Geometric Analysis of the Landau–Lifshitz Equation in Minkowski Space

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Differential geometry of surfaces deals with the smooth surfaces, which includes a variety of different structures, usually a Riemann metric. Mostly, surfaces have been investigated from two mainly perspectives. The first concerns their embedding in externally, Euclidean or non-Euclidean space.

The features measured along curves on the surface, reflecting their properties determined by distance within the surface. The most well known concepts investigated is the Gaussian curvature which is first defined by Carl Friedrich Gauss *(Gauss, 1965).*

By studying the intrinsic properties of surfaces, we are familiar with the concept of curves lying on a surface. In this respect, it is the most preferred way of examining the local differential geometric structure of the curve. In many studies dealing with differential geometric properties of curves, some methods and tools of differential calculus are used.

Here it uses the more well known Frenet-Serret frame. Analyzing the geometric structures of the curves with the help of vector analysis reveals valuable results. Let $\sigma(s, n, b)$ be a space curve in three-dimensional Euclidean space. Here s, n and b are the distance along the t -lines, n -lines and b -lines *(Marris and Passman; 1969)*.

Minkowski 3-space is the Euclidean space provided with Lorentzian product

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle_L = -u_1 v_1 + u_2 v_2 + u_3 v_3$$

where $\overrightarrow{u} = (u_1, u_2, u_3)$, $\overrightarrow{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$. By definition, this product is not positively defined. Instead, this product classifies the vectors in $\mathbb{E}_1^3$ as follows:
i)If $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_L > 0$ or $(\overrightarrow{u} = 0)$ then $\overrightarrow{u}$ is called a **spacelike** vector**;**
ii) If $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_L < 0$ then $\overrightarrow{u}$ is called a **timelike** vector**;**
iii) If $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_L = 0$ then $\overrightarrow{u}$ is called a **lightlike (null)** vector.

For each $\overrightarrow{u} \in \mathbb{E}_1^3$, the norm of $\overrightarrow{u}$ vector is defined

$$\|\overrightarrow{u}\|_L = \sqrt{|\langle \overrightarrow{u}, \overrightarrow{u} \rangle_L|}.$$

If $\langle \overrightarrow{u}, \overrightarrow{v} \rangle_L = 0$ then $\overrightarrow{u}$ and $\overrightarrow{v}$ vectors are said to be orthogonal. For each $\overrightarrow{u}$, $\overrightarrow{v} \in \mathbb{E}_1^3$, we may write

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle_L = -u_1 v_1 + u_2 v_2 + u_3 v_3 = u^T I^* v$$

where

$$I^* = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Walrave; 1995).

Let $\alpha : I \rightarrow \mathbb{E}_1^3$ be a regular curve. If $\left\langle \alpha'(s), \alpha'(s) \right\rangle_L > 0$ and $\left\langle \alpha''(s), \alpha''(s) \right\rangle_L = 0$ for $\forall s \in I$, then α is called pseudo-null curve. And if $\left\langle \alpha'(s), \alpha'(s) \right\rangle_L = 1$, then α is called a pseudo null curve given by pseudo arc length parameter *(Walrave; 1995).*

Let $\alpha : I \rightarrow \mathbb{E}_1^3$ be a pseudo-null curve given by pseudo arc length parameter. We know that

$$\overrightarrow{t}(s) = \alpha^{'}(s)$$

is the tangent vector of α and $\overrightarrow{n}(s) = \alpha^{''}(s)$ is a null vector. The binormal vector field $\overrightarrow{b}$ is the unique null vector field orthogonal to $\overrightarrow{t}$ such that

$$\left\langle \overrightarrow{n}(s), \overrightarrow{b}(s) \right\rangle_L = 1.$$

If α is straight line then $\kappa(s) = 0$ and in other cases $\kappa(s) = 1$. Furthermore

$$\tau(s) = \left\langle \overrightarrow{n}'(s), \overrightarrow{b}(s) \right\rangle_L.$$

The Frenet formulas for pseudo-null curve $\alpha : I \to E_1^3$ given by pseudo arc length parameter are as follows

$$\frac{d}{ds}\begin{bmatrix} \overrightarrow{t}(s) \\ \overrightarrow{n}(s) \\ \overrightarrow{b}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ 0 & \tau(s) & 0 \\ -\kappa(s) & 0 & -\tau(s) \end{bmatrix} \begin{bmatrix} \overrightarrow{t}(s) \\ \overrightarrow{n}(s) \\ \overrightarrow{b}(s) \end{bmatrix}.$$

Frenet vector fields of α satisfying following relations:

$$\left\langle \overrightarrow{n}(s), \overrightarrow{n}(s) \right\rangle_L = \left\langle \overrightarrow{b}(s), \overrightarrow{b}(s) \right\rangle_L = 0,$$
$$\left\langle \overrightarrow{t}(s), \overrightarrow{n}(s) \right\rangle_L = \left\langle \overrightarrow{t}(s), \overrightarrow{b}(s) \right\rangle_L = 0,$$
$$\left\langle \overrightarrow{t}(s), \overrightarrow{t}(s) \right\rangle_L = \left\langle \overrightarrow{n}(s), \overrightarrow{b}(s) \right\rangle_L = 1$$

In this context, we consider that $\Im = \Im(s, n, b)$ is a given pseodu null curve in Minkowski Space. As known, s denotes the distance along $t - lines$ where the spacelike tangent vector of $t - lines$ is defined by

$$\overrightarrow{t} = \overrightarrow{t}(s, n, b) = \frac{\partial \Im}{\partial s}.$$

Then the distance along $n - lines$ is denoted by n where the null tangent vector of $n - lines$ is

$$\overrightarrow{n} = \overrightarrow{n}(s, n, b) = \frac{\partial \Im}{\partial n}.$$

Moreover, b denotes the distance along $b - lines$ where null tangent vector of $b - lines$ is

$$\overrightarrow{b} = \overrightarrow{b}(s, n, b) = \frac{\partial \Im}{\partial b}.$$

Directional derivatives of $\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\}$ with respect to parameter s are given as follows:

$$\frac{\partial \overrightarrow{t}}{\partial s} = \kappa \overrightarrow{n},$$

$$\frac{\partial \overrightarrow{n}}{\partial s} = \tau \overrightarrow{n},$$

$$\frac{\partial \overrightarrow{b}}{\partial s} = -\kappa \overrightarrow{t} - \tau \overrightarrow{b}.$$

Directional derivatives of the $\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\}$ with respect to parameter n are obtained as follows:

$$\frac{\partial \overrightarrow{t}}{\partial n} = \mu_b \overrightarrow{n} + \xi_{ns} \overrightarrow{b},$$

$$\frac{\partial \overrightarrow{n}}{\partial n} = -\xi_{ns} \overrightarrow{t} - (\operatorname{div} b + \kappa) \overrightarrow{n},$$

$$\frac{\partial \overrightarrow{b}}{\partial n} = -\mu_b \overrightarrow{t} + (\operatorname{div} b + \kappa) \overrightarrow{b}.$$

Moreover, directional derivatives of $\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\}$ with respect to parameter b are obtained as follows:

$$\frac{\partial \overrightarrow{t}}{\partial b} = \xi_{bs}\overrightarrow{n} - \mu_n\overrightarrow{b},$$

$$\frac{\partial \overrightarrow{n}}{\partial b} = \mu_n\overrightarrow{t} + \operatorname{div} n\,\overrightarrow{n},$$

$$\frac{\partial \overrightarrow{b}}{\partial b} = -\xi_{bs}\overrightarrow{t} - \operatorname{div} n\,\overrightarrow{b},$$

Here,

$$\xi_{ns} = \left\langle \frac{\partial \overrightarrow{t}}{\partial n}, \overrightarrow{n} \right\rangle_L, \quad \xi_{bs} = \left\langle \frac{\partial \overrightarrow{t}}{\partial b}, \overrightarrow{b} \right\rangle_L$$

are the normal deformations of the directions $\overrightarrow{n}$ and $\overrightarrow{b}$, respectively. Also, we have

$$\operatorname{div} \overrightarrow{t} = \xi_{ns} + \xi_{bs},$$

$$\operatorname{div} \overrightarrow{n} = \left\langle \overrightarrow{b}, \frac{\partial \overrightarrow{n}}{\partial b} \right\rangle_L,$$

$$\operatorname{div} \overrightarrow{b} = -\kappa + \left\langle \overrightarrow{n}, \frac{\partial \overrightarrow{b}}{\partial n} \right\rangle_L.$$

Then, we obtain

$$\operatorname{curl} \overrightarrow{t} = (\xi_{ns} - \xi_{bs}) \overrightarrow{t} + \kappa \overrightarrow{n},$$

$$\operatorname{curl} \overrightarrow{n} = -\operatorname{div} n \overrightarrow{t} + (\xi_{ns} + \tau) \overrightarrow{n} + \left\langle \frac{\partial \overrightarrow{n}}{\partial b}, \overrightarrow{t} \right\rangle_L \overrightarrow{b},$$

$$\operatorname{curl} \overrightarrow{b} = (\operatorname{div} n + \kappa) \overrightarrow{t} + \left\langle \frac{\partial \overrightarrow{t}}{\partial n}, \overrightarrow{b} \right\rangle_L \overrightarrow{n} + (\tau - \xi_{bs}) \overrightarrow{b}.$$

Abnormalities of $\overrightarrow{t}$, $\overrightarrow{n}$ and $\overrightarrow{b}$ are

$$\mu_s = \left\langle \text{curl } \overrightarrow{t}, \overrightarrow{t} \right\rangle_L = \xi_{ns} - \xi_{bs},$$

$$\mu_n = \langle \text{curl } \overrightarrow{n}, \overrightarrow{n} \rangle_L = \left\langle \frac{\partial \overrightarrow{n}}{\partial b}, \overrightarrow{t} \right\rangle_L,$$

$$\mu_b = \left\langle \text{curl } \overrightarrow{b}, \overrightarrow{b} \right\rangle_L = \left\langle \frac{\partial \overrightarrow{t}}{\partial n}, \overrightarrow{b} \right\rangle_L,$$

respectively. Here $\kappa = \kappa(s, n, b)$ is the curvature function and $\tau = \tau(s, n, b)$ is the torsion function of the pseudo null curve $\Im = \Im(s, n, b)$.

Since curl $\overrightarrow{t}$ does not include any component in the direction of binormal $\overrightarrow{b}$, then there exists a surface which contains both $s-lines$ and $n-lines$. This implies

$$\mu_b = 0.$$

There exists a binormal congruence of surfaces containing the $s-lines$ and $n-lines$ if and only if the abnormality of $\overrightarrow{b}$ is zero. We will investigate one-parameter family of degenerate surfaces $\Gamma = \Gamma(s, n)$ containing the $s-lines$ and $n-lines$ i.e.

$$\frac{\partial \Gamma}{\partial s} = \overrightarrow{t}, \quad \frac{\partial \Gamma}{\partial n} = \overrightarrow{n}.$$

Now, we consider the compatibility conditions. These equations reduces to the nonlinear system, which is called Gauss-Mainardi-Codazzi equations for this binormal congruence of surfaces.

Theorem 4.1

The Gauss-Mainardi-Codazzi equations are obtained as follows:

$$\begin{aligned}
\frac{\partial \mu_b}{\partial s} &= -\mu_b \tau - \operatorname{div} b - 1, \\
\frac{\partial \xi_{ns}}{\partial s} &= \xi_{ns} \tau, \\
\frac{\partial \tau}{\partial n} + \frac{\partial \operatorname{div} b}{\partial s} &= -\xi_{ns}.
\end{aligned}$$

Theorem 4.2

The Gaussian curvature K and mean curvature H of one-parameter family of surfaces $\Gamma = \Gamma(s, n)$ containing the $s-$ lines and $n-$ lines are

$$K = -\xi_{ns}\tau, \quad H = -\frac{\operatorname{div} b + 1}{2},$$

respectively.

Similarly, there exists a normal congruence of surfaces containing the $s-lines$ and $b-lines$ if and only if the abnormality of $\overrightarrow{n}$ is zero, i.e.

$$\mu_n = 0.$$

For details of this case, see (Yavuz and Erdoğdu, 2022).

In this section, general geometric properties of Landau–Lifshitz equation is discussed. The following equation is called Landau–Lifshitz

$$\frac{\partial \overrightarrow{t}}{\partial t} = \overrightarrow{t} \times \frac{\partial^2 \overrightarrow{t}}{\partial s^2}$$

where t is time, s is pseudo arc length parameter and the spacelike tangent vector $\overrightarrow{t}$ of the pseudo null curve $\Im$ is the unit spin vector.

Theorem 5.1

Let $\Im$ be a pseudo null curve lying in three dimensional Minkowski space. Angular velocity vectors of are given as

$$
\begin{aligned}
D^{s} &= \tau \overrightarrow{t} - \kappa \overrightarrow{n}, \\
D^{n} &= -(\operatorname{div} b + 1) \overrightarrow{t} - \mu_{b} \overrightarrow{n} + \xi_{ns} \overrightarrow{b}, \\
D^{b} &= -\operatorname{div} n \overrightarrow{t} + \xi_{bs} \overrightarrow{n} - \mu_{n} \overrightarrow{b},
\end{aligned}
$$

respectively.

Derivative formula of time evolution equation Spin vector $\overrightarrow{t}$ is obtained as

$$\frac{\partial \overrightarrow{t}}{\partial t} = \overrightarrow{t} \times \frac{\partial^2 \overrightarrow{t}}{\partial s^2} = (\frac{\partial \kappa}{\partial s} + \kappa\tau)\overrightarrow{n}.$$

On the other hand, derivative formula of time evolution equation for null principal normal vector $\overrightarrow{n}$ and null binormal vector $\overrightarrow{b}$ are obtained as follows:

$$\frac{\partial \overrightarrow{n}}{\partial t} = D^s \times_L \overrightarrow{n} = \tau \overrightarrow{n}$$

$$\frac{\partial \overrightarrow{b}}{\partial t} = D^s \times_L \overrightarrow{b} = -\kappa \overrightarrow{t} - \tau \overrightarrow{b}.$$

We may also write

$$\frac{\partial}{\partial s}\left(\frac{\partial \overrightarrow{t}}{\partial t}\right) = \left(\frac{\partial^2 \kappa}{\partial s^2} + 2\frac{\partial \kappa}{\partial s}\tau + \frac{\partial \tau}{\partial s}\kappa + \kappa\tau^2\right)\overrightarrow{n}$$
$$\frac{\partial}{\partial t}\left(\frac{\partial \overrightarrow{t}}{\partial s}\right) = \frac{\partial}{\partial t}(\kappa\overrightarrow{n}) = \left(\frac{\partial \kappa}{\partial t} + \kappa\tau\right)\overrightarrow{n}.$$

By the compatibility conditions $\frac{\partial}{\partial s}(\frac{\partial \overrightarrow{t}}{\partial t}) = \frac{\partial}{\partial t}(\frac{\partial \overrightarrow{t}}{\partial s})$, we get

$$\frac{\partial^2 \kappa}{\partial s^2} + 2\frac{\partial \kappa}{\partial s}\tau + \frac{\partial \tau}{\partial s}\kappa + \kappa\tau^2 = \frac{\partial \kappa}{\partial t} + \kappa\tau.$$

On the other hand, we have

$$\frac{\partial}{\partial s}\left(\frac{\partial \overrightarrow{n}}{\partial t}\right) = \frac{\partial}{\partial s}(\tau \overrightarrow{n}) = \left(\frac{\partial \tau}{\partial s} + \tau^2\right)\overrightarrow{n}$$
$$\frac{\partial}{\partial t}\left(\frac{\partial \overrightarrow{n}}{\partial s}\right) = \frac{\partial}{\partial t}(\tau \overrightarrow{n}) = \left(\frac{\partial \tau}{\partial t} + \tau^2\right)\overrightarrow{n}.$$

By the compatibility conditions $\frac{\partial}{\partial s}\left(\frac{\partial \overrightarrow{n}}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \overrightarrow{n}}{\partial s}\right)$, we get

$$\frac{\partial \tau}{\partial s} = \frac{\partial \tau}{\partial t}.$$

Finally, we may write

$$\frac{\partial}{\partial s}\left(\frac{\partial \overrightarrow{b}}{\partial t}\right) = \left(-\frac{\partial \kappa}{\partial s} + \kappa\tau\right)\overrightarrow{t} + \left(-\kappa^2\right)\overrightarrow{n} + \left(-\frac{\partial \tau}{\partial s} + \tau^2\right)\overrightarrow{b}$$

and

$$\frac{\partial}{\partial t}\left(\frac{\partial \overrightarrow{b}}{\partial s}\right) = \left(-\frac{\partial \kappa}{\partial t} + \kappa\tau\right) + \left(-\kappa\tau\right)\overrightarrow{n} + \left(-\frac{\partial \tau}{\partial t} + \tau^2\right)\overrightarrow{b}.$$

Again by compatibility condition $\frac{\partial}{\partial s}(\frac{\partial \overrightarrow{b}}{\partial t}) = \frac{\partial}{\partial t}(\frac{\partial \overrightarrow{b}}{\partial s})$, we get

$$\frac{\partial \kappa}{\partial s} = \frac{\partial \kappa}{\partial t}, \quad \kappa^2 = \kappa\tau.$$

Corollary 5.2

The pseudo null curve $\Im$ is always a straight line or helix, if the tangent vector $\overrightarrow{t}$ satisfies Landau–Lifshitz equation, i.e.

$$\frac{\partial \overrightarrow{t}}{\partial t} = \overrightarrow{t} \times \frac{\partial^2 \overrightarrow{t}}{\partial s^2}.$$

For examle, consider the surface $\Gamma = \Gamma(s, n)$ containing the $s - lines$ and $n - lines$ be given as follows

$$\Gamma(s, n) = (\sinh(s+n) + \cosh(s+n), s, \sinh(s+n) + \cosh(s+n))$$

where

$$\frac{\partial \Gamma}{\partial s} = (\sinh(s+n) + \cosh(s+n), 1, \sinh(s+n) + \cosh(s+n)),$$
$$\frac{\partial \Gamma}{\partial n} = (\sinh(s+n) + \cosh(s+n), 0, \sinh(s+n) + \cosh(s+n)).$$

And binormal vector of one-parameter family of surfaces $\Gamma = \Gamma(s, n)$ is given by

$$\overrightarrow{b}(s,n) = \Big(-\frac{\left(\sinh(s+n)+\cosh(s+n)\right)^2+1}{2\left(\sinh(s+n)+\cosh(s+n)\right)}, -1, \frac{1-\left(\sinh(s+n)+\cosh(s+n)\right)^2}{2\left(\sinh(s+n)+\cosh(s+n)\right)}\Big).$$

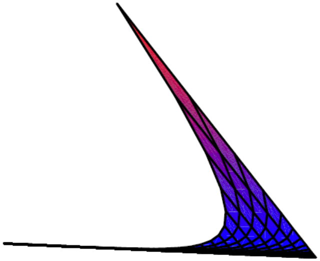
Furthermore, Gaussian and mean curvature of one-parameter family of the surfaces $\Gamma = \Gamma(s, n)$ are

$$K = 0, \quad H = \frac{1}{2}.$$

The Weingarten matrix of the surface $\Gamma = \Gamma(s, n)$ is found as follows

$$\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}.$$

The image of one-parameter family of surface Γ is illustrated in the following figure



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