# A Study of Special Weingarten Hypersurfaces in $\mathbb{R}^4$ with Spherical Foliations

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# Regards form TTU

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# Brief History

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- Aulisa, Gruber, and Toda [2] discussed generalized Willmore energies and their applications. The geometry of hypersurfaces in four-dimensional spaces has also been explored through different constructions, such as the Laplace–Beltrami operator of helicoidal hypersurfaces in $\mathbb{R}^4$ by Güler, Magid, and Yaylı [5].

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- Li, Güler, and Toda [8] analyzed a family of right conoid hypersurfaces with light-like axis in Minkowski four-space.
- Related variational aspects involving Willmore-like energies and elastic curves with potential were also studied by Pámpano [11].
- The geometry of rotational hypersurfaces in pseudo-Euclidean spaces has also been investigated; for example, Güler, Yaylı, and Hacısalihoğlu [6] studied a particular class of space-like rotational hypersurfaces in the pseudo-Euclidean space $\mathbb{E}^4_2$.

# Weingarten Surfaces

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# Weingarten Surfaces

- The study of Weingarten surfaces was first initiated by Weingarten, who investigated special classes of surfaces satisfying relations between their principal curvatures in the three-dimensional space $\mathbb{R}^3$ [12, 13].
- This concept naturally extends to four-dimensional space, where hypersurfaces may satisfy relations among their principal curvatures.

# Weingarten Hypersurfaces

**Definition**

A hypersurface $S$ in the Euclidean space $\mathbb{R}^4$ is called a *Weingarten hypersurface* if its three principal curvatures $\kappa_1, \kappa_2, \kappa_3$ satisfy a functional relation. More precisely, there exists a smooth function $\mathbf{W}$ of three variables such that $\mathbf{W}(\kappa_1, \kappa_2, \kappa_3) = 0$.

- Let $\mathbb{K}_1$, $\mathbb{K}_2$, and $\mathbb{K}_3$ denote the mean curvature, the second curvature, and the Gauss–Kronecker curvature of $S$, respectively. In this case, the relation $\mathbf{W} = 0$ can equivalently be expressed as a relation of the form $\mathbf{U}(\mathbb{K}_1, \mathbb{K}_2, \mathbb{K}_3) = 0$.

# Special Weingarten Hypersurfaces

- In this talk, we restrict our attention to the simplest situation in which the function **U** is linear. Hence, we consider hypersurfaces satisfying

$$a\mathbb{K}_1 + b\mathbb{K}_2 + c\mathbb{K}_3 = d \tag{1}$$

where $a$, $b$, $c$, and $d$ are real constants.

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- Hypersurfaces fulfilling condition (1) will be referred to as *special Weingarten hypersurfaces*, or briefly *SW-hypersurfaces*.
- Important examples arise when one of the curvature quantities is constant, such as hypersurfaces with constant mean curvature ($b = c = 0$), constant second curvature ($a = c = 0$), or constant Gauss–Kronecker curvature ($a = b = 0$).

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- Important examples arise when one of the curvature quantities is constant, such as hypersurfaces with constant mean curvature $(b = c = 0)$, constant second curvature $(a = c = 0)$, or constant Gauss–Kronecker curvature $(a = b = 0)$.
- Despite various studies, a complete classification of Weingarten hypersurfaces in $\mathbb{R}^4$ is still an open problem. For further background on Weingarten surfaces, we refer the reader to [7, 9, 10].

# Cyclic Hypersurfaces

- A natural starting point in the study of hypersurfaces subject to the Weingarten condition is the family of rotational hypersurfaces.

### Definition

A cyclic hypersurface in the Euclidean space $\mathbb{R}^4$ is a hypersurface generated by a smooth one-parameter family of portions of two-spheres.

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**Definition**

A cyclic hypersurface in the Euclidean space $\mathbb{R}^4$ is a hypersurface generated by a smooth one-parameter family of portions of two-spheres.

- For cyclic surfaces in $\mathbb{R}^3$, we refer the reader to Lopez [9].

# Overview

- In this talk, we investigate whether cyclic SW-hypersurfaces exist other than hypersurfaces of revolution. This question is inspired by known results for hypersurfaces with constant mean curvature or constant Gauss curvature. In these situations, the hypersurface is either an open part of a hypersphere or the hyperplanes containing the spherical foliation are parallel.

# Overview

- If the foliation planes are parallel, the hypersurface must either belong to a hypersurface of revolution or to one of the following cyclic hypersurfaces that are not rotational:

1. A class of periodic minimal hypersurfaces, which arises in the case $\mathbb{K}_1 = 0$.

2. A generalized hypercone. In this situation the cyclic hypersurface is formed by spheres whose centres lie along a straight line, while the radius varies linearly. This corresponds to the case $\mathbb{K}_3 = 0$.

# Overview

- Locally, such a hypersurface admits the parametrization

$$\begin{aligned}\mathfrak{x}(u,v,w) &= (f(u), g(u), h(u), u) \\ &\quad + r(u)\,(\cos v \cos w, \sin v \cos w, \sin w, 0)\end{aligned} \tag{2}$$

where $f$, $g$, $h$, and $r > 0$ are linear functions depending on $u$.

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  where $f$, $g$, $h$, and $r > 0$ are linear functions depending on $u$.
- Our first main result shows that for a cyclic SW-hypersurface the hyperplanes that determine the foliation must necessarily be parallel, unless the hypersurface is a hypersphere.

# Notions of Four-Dimension

- We introduce the notation used for the local differential geometry of hypersurfaces in $\mathbb{R}^4$. For further details we refer to [4].

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- Let $S$ be a hypersurface in $\mathbb{R}^4$ and consider a local parametrization $\mathfrak{x} = \mathfrak{x}(u, v, w)$ defined on an open domain in the $(u, v, w)$-parameter space.

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- Let $S$ be a hypersurface in $\mathbb{R}^4$ and consider a local parametrization $\mathfrak{x} = \mathfrak{x}(u, v, w)$ defined on an open domain in the $(u, v, w)$-parameter space.
- Let $\mathbf{N}$ be a unit normal vector field along $S$, defined by

$$\mathbf{N} = \frac{\mathfrak{x}_u \wedge \mathfrak{x}_v \wedge \mathfrak{x}_w}{\|\mathfrak{x}_u \wedge \mathfrak{x}_v \wedge \mathfrak{x}_w\|}$$

where the symbol $\wedge$ represents the triple vector product in $\mathbb{R}^4$.

# Notions of Four-Dimension

- The metric induced by the Euclidean inner product $\langle\cdot,\cdot\rangle$ gives rise to the first fundamental form

$$\begin{aligned} \mathbf{I} &= \langle d\mathfrak{x}, d\mathfrak{x}\rangle \\ &= E\,du^2 + 2F\,du\,dv + G\,dv^2 + 2A\,du\,dw + 2B\,dv\,dw + C\,dw^2. \end{aligned}$$

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- The coefficients of this form are arranged as $(E, F, G)$ and $(A, B, C)$ and are defined by

$$\begin{aligned} (E, F, G) &= \left(\langle \mathfrak{x}_u, \mathfrak{x}_u \rangle, \langle \mathfrak{x}_u, \mathfrak{x}_v \rangle, \langle \mathfrak{x}_v, \mathfrak{x}_v \rangle\right) \\ (A, B, C) &= \left(\langle \mathfrak{x}_u, \mathfrak{x}_w \rangle, \langle \mathfrak{x}_v, \mathfrak{x}_w \rangle, \langle \mathfrak{x}_w, \mathfrak{x}_w \rangle\right). \end{aligned}$$

# Notions of Four-Dimension

- The shape operator associated with the immersion can be described through the second fundamental form

$$\begin{aligned} \mathbf{II} &= -\langle \mathrm{d}\mathbf{N}, \mathrm{d}\mathfrak{x} \rangle \\ &= L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\,\mathrm{d}v + N\,\mathrm{d}v^2 + 2P\,\mathrm{d}u\,\mathrm{d}w + 2T\,\mathrm{d}v\,\mathrm{d}w + V\,\mathrm{d}w^2. \end{aligned}$$

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- The coefficients are grouped as $(L, M, N)$ and $(P, T, V)$ and are given by

$$\begin{aligned} (L, M, N) &= \left(\langle \mathbf{N}, \mathfrak{x}_{uu}\rangle, \langle \mathbf{N}, \mathfrak{x}_{uv}\rangle, \langle \mathbf{N}, \mathfrak{x}_{vv}\rangle\right) \\ (P, T, V) &= \left(\langle \mathbf{N}, \mathfrak{x}_{uw}\rangle, \langle \mathbf{N}, \mathfrak{x}_{vw}\rangle, \langle \mathbf{N}, \mathfrak{x}_{ww}\rangle\right). \end{aligned}$$

# Notions of Four-Dimension

- With respect to the parametrization $\mathfrak{x}$, the mean curvature $\mathbb{K}_1$, can be written as

$$\mathbb{K}_1 = \frac{\begin{pmatrix} \left(EN+GL-2FM\right)C+\left(EG-F^2\right)V \\ -B^2L+2ABM-A^2N \\ +2\left(A\left(FT-GP\right)+B\left(FP-ET\right)\right) \end{pmatrix}}{3\left(\left(EG-F^2\right)C-EB^2+2FAB-GA^2\right)} \tag{3}$$

# Notions of Four-Dimension

- the second curvature $\mathbb{K}_2$

$$\mathbb{K}_2 = \frac{\begin{pmatrix} (EN + GL - 2FM)\, V + \left(LN - M^2\right) C \\ -ET^2 + 2FPT - GP^2 \\ +2\left(A\left(MT - NP\right) + B\left(MP - LT\right)\right) \end{pmatrix}}{3\left(\left(EG - F^2\right) C - EB^2 + 2FAB - GA^2\right)} \tag{4}$$

# Notions of Four-Dimension

- and the Gauss–Kronecker curvature $\mathbb{K}_3$

$$\mathbb{K}_3 = \frac{\left(LN - M^2\right) V - LT^2 + 2MPT - NP^2}{\left(EG - F^2\right) C - EB^2 + 2FAB - GA^2}. \tag{5}$$

# Notions of Four-Dimension

- Let $[\,\cdot,\cdot,\cdot,\cdot\,]$ denote the determinant in $\mathbb{R}^4$ and define

$$\Delta := \det \mathbf{I} = \left(EG - F^2\right) C - EB^2 + 2FAB - GA^2.$$

# Notions of Four-Dimension

- Using this notation, the curvature functions can also be expressed in terms of determinants as

$$\begin{aligned}\mathbb{K}_1 &= \frac{\begin{pmatrix} (E\mathcal{N}+G\mathcal{L}-2F\mathcal{M})\,C+\left(EG-F^2\right)\mathcal{V} \\ -B^2\mathcal{L}+2AB\mathcal{M}-A^2\mathcal{N} \\ +2\,(A\,(F\mathcal{T}-G\mathcal{P})+B\,(F\mathcal{P}-E\mathcal{T})) \end{pmatrix}}{3\Delta^{3/2}} \qquad (6)\\ &=: \frac{K_1}{3\Delta^{3/2}}\end{aligned}$$

# Notions of Four-Dimension

- $$\mathbb{K}_2 = \frac{\begin{pmatrix} \left(E\mathcal{N} + G\mathcal{L} - 2F\mathcal{M}\right)\mathcal{V} + \left(\mathcal{L}\mathcal{N} - \mathcal{M}^2\right)C \\ -E\mathcal{T}^2 + 2F\mathcal{P}\mathcal{T} - G\mathcal{P}^2 \\ +2\left(A\left(\mathcal{M}T - \mathcal{N}\mathcal{P}\right) + B\left(\mathcal{M}\mathcal{P} - \mathcal{L}\mathcal{T}\right)\right) \end{pmatrix}}{3\Delta^2} \tag{7}$$
$$= : \frac{K_2}{3\Delta^2}$$

## Notions of Four-Dimension

- $$\begin{aligned} \mathbb{K}_3 &= \frac{\left(\mathcal{L}\mathcal{N}-\mathcal{M}^2\right)\mathcal{V}-\mathcal{L}\mathcal{T}^2+2\mathcal{M}\mathcal{P}\mathcal{T}-\mathcal{N}\mathcal{P}^2}{\left(EG-F^2\right)C-EB^2+2FAB-GA^2} \\ &=: \frac{K_3}{\Delta^{5/2}}. \end{aligned} \tag{8}$$

# Notions of Four-Dimension

- Here the quantities $\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{P}, \mathcal{T}, \mathcal{V}$ are defined through determinants as

$$\begin{aligned}
\mathcal{L} &= \left[\mathfrak{x}_u, \mathfrak{x}_v, \mathfrak{x}_w, \mathfrak{x}_{uu}\right], \quad \mathcal{P} = \left[\mathfrak{x}_u, \mathfrak{x}_v, \mathfrak{x}_w, \mathfrak{x}_{uw}\right], \\
\mathcal{M} &= \left[\mathfrak{x}_u, \mathfrak{x}_v, \mathfrak{x}_w, \mathfrak{x}_{uv}\right], \quad \mathcal{T} = \left[\mathfrak{x}_u, \mathfrak{x}_v, \mathfrak{x}_w, \mathfrak{x}_{vw}\right], \\
\mathcal{N} &= \left[\mathfrak{x}_u, \mathfrak{x}_v, \mathfrak{x}_w, \mathfrak{x}_{vv}\right], \quad \mathcal{V} = \left[\mathfrak{x}_u, \mathfrak{x}_v, \mathfrak{x}_w, \mathfrak{x}_{ww}\right].
\end{aligned}$$

# Cyclic Parametrization

- For the cyclic hypersurface given in $(2)$, it is convenient to separate the dependence on the spherical parameters from the variable $u$.

# Cyclic Parametrization

- We introduce the notation

$$n(v, w) := (\cos v \cos w,\ \sin v \cos w,\ \sin w) \in \mathbb{S}^2 \subset \mathbb{R}^3$$

which represents the outward unit normal direction on the unit 2-sphere expressed in spherical coordinates.

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- Throughout this section we use the notation $f_u = f'(u)$ and $f_{uu} = f''(u)$ (and similarly for $g, h, r$). Accordingly, define

$$p(u) := (f'(u), g'(u), h'(u)), \qquad q(u) := (f''(u), g''(u), h''(u)).$$

# Cyclic Parametrization

- Next, consider the scalar functions

$$\begin{aligned}\beta(u,v,w) &:= \langle p(u), n(v,w)\rangle + r'(u)\\ &= f'(u)\cos v\cos w + g'(u)\sin v\cos w\\ &+ h'(u)\sin w + r'(u)\\ \alpha(u,v,w) &:= \langle q(u), n(v,w)\rangle + r''(u)\\ &= f''(u)\cos v\cos w + g''(u)\sin v\cos w\\ &+ h''(u)\sin w + r''(u)\end{aligned}$$

and set

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- Then, define

$$W(u,v,w) := 1 + \beta(u,v,w)^2.$$

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- Then, define

$$W(u,v,w) := 1 + \beta(u,v,w)^2.$$

- Finally, the determinant of the first fundamental form is

$$\det \mathbf{I} = r^4 W \cos^2 w.$$

# Curvatures

- Next, we determine the curvature functions of the hypersurface given by (2).

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- For convenience, and in order to clarify the terms that are implicit in the compressed notation, the curvature functions can be rewritten in terms of the quantities $\alpha$, $\beta$, $\sigma$, and $W$ as follows

$$\begin{aligned}
\mathbb{K}_1 &= \frac{1}{3rW^{3/2}}\left(r\alpha - \beta^2 - \sigma - 2r_u\beta + r_u^2 - 2\right) \\
\mathbb{K}_2 &= \frac{1}{3r^2W^2}\left(-2r\alpha + \sigma - r_u^2 + 2r_u\beta + 1\right) \\
\mathbb{K}_3 &= \frac{1}{r^2W^{5/2}}\alpha
\end{aligned}$$

# Curvatures

- where

$$\begin{aligned}
\alpha &= f_{uu}c_v c_w + g_{uu}s_v c_w + h_{uu}s_w + r_{uu}\\
\beta &= f_u c_v c_w + g_u s_v c_w + h_u s_w + r_u\\
\sigma &= f_u^2 + g_u^2 + h_u^2\\
W &= 1 + \beta^2.
\end{aligned}$$

# Weingarten Condition

- Here we use the notation $\mathbb{K}_1 = H$ for the mean curvature and $\mathbb{K}_3 = K$ for the Gauss–Kronecker curvature. In addition, we set $c_v = \cos v$, $c_w = \cos w$, $s_v = \sin v$, and $s_w = \sin w$.

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- By using (6), (7), and (8), the Weingarten condition for a SW-hypersurface can be written as

$$a\frac{K_1}{3\Delta^{3/2}} + b\frac{K_2}{3\Delta^2} + c\frac{K_3}{\Delta^{5/2}} = d.$$

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$$a\frac{K_1}{3\Delta^{3/2}} + b\frac{K_2}{3\Delta^2} + c\frac{K_3}{\Delta^{5/2}} = d.$$

- Rearranging the above relation gives

$$a\Delta K_1 + 3cK_3 = \Delta^{1/2}\left(3d\Delta^2 - bK_2\right).$$

# Weingarten Condition

- Squaring both sides leads to

$$a^2\Delta^2 K_1^2 + 6ac\Delta K_1 K_3 + 9c^2 K_3^2 - \Delta\left(3d\Delta^2 - bK_2\right)^2 = 0. \qquad (9)$$

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- From a trigonometric viewpoint, substituting the expressions of the curvatures into (9), it can be written as

$$\mathcal{A}_{0,0} + \sum_{j,k=1}^{10}\left[\mathcal{A}_{j,k}(u)\cos(jv+kw) + \mathcal{B}_{j,k}(u)\sin(jv+kw)\right] = 0$$

where $u \in I$ and $v, w \in [0, 2\pi)$.

# Weingarten Condition

- Assuming $a = 1$, we obtain $\mathcal{A}_{j,k}(u) = 0$ and $\mathcal{B}_{j,k}(u) = 0$ whenever $j, k > 8$. In particular, the highest-order coefficients $\mathcal{A}_{8,8}$ and $\mathcal{B}_{8,8}$ are given by

$$\begin{aligned}\mathcal{A}_{8,8} &= \frac{1}{2^{15}} r^2 \left(f'^4 + 4f'^3 g' - 6f'^2 g'^2 - 4f' g'^3 + g'^4\right) \\ &\quad \times \left(f'^4 - 4f'^3 g' - 6f'^2 g'^2 + 4f' g'^3 + g'^4\right)\end{aligned}$$

$$\begin{aligned}\mathcal{B}_{8,8} &= \frac{1}{2^{12}} r^2 f' g' \left(f'^2 - g'^2\right) \\ &\quad \times \left(f'^2 - 2f' g' - g'^2\right) \left(f'^2 + 2f' g' - g'^2\right).\end{aligned}$$

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- We generalize the Weingarten relation known for surfaces [12, 13] to the hypersurface given by (2).

# Weingarten Condition

Consider the hypersurface

$$\mathfrak{x}(u,v,w) = (f(u), g(u), h(u), u) + r(u)\,(\cos v \cos w, \sin v \cos w, \sin w, 0)\,.$$

**Theorem**. Let $\mathbb{K}_1$, $\mathbb{K}_2$, and $\mathbb{K}_3$ denote its mean, second, and Gauss–Kronecker curvatures, respectively. Then these curvature functions satisfy the Weingarten condition

$$\det\left(\frac{\partial(\mathbb{K}_1, \mathbb{K}_2, \mathbb{K}_3)}{\partial(u,v,w)}\right) = 0$$

# Weingarten Condition

**Theorem (continue)** if and only if the following partial differential equation is satisfied

$$
\begin{aligned}
0 \;=\; & -2(f_u c_v c_w + g_u s_v c_w + h_u s_w + r_u)c_w \\
& \times (f_u^2(-1 + c_v^2 c_w^2) + g_u^2(-1 + s_v^2 c_w^2) + h_u^2\left(-1 + s_w^2\right) \\
& - \left(2 f_u g_u s_v c_v c_w + 2 f_u h_u c_v s_w + 2 g_u h_u s_v s_w\right) c_w - 2 r_u^2) \\
& \times (h_u (f_{uu} s_v c_w - g_{uu} c_v c_w) \\
& + g_u(-f_{uu} s_w + h_{uu} c_v c_w) + f_u(g_{uu} s_w - h_{uu} s_v c_w)) \\
& \times (f_u f_{uu} + g_u g_{uu} + h_u h_{uu} \\
& + \left(f_u c_v c_w + g_u s_v c_w + h_u s_w + 2 r_u\right) r_{uu})
\end{aligned} \tag{10}
$$

where $r \neq 0$, $c_v = \cos v$, $c_w = \cos w$, $s_v = \sin v$, and $s_w = \sin w$.

# Explicit Examples

- In this section we provide several explicit examples of cyclic hypersurfaces of the form

$$\begin{aligned} \mathfrak{x}(u,v,w) &= (f(u), g(u), h(u), u) \\ &\quad + r(u)\left(\cos v \cos w, \sin v \cos w, \sin w, 0\right) \end{aligned}$$

which illustrate the geometric configurations discussed in the previous sections.

# Example 1 (A nontrivial cyclic hypersurface)

- Consider the functions

$$f(u) = \cos u, \quad g(u) = \sin u, \quad h(u) = \frac{1}{2}u, \quad r(u) = 1 + \frac{1}{4}\sin u.$$

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- Then the parametrization becomes

$$\begin{aligned} \mathfrak{x}(u, v, w) &= \left(\cos u, \sin u, \frac{1}{2}u, u\right) \\ &\quad + \left(1 + \frac{1}{4}\sin u\right)(\cos v \cos w, \sin v \cos w, \sin w, 0) . \end{aligned}$$

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- In this case, the centers of the spheres follow a spatial curve, while the radius varies with the parameter $u$. As a result, the generated hypersurface constitutes a nontrivial cyclic example. Its projection onto the $X_1X_2X_3$-space is illustrated in Figure 1 (left), where the geometric configuration can be observed in three dimensions.

# Example 2 (A hypersurface of revolution)

- Assume that

$$f(u) = g(u) = h(u) = 0, \quad r(u) = r(u).$$

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- In this situation the centers of the spheres lie on the $X_4$-axis, and the resulting hypersurface possesses rotational symmetry. The projection onto the $X_1X_3X_4$-space is depicted in Figure 1 (right), illustrating the geometry with respect to the $X_4$-axis.

# Figure

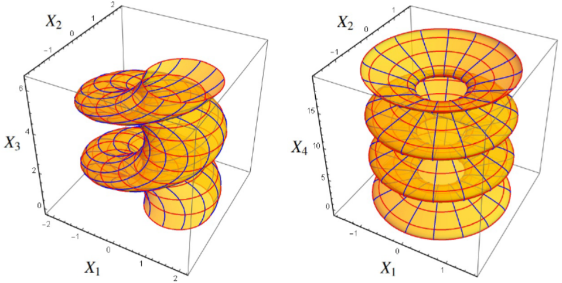


Figure 1. Projection of the surface from Example 1 onto $X_1X_2X_3$-space (left), from Example 2 onto $X_1X_3X_4$-space (right).

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# Thank You

Thank You Very Much!