq 0$: $T(x,t) = 0$, $I(x,t) = 1$, and $F(x,t) = 1$

A Novel Exponential Neutrosophic Norm

While existing research heavily relies on a standard fractional norm architecture [1, 2], we introduce a novel exponential alternative to address structural limitations.

Standard Literature Norm

For a classical NLS $(X, \|\cdot\|)$ and a parameter $t > \|x\|$:

$$T(x,t) = \frac{t}{t + \|x\|}$$

$$I(x,t) = \frac{\|x\|}{t + \|x\|}$$

$$F(x,t) = \frac{\|x\|}{t}$$

For $t \leq \|x\|$: $T = 0,\ I = 1,\ F = 1.$

Proposed Exponential Norm

For a classical NLS $(X, \|\cdot\|)$, constants $\alpha, \beta, \gamma > 0$, and $t > 0$:

$$T(x,t) = e^{-\alpha\|x\|/t}$$

$$I(x,t) = 1 - e^{-\beta\|x\|/t}$$

$$F(x,t) = 1 - e^{-\gamma\|x\|/t}$$

For $t \leq 0$: $T = 0,\ I = 1,\ F = 1.$

Note: Both systems utilize the continuous t-norm $a * b = ab$ and continuous t-conorm $a \diamond b = a + b - ab$. The exponential model introduces a alternative setting absent from current literature.

Neutrosophic Convergence & Boundedness

Definition ([1, 2]Neutrosophic Convergence)
A sequence (x_n) in an NNLS converges to $L \in X$ if $\forall \epsilon \in (0,1)$ and $\forall t > 0, \exists N_0 \in \mathbb{N}$ such that $\forall n \geq N_0$:

$$T(x_n - L, t) > 1 - \epsilon, \quad I(x_n - L, t) < \epsilon, \quad F(x_n - L, t) < \epsilon.$$

Definition ([1, 2]Neutrosophic Boundedness)
A sequence (x_n) in X is neutrosophically bounded if $\exists t > 0$ and $\epsilon \in (0,1)$ such that $\forall n \in \mathbb{N}$:

$$T(x_n, t) \geq 1 - \epsilon, \quad I(x_n, t) \leq \epsilon, \quad F(x_n, t) \leq \epsilon.$$

Neutrosophic Almost Convergence (NAC)

Definition
Let (x_n) be a neutrosophically bounded sequence in an NNLS. It is **neutrosophically almost convergent (NAC)** to $L \in X$ if $\forall t > 0$:

$$\lim_{n\to\infty} \inf_{m\geq 0} T\left(\frac{1}{n}\sum_{k=1}^{n}(x_{k+m} - L), t\right) = 1$$

$$\lim_{n\to\infty} \sup_{m\geq 0} I\left(\frac{1}{n}\sum_{k=1}^{n}(x_{k+m} - L), t\right) = 0$$

$$\lim_{n\to\infty} \sup_{m\geq 0} F\left(\frac{1}{n}\sum_{k=1}^{n}(x_{k+m} - L), t\right) = 0$$

Example: The sequence $x_n = \cos(2n\pi/3)$ is NAC to 0 under the exponential neutrosophic norm, despite not converging in the classical or standard neutrosophic sense.

Characterisations of $\mathcal{NAC}(X)$

Theorem (Uniqueness)
If (x_n) neutrosophically almost converges to L_1 and L_2, then $L_1 = L_2$.

Theorem (Linearity)
If $(x_n), (y_n) \in \mathcal{NAC}(X)$ converging to L and M, respectively, then for any $\alpha, \beta \in \mathbb{K}$, $(\alpha x_n + \beta y_n)$ neutrosophically almost converges to $\alpha L + \beta M$.

Structural Result: The space $\mathcal{NAC}(X)$ forms a vector space over $\mathbb{R}$ or $\mathbb{C}$, but *does not* form an algebra (it is not closed under sequence multiplication). Here, being closed under sequence multiplication means the multiplication of two neutrosophic almost convergent sequences may not be almost convergent.

Inclusions: Convergence $\implies$ NAC

Theorem (Convergence implies NAC)
In an NNLS with $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$, if (x_n) neutrosophically converges to L, then it is neutrosophically almost convergent to L.

Proof.
Suppose, $\{x_n\}$ neutrosophically converges to L. Let $\epsilon > 0$ and $0 < t < 1$ be given.

Clearly $\{x_j - L\}$ neutrosophically converges to 0. Hence $\{x_j - L\}$ is neutrosophically bounded. Then $\exists t_B > 0$ such that

$$T(x_j - L, t_B) > 1 - \epsilon,\ I(x_j - L, t_B) < \epsilon,\ F(x_j - L, t_B) < \epsilon$$

Also from Definition, $\exists N_1 \in \mathbb{N}$ satisfying

$$\begin{aligned} T(x_j - L, t/2) &> 1 - \epsilon, \\ I(x_j - L, t/2) &< \epsilon, \\ F(x_j - L, t/2) &< \epsilon \quad \forall j > N_1. \end{aligned}$$

Proof: Mean Decomposition

Proof (continued).
Now, consider the mean difference:

$$S_{n,m} - L = \frac{1}{n}\sum_{k=1}^{n}(x_{k+m} - L) = \frac{1}{n}(B_{n,m} + G_{n,m})$$

where we decompose the sum into a bounded part and a tail part:

$$B_{n,m} = \sum_{k+m \leq N_1} (x_{k+m} - L)$$

$$G_{n,m} = \sum_{k+m > N_1} (x_{k+m} - L)$$

□

Proof (continued).
Using (NN8), $\exists t^\star = N_1 t_B > 0$ (independent of n & m) such that $T(B_{n,m}, t^\star) > 1 - \epsilon$.

Also, using (NN7) we get:

$$T\left(\frac{1}{n}B_{n,m}, \frac{t}{2}\right) = T\left(B_{n,m}, \frac{nt}{2}\right).$$

Clearly, $\frac{nt}{2} \to \infty$ as $n \to \infty$. Then $\exists N_2 \in \mathbb{N}$ s.t. $\frac{nt}{2} > t^\star \quad \forall n \geq N_2$.

From (NN5), $\lim_{\tau \to \infty} T(x, \tau) = 1$. Since T is non-decreasing in t, then $\forall n \geq N_2$,

$$T\left(\frac{1}{n}B_{n,m}, \frac{t}{2}\right) = T\left(B_{n,m}, \frac{nt}{2}\right) > T(B_{n,m}, t^\star) > 1 - \epsilon.$$

$\square$

Proof (continued).
Again, $\frac{1}{n}G_{n,m} = \sum_{k+m>N_1} \frac{1}{n}(x_{k+m} - L)$. Suppose, $G_{n,m}$ contain P terms.

If $P = 0$ then $G_{n,m} = 0$ and $T(0, t/2) = 1 > 1 - \epsilon$.

If $P > 0$ then repeated application of (NN8) gives:

$$\begin{aligned} T\left(\frac{1}{n}G_{n,m}, \frac{t}{2}\right) &\geq \min_{k+m>N_1} T\left(\frac{1}{n}(x_{k+m} - L), \frac{t}{2P}\right) \\ &= \min_{k+m>N_1} T\left(x_{k+m} - L, \frac{nt}{2P}\right) \\ &\geq \min_{k+m>N_1} T\left(x_{k+m} - L, \frac{t}{2}\right) \ \left(\because \frac{n}{P} > 1 \text{ as } P \leq n\right) \\ &> 1 - \epsilon. \end{aligned}$$

$\square$

Proof: Consolidating the Estimates

Proof (continued).
Take $N = \max\{N_1, N_2\}$. Then $\forall n \geq N$, and $\forall m \geq 0$:

$$T(S_{n,m} - L, t) \geq \min\left\{T\left(\frac{1}{n}B_{n,m}, \frac{t}{2}\right), T\left(\frac{1}{n}G_{n,m}, \frac{t}{2}\right)\right\}$$
$$> \min\{1-\epsilon, 1-\epsilon\} = 1 - \epsilon$$
$$\implies \inf_{m\geq 0} T(S_{n,m} - L, t) \geq 1 - \epsilon.$$

Similarly using (NN9), (NN10) we get $\sup_{m\geq 0} I(S_{n,m} - L, t) \leq \epsilon$ and $\sup_{m\geq 0} F(S_{n,m} - L, t) \leq \epsilon$.

From the arbitrariness of $\epsilon > 0$, we find

$$\lim_{n,m\to\infty} T(S_{n,m} - L, t) = 1,$$
$$\lim_{n,m\to\infty} I(S_{n,m} - L, t) = 0 = \lim_{n,m\to\infty} F(S_{n,m} - L, t).$$

$\therefore \{x_n\}$ neutrosophically almost converges to L. $\square$

Limits of the Theorems

Important Remarks on the Framework

- **Theorem 1 Limitation:**
  The previous result (Convergence $\implies$ NAC) **fails** if we use the product t-norm $a * b = ab$ due to a lack of local convexity in the resulting space.

- **Irreversibility:**
  The converses of both main theorems are generally false.
  - A sequence can be **NAC without being convergent** (e.g., oscillating sequences like $x_n = (-1)^n$).
  - A sequence can be **bounded without being NAC**.

NAC implies Neutrosophic Boundedness

Theorem

Let $(X, \mathcal{N}, *, \diamond)$ be an NNLS. If $(x_n) \subset X$ is neutrosophically almost convergent, then (x_n) is neutrosophically bounded.

Proof.
Suppose that (x_n) is neutrosophically almost convergent to some $L \in X$. Let $S_{n,m} = \frac{1}{n}\sum_{k=1}^{n} x_{k+m}$. By Definition 3.3, for any fixed $\tau > 0$, $(S_{n,m})$ converges uniformly in m to L as $n \to \infty$.

Let $\epsilon \in (0,1)$ be given. Since the ${}^{C}TN$ $*$ satisfies $1 * 1 = 1$, we find a $\delta \in (0,1)$ satisfying $(1-\delta) * (1-\delta) > 1 - \epsilon$. Similarly, since the ${}^{C}TCN$ $\diamond$ satisfies $0 \diamond 0 = 0$, we find a $\gamma \in (0, \epsilon)$ satisfying $\gamma \diamond \gamma < \epsilon$. □

Proof (continued).
Fix $\tau = 1$. By Definition, we can find $N \in \mathbb{N}$ satisfying $\forall n \geq N$ and $\forall m \geq 0$:

$$T(S_{n,m}-L,\tau) > 1-\delta, \quad I(S_{n,m}-L,\tau) < \gamma, \quad F(S_{n,m}-L,\tau) < \gamma. \tag{1}$$

Take $n_0 = \max(N+1, 2)$. Then $n_0 > N$ and $n_0 - 1 \geq N$. Now

$$x_{m+1} - L = n_0(S_{n_0,m} - L) - (n_0 - 1)(S_{n_0-1,m+1} - L).$$

Choose $t_0 = n_0\tau + (n_0 - 1)\tau = (2n_0 - 1)\tau$. Since $n_0 \geq 2$ and $\tau = 1$, it follows that $t_0 > 0$. □

Proof: Applying Neutrosophic Norms

Proof (continued).
Applying axiom (NN8) to t_0, we obtain:

$$T(x_{m+1}-L, t_0) \geq T\Big(n_0(S_{n_0,m}-L), n_0\tau\Big) * T\Big(-(n_0-1)(S_{n_0-1,m+1}-L), (n_0-1)$$

By applying (NN6) and (NN7) we get,

$$T(x_{m+1} - L, t_0) \geq T(S_{n_0,m} - L, \tau) * T(S_{n_0-1,m+1} - L, \tau).$$

Using (1), we find

$$T(x_{m+1} - L, t_0) \geq (1 - \delta) * (1 - \delta) > 1 - \epsilon.$$

$\square$

Proof (continued).
A parallel application of Axioms (NN9) and (NN10) for indeterminacy and falsity gives:

$$I(x_{m+1} - L, t_0) \leq I(S_{n_0,m} - L, \tau) \diamond I(S_{n_0-1,m+1} - L, \tau) < \gamma \diamond \gamma < \epsilon,$$
$$F(x_{m+1} - L, t_0) \leq F(S_{n_0,m} - L, \tau) \diamond F(S_{n_0-1,m+1} - L, \tau) < \gamma \diamond \gamma < \epsilon.$$

These strict bounds hold uniformly $\forall m \geq 0$. Thus, for the fixed t_0, $(x_k - L)$ is neutrosophically bounded for all $k \geq 1$. These strict bounds hold uniformly for all $m \geq 0$, showing that $(x_k - L)$ is neutrosophically bounded for t_0. we show that (x_n) is bounded. □

Proof: Final Bounding Constant

Proof (continued).
Now, $x_n = (x_n - L) + L$. Using axiom (NN5), we can find $t_L > 0$ satifying $T(L, t_L) > 1 - \epsilon$, $I(L, t_L) < \epsilon$, and $F(L, t_L) < \varepsilon$. Let $t^* = t_0 + t_L$. Applying axioms (NN8)–(NN10) to the sum $(x_n - L) + L$ we get,

$$T(x_n, t^*) \geq T(x_n - L, t_0) * T(L, t_L) \geq (1 - \varepsilon) * (1 - \varepsilon),$$
$$I(x_n, t^*) \leq I(x_n - L, t_0) \diamond I(L, t_L) \leq \varepsilon \diamond \varepsilon,$$
$$F(x_n, t^*) \leq F(x_n - L, t_0) \diamond F(L, t_L) \leq \varepsilon \diamond \varepsilon.$$

Since $\varepsilon \in (0, 1)$ and $*$ and $\diamond$ yield values strictly within $(0, 1)$, we can define a single bounding constant:

$$\alpha = \max\Big\{1 - \big((1 - \epsilon) * (1 - \epsilon)\big),\ \epsilon \diamond \epsilon\Big\}.$$

It follows that $0 < \alpha < 1$ and $n \geq 1$:

$$T(x_n, t^*) \geq 1 - \alpha, \quad I(x_n, t^*) \leq \alpha, \quad F(x_n, t^*) \leq \alpha.$$

Proof: Defining the Bounding Constant

Proof (continued).
Since $\varepsilon \in (0,1)$ and $*$ and $\diamond$ yield values strictly within $(0,1)$, we can define a single bounding constant:

$$\alpha = \max\Big\{1 - \big((1-\epsilon) * (1-\epsilon)\big),\ \epsilon \diamond \epsilon\Big\}.$$

It follows that $0 < \alpha < 1$ and $n \geq 1$:

$$T(x_n, t^*) \geq 1 - \alpha, \quad I(x_n, t^*) \leq \alpha, \quad F(x_n, t^*) \leq \alpha.$$

Hence, (x_n) is neutrosophically bounded. $\square$

Norm-Compatible NNLS

Definition (Norm-Compatibility)
An NNLS is compatible with a classical norm $\|\cdot\|$ if $\|y_n\| \to 0$ implies $T(y_n, t) \to 1, I(y_n, t) \to 0, F(y_n, t) \to 0$ as $n \to \infty$.

Theorem (Bridge to Classical Theory)
In a norm-compatible NNLS:

1. *Neutrosophic convergence implies NAC for any arbitrary t-norm and t-conorm.*
2. *If (x_n) classically almost converges to L uniformly, then (x_n) neutrosophically almost converges to L.*

Concluding Remarks & Future Outlines

Conclusions

- Successfully extended Lorentz's summability theory to the neutrosophic environment to handle deep indeterminacy.
- Introduced a novel exponential neutrosophic norm.
- Established $\mathcal{NAC}(X)$ as a vector space and proved foundational topological relations (uniqueness, boundedness, inclusion).

Future Directions

- **Matrix Transformations:** Conditions for infinite matrices between NAC spaces.
- **Statistical & Ideal Convergence:** Formulating neutrosophic statistically almost convergent spaces and $\mathcal{I}$-almost convergence.
- **Applications:** Decision-making algorithms and dynamical models with highly noisy or inconsistent data.

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Thank You!

Questions and Discussions are Welcome.