Non-Commutative (NC) Integrable Systems (Part 1)

← NC spaces

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Sponsored by Asahipen Hikari Foundation
← paint company

§1 Introduction

Non-Commutative (NC) Space :

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \quad \cdots (*) \qquad (\theta^{\mu\nu}: \text{NC parameter, real})$$

Three merits :

① Resolution of singularity

② description of bkg magnetic fields

③ easy treatments

§1 Introduction

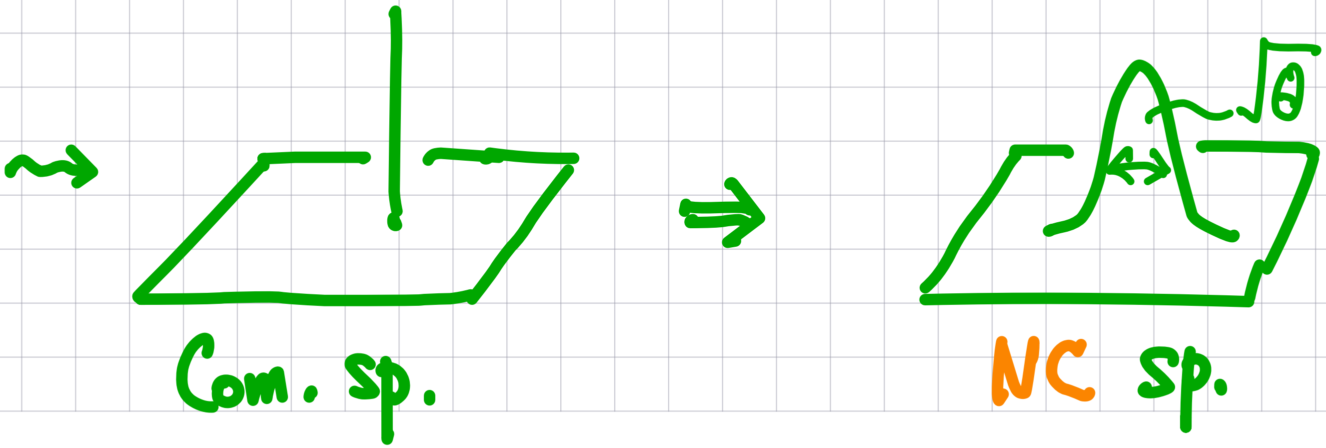
Non-Commutative (NC) Space :

$$[x^\mu, x^\nu] = i\,\theta^{\mu\nu} \quad \cdots (*) \qquad (\theta^{\mu\nu} : \text{NC parameter, real})$$

Three merits :

① **Resolution of singularity**

☺ (*) looks like CCR in QM ⇝ ("space-space uncertainty")



§1 Introduction

Non-Commutative (NC) Space :

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Three merits :

① Resolution of singularity

② description of bkg magnetic fields

③ easy treatments

due to $\begin{cases} \text{the merit ①} \\ \text{Quasideterminant descriptions} \end{cases}$

Focusing on :

- NC instantons (to explain ① & ②)
- NC KdV eqs. (to mention ③)
- Quasideterminants (")

Today

Tomorrow

Starting with Anti-Self-Dual Yang-Mills eqs.

(via Ward conjecture) (= ASDYM)

Plan of Talk (Today)

§1 Introduction (5 min)

§2 NC integrable systems (8 min)

§3 ADHM Construction of instantons (10 ~ 15 min)

§4 NC KdV equations (0 ~ 10 min)

tomorrow?

§2. NC Integrable Systems

ASDYM eq. (in 4 dim. $\mathbb{R}^4$, G: gauge gp.)

$$\underbrace{*}_{\text{Hodge dual}} F_{\mu\nu} = -F_{\mu\nu} \qquad \underbrace{F_{\mu\nu}}_{\text{field strength}} = \partial_\mu \underbrace{A_\nu}_{\text{gauge field}} - \partial_\nu A_\mu + \underbrace{[A_\mu, A_\nu]}_{\text{commutator}}$$

$\mu, \nu \in \{1, 2, 3, 4\}$

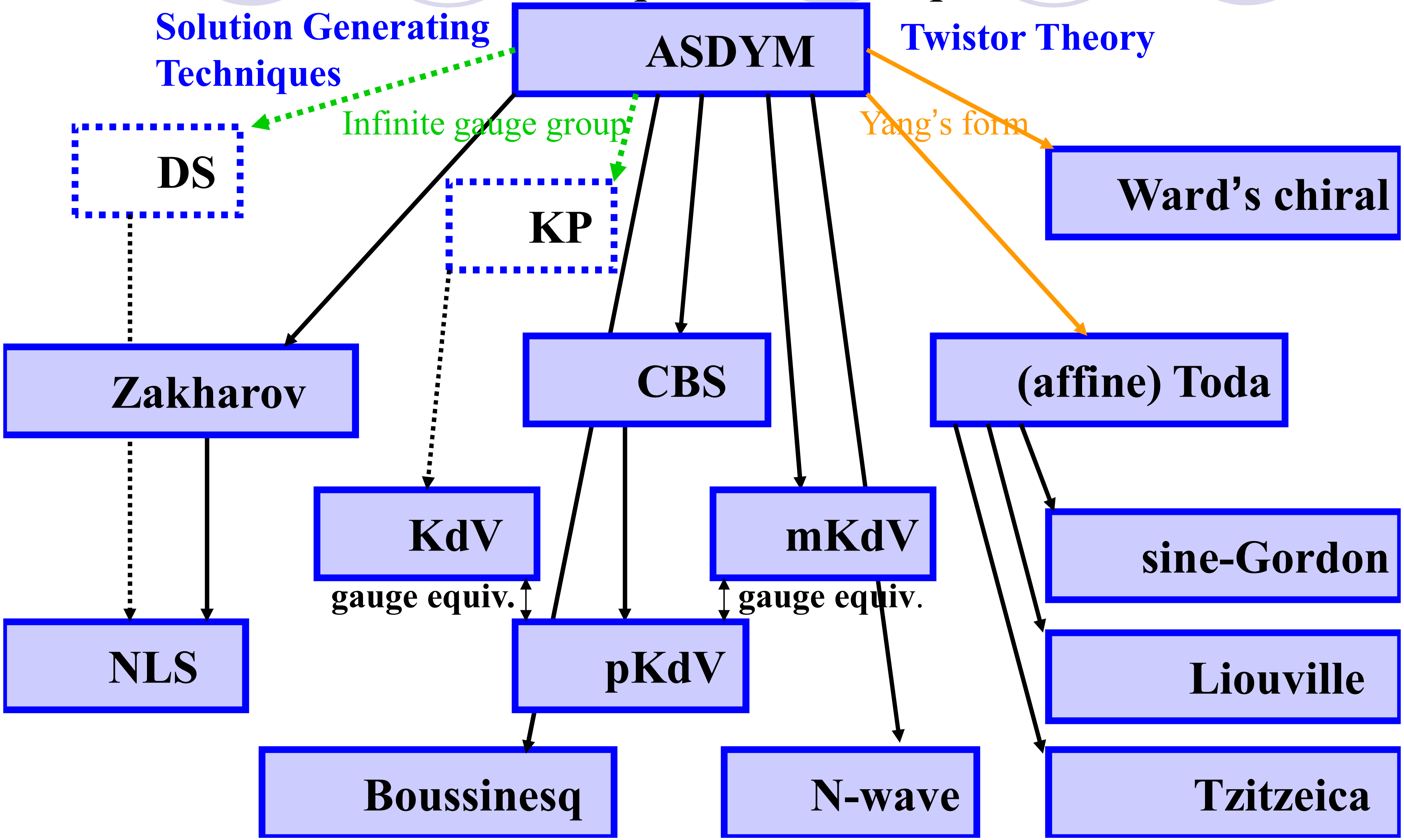
$$\Leftrightarrow \quad F_{12} = -F_{34}, \quad F_{13} = -F_{42}, \quad F_{14} = -F_{23}$$

Note A_μ, $F_{\mu\nu}$ take value in $\mathfrak{g}$ (Lie alg. of G)

e.g. $G = U(N)$, A_μ, $F_{\mu\nu}$ are $N \times N$ anti-hermitian.

Ward's conjecture: Many (perhaps all?) integrable equations are reductions of the ASDYM eqs.

ASDYM eq. is a master eq. !



Reduction to KdV from ASDYM $(G=SL(2,\mathbb{C}))$

ASDYM : $F_{zw}=0,\ F_{\tilde z\tilde w}=0,\ F_{z\tilde z}-F_{w\tilde w}=0$

$\downarrow$

① $\partial_w-\partial_{\tilde w}=0,\ \partial_{\tilde z}=0$ (dim. reduction)

② $A_{\tilde w}=\begin{pmatrix}0&0\\ \frac{u}{2}&0\end{pmatrix},\ A_{\tilde z}=\begin{pmatrix}0&0\\ 1&0\end{pmatrix},\ A_w=\begin{pmatrix}0&-1\\ u&0\end{pmatrix}$

$A_z=\frac{1}{4}\begin{pmatrix}u' & -2u\\ u''+2u^2 & -u'\end{pmatrix}$ $\qquad u=u(z,\underbrace{x}_{w+\tilde w})$

$\qquad u'=\partial_x u$

$u_{\underset{t}{z}}-\frac{1}{4}u_{xxx}-\frac{3}{2}uu_x=0$: **KdV eq.!**

(t,x) are real $\Rightarrow$ not $(\underset{(E)}{++++})$ but $(\underset{(U)}{++--})$

Reduction to NLS from ASDYM $(G=SL(2,\mathbb{C}))$

ASDYM : $F_{zw}=0,\ F_{\tilde{z}\tilde{w}}=0,\ F_{z\tilde{z}}-F_{w\tilde{w}}=0$

① $\partial_w-\partial_{\tilde{w}}=0,\ \partial_{\tilde{z}}=0$ (dim. reduction)

② $A_{\tilde{w}}=O,\ A_{\tilde{z}}=\frac{i}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ A_w=\begin{pmatrix}0&\psi\\ \varepsilon\bar{\psi}&0\end{pmatrix}$ $\qquad \varepsilon=\pm1$

$$A_z=i\varepsilon\begin{pmatrix}-|\psi|^2 & -\varepsilon\psi'\\ \bar{\psi}' & |\psi|^2\end{pmatrix}$$

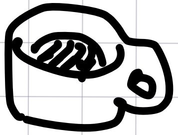
$\psi=\psi(z,\underset{\sim}{x})$ $\quad \underset{\sim}{w+\tilde{w}}$

$\psi'=\partial_x\psi$

$$i\psi_z-\psi_{xx}+2\varepsilon\psi^2\bar{\psi}=0$$: **NLS eq.!**

t

(t,x) are real $\Rightarrow$ not $(++++)$ (E) but $(++--)$ (U)

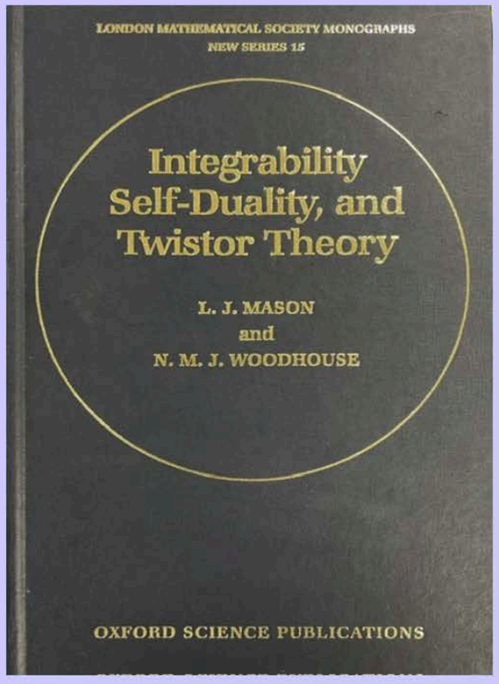


TEA BREAK (annoncement) Welcome!

We'll have a workshop on ASDYM & Twistor:

Integrability, Self-Duality, and Twistor Theory

27〜28 August, 2026 at Nagoya University and Online



This workshop is co-sponsored by Daiko Foundation and ...

(Q) How to obtain NC theory from a com. theory?

(A) NC field eqs. are defined from commutative field eqs. by replacement of all products with Moyal products

$$f(x) \star g(x) := f(x)\, e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\,\overrightarrow{\partial_\nu}}\, g(x)$$

c-number

$$= f(x)\cdot g(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f \cdot \partial_\nu g + O(\theta^2)$$

$$[x^\mu, x^\nu]_\star := x^\mu \star x^\nu - x^\nu \star x^\mu = x^\mu \cdot x^\nu + \frac{i}{2}\theta^{\mu\nu} - \left(x^\nu \cdot x^\mu + \frac{i}{2}\theta^{\nu\mu}\right) = i\theta^{\mu\nu}$$

§2. NC Integrable Systems

NC ASDYM eq. (in 4 dim., G: gauge gp.)

$$*F^\star_{\mu\nu} = -F^\star_{\mu\nu} \qquad F^\star_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_\star$$

must be kept even for $G = U(1)$

$$\mu, \nu \in \{1, 2, 3, 4\}$$

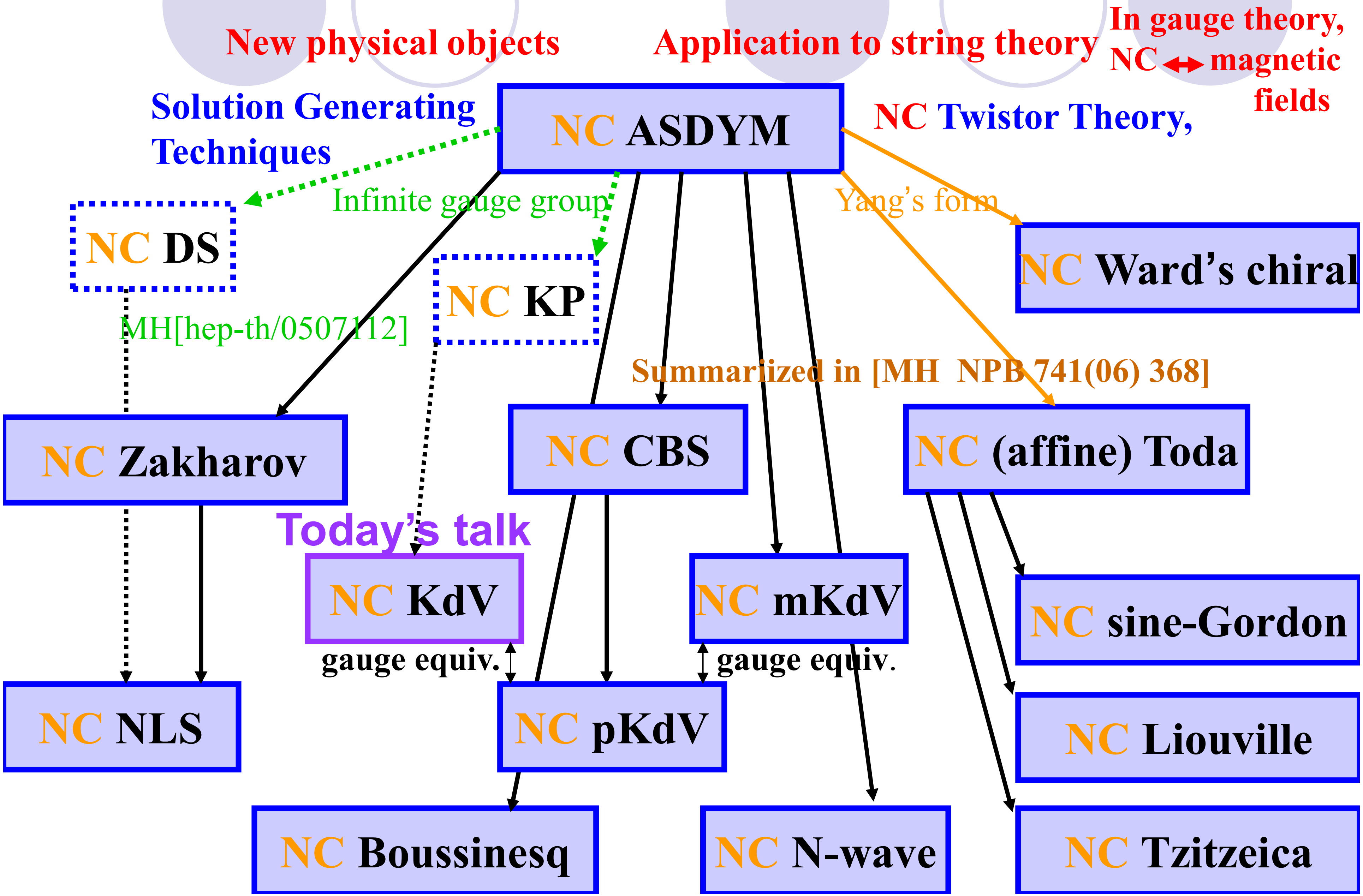
$$\Leftrightarrow \quad F^\star_{12} = -F^\star_{34}, \quad F^\star_{13} = -F^\star_{42}, \quad F^\star_{14} = -F^\star_{23}$$

Note A_μ, $F_{\mu\nu}$ take value in $\mathfrak{g}$ (Lie alg. of G)

e.g. $G = U(N)$, A_μ, $F_{\mu\nu}$ are $N \times N$ anti-hermitian.

NC Ward's conjecture: Many (perhaps all?) NC integrable eqs are reductions of the NC ASDYM eqs.

MH &K.Toda, PLA316 (03)77 [hepth/0211148]



Reduction to KdV from NC ASDYM $(G = SL(2,\mathbb{C}))$

NC ASDYM : $F^{\star}_{zw} = 0,\ F^{\star}_{\tilde{z}\tilde{w}} = 0,\ F^{\star}_{z\tilde{z}} - F^{\star}_{w\tilde{w}} = 0$

① $\partial_w - \partial_{\tilde{w}} = 0,\ \partial_{\tilde{z}} = 0$ (dim. reduction)

② $A_{\tilde{w}} = \begin{pmatrix} 0 & 0 \\ \frac{u}{2} & 0 \end{pmatrix},\ A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ A_w = \begin{pmatrix} 0 & -1 \\ u & 0 \end{pmatrix}$

$A_z = \frac{1}{4}\begin{pmatrix} u' & -2u \\ u'' + 2u^2 & -u' \end{pmatrix}$

$u = u(z, \underset{\sim}{x}),\quad x = \underbrace{w + \tilde{w}}$

$u' = \partial_x u$

$\underset{t}{u_z} - \frac{1}{4}u_{xxx} - \frac{3}{4}\underbrace{(u \star u_x + u_x \star u)}_{\text{Symmetric}} = 0$: NC KdV eq.!

Reduction to $\overset{\text{NC}}{\text{NLS}}$ from $\overset{\text{NC}}{\text{ASDYM}}$ ($G = GL(2,\mathbb{C})$)

NC ASDYM : $F_{zw} = 0,\ F_{\tilde{z}\tilde{w}} = 0,\ F_{z\tilde{z}} - F_{w\tilde{w}} = 0$

① $\partial_w - \partial_{\tilde{w}} = 0,\ \partial_{\tilde{z}} = 0$ (dim. reduction)

② $A_{\tilde{w}} = 0,\ A_{\tilde{z}} = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ A_w = \begin{pmatrix} 0 & \psi \\ \varepsilon\bar{\psi} & 0 \end{pmatrix}$ $\quad \varepsilon = \pm 1$

$$A_z = i\varepsilon \begin{pmatrix} -\bar{\psi} \star \psi & -\varepsilon\psi' \\ \bar{\psi}' & \psi \star \bar{\psi} \end{pmatrix} \qquad \begin{array}{l} \psi = \psi(z, x^{w+\tilde{w}}) \\ \psi' = \partial_x \psi \end{array}$$

$$i\psi_z - \psi_{xx} + 2\varepsilon\, \psi \star \bar{\psi} \star \psi = 0$$

: NC NLS eq. !

z = t

★ trace part ($U(1)$ part in gauge gp) is crucial !

- NC Bogomol'nyi eq. $B_i = D_i \Phi$ (products are all Moyal)

- NC KP eq. $$u_t = u_{xxx} + u_x u + u u_x + \partial_x^{-1} u_{yy} - [u, \partial_x^{-1} u_y]$$

- NC Zakhalov system $(\varepsilon = \pm 1)$

$$i\psi_t = \psi_{xy} - \varepsilon \psi \partial_x^{-1} \partial_y (\bar{\psi}\psi) - \varepsilon \partial_x^{-1} \partial_y (\psi\bar{\psi})\psi$$

- NC Bogoyavlenskii-Calogero-Schiff eq.

$$u_t = u_{xxy} + u u_y + u_y u + u_y \partial_x^{-1} u_y + (\partial_x^{-1} u_y) u_y + \partial_x^{-1}[u, \partial_x^{-1}[u, \partial_x^{-1} u_y]]$$

- NC Davey-Stewartson eq.

$$\begin{cases} 2\kappa q_t = (q_{xx} + \kappa^2 q_{yy}) + R_1 q - q R_2 \\ 2\kappa r_t = -(r_{xx} + \kappa^2 r_{yy}) + R_2 r - r R_1 \end{cases} \quad \text{w/} \quad \begin{cases} R_{1,z} = -(qr)_{\bar{z}} \\ R_{2,\bar{z}} = (rq)_z \end{cases}$$

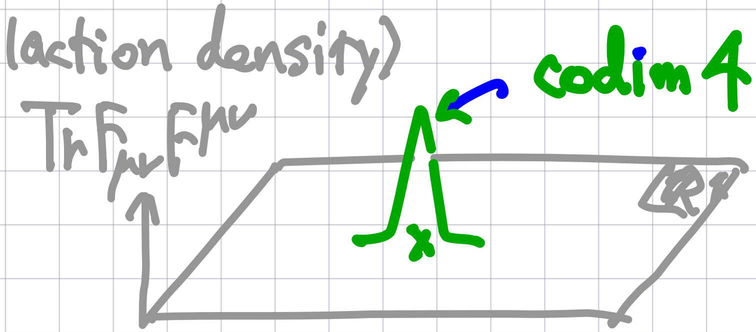
- NC KdV eq. $u_t = u_{xxx} + u u_x + u_x u$
- NC Boussinesq. $u_{tt} = u_{xxx} + (u^2)_x - [u, \partial_x^{-1} u_t]_x$
- NC NLS eq. $i \psi_t = \psi_{xx} - \varepsilon \psi \bar{\psi} \psi$
- NC mKdV eq. $v_t = v_{xxx} - v^2 v_x - v_x v^2$
- NC Sawada-Kotera eq. summarized in [H, hep-th/0601209, H, hep-th/0507112]

$$u_t + u_{5x} + u_{xxx} u + u_{xx} u_x + u u_x u = 0$$

- NC Toda, sine-Gordon, Tzitzéica, ...

§3 ADHM Construction of Instantons

Instantons are finite-action solutions of ASDYM



codim 4 solitons in 4-dim = instantons

ADHM construction is a powerful method

- to give any instantons
- based on 1:1 correspondence between

instanton moduli sp. & ADHM moduli sp.

$G = U(N)$ k-instanton

A_μ: $N\times N$ $\quad \frac{1}{8\pi^2}\int F\wedge F = k$

sol. of ASDYM eq.

(mod $U(N)$ gauge trf.)

$\longleftrightarrow$ 1:1

"$G = U(k)$" ADHM data

$B_{1,2}$, I, J

$k\times k$, $k\times N$, $N\times k$

sol. of ADHM eq.

(mod "$U(k)$ gauge trf")

$$*F_{\mu\nu} = -F_{\mu\nu}$$

(PDE in 4dim)

$$\begin{cases} [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + I I^\dagger - J^\dagger J = 0 \\ [B_1, B_2] + IJ = 0 \end{cases}$$

(Matrix eq.)

Difficult

easy

[Atiyah-Drinfeld-Hitchin-Manin, PLA65('78)185]

$G = U(2)$ 1-instanton

$$*F_{\mu\nu} = -F_{\mu\nu}$$

$\longleftrightarrow$ 1:1

"$G = U(1)$" ADHM data

$$\begin{array}{ccc} B_{1,2}, & I, & J \\ k\times k & k\times N & N\times k \\ \shortparallel & \shortparallel & \shortparallel \\ 1\times 1 & 1\times 2 & 2\times 1 \end{array}$$

$$\begin{cases} \underbrace{[B_1, B_1^\dagger] + [B_2, B_2^\dagger]} + II^\dagger - J^\dagger J = 0 & k\times k \\ \underbrace{[B_1, B_2]} + IJ = 0 & k\times k \end{cases}$$

dropped $\quad k=1$

Difficult

easy

$G = U(2)$ 1-instanton

$*F_{\mu\nu} = -F_{\mu\nu}$

$\longleftrightarrow$

1:1

"$G = U(1)$" ADHM data

$B_{1,2}$ = arbitrary

$$\begin{cases} II^\dagger - J^\dagger J = 0 \\ IJ = 0 \end{cases}$$

$\Downarrow$

$$\begin{cases} I = \begin{pmatrix} \rho & 0 \end{pmatrix} \\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix} \end{cases} \qquad \rho \in \mathbb{R}$$

Difficult

easy

$G = U(2)$ 1-instanton

$$*F_{\mu\nu} = -F_{\mu\nu}$$

$\longleftrightarrow$ 1:1

"$G = U(1)$" ADHM data

$$B_{1,2} = \alpha_{1,2} \in \mathbb{C}$$

$$\begin{cases} II^\dagger - J^\dagger J = 0 \\ IJ = 0 \end{cases}$$

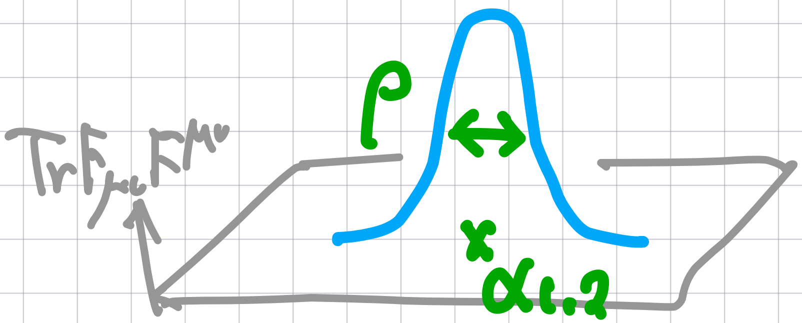
$$\Downarrow$$

$$\begin{cases} I = \begin{pmatrix} \rho & 0 \end{pmatrix} \\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix} \end{cases} \quad \rho \in \mathbb{R}$$

$\longleftarrow$

$$A_\mu = \frac{i(x-b)^\nu \eta^{(-)a}_{\mu\nu} \sigma^a}{|z-\alpha|^2 + \rho^2}$$

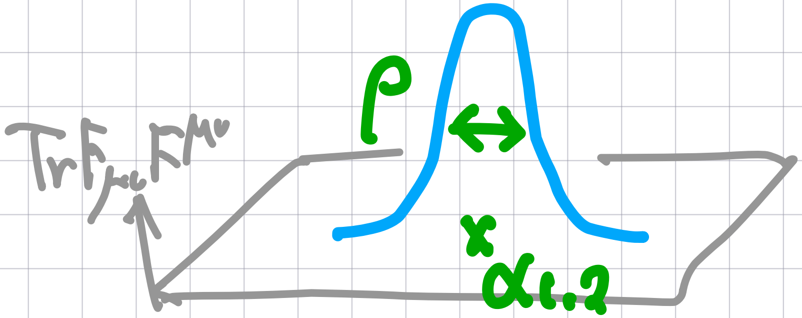
BPST instanton!



$G = U(2)$ 1-instanton

"$G = U(1)$" ADHM data

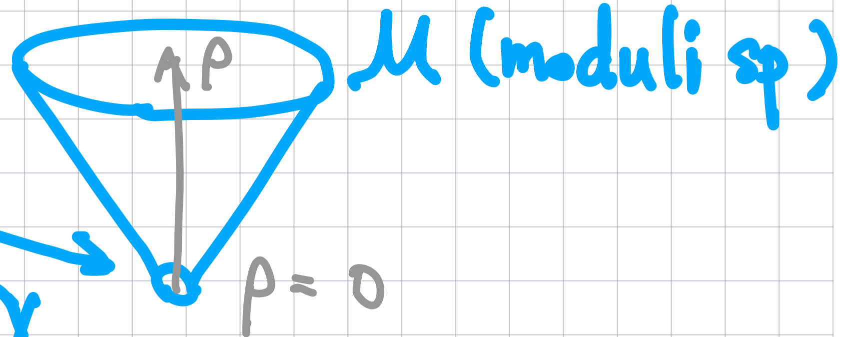
$$\begin{cases} II^\dagger - J^\dagger J = 0 \\ IJ = 0 \end{cases}$$



$B_{1,2} = \alpha_{1,2} \in \mathbb{C}$

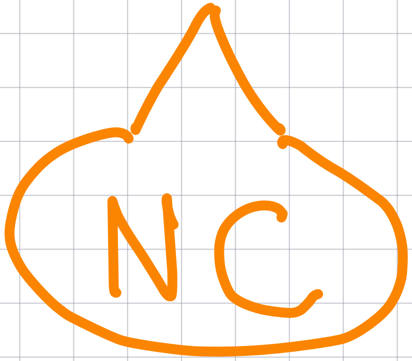
$I = \begin{pmatrix} \rho & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$

· Let's take $\rho \to 0$ limit:



small instanton singularity

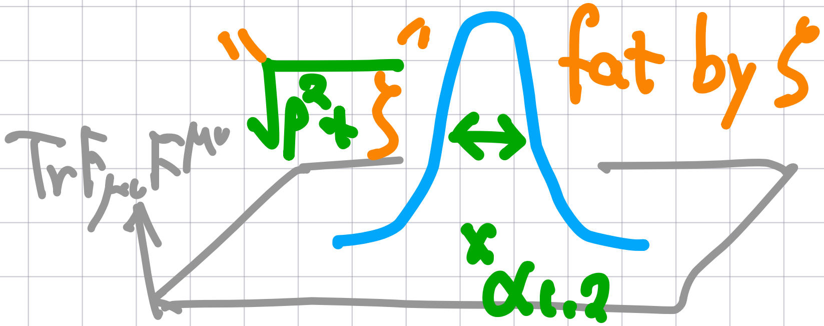
$G=U(2)$ 1-instanton



"$G=U(1)$" ADHM data

$[B_1, B_1^\dagger] + [z_1, \bar{z}_1] + \cdots$

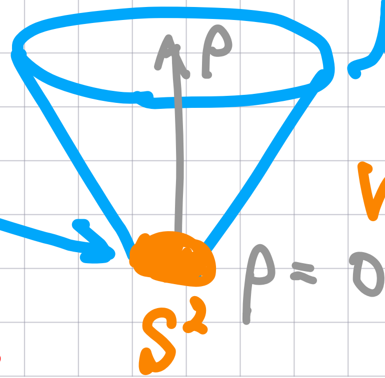
$$\begin{cases} II^\dagger - J^\dagger J = \zeta \ (>0) \\ IJ = 0 \end{cases}$$



$B_{1,2} = \alpha_{1,2} \in \mathbb{C}$

$I = \boxed{\sqrt{\rho^2+\zeta} \;\; 0}$, $\quad J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$

- Even in $\rho \to 0$ limit, configuration is smooth (U(1) instanton) special to NC space!



$\mathcal{M}$ (moduli sp)

resolution of singularity

D-brane interpretation = kD0-ND4 BPS branes

$G=U(2)$ 1-instanton

$$*F_{\mu\nu} = -F_{\mu\nu}$$

$\longleftrightarrow$ 1:1 (trivial!?)

"$G=U(1)$" ADHM data

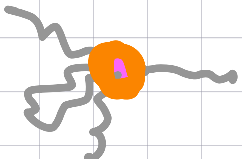
$B_{1,2}$ = arbitrary

$$\begin{cases} II^\dagger - J^\dagger J = 0 \\ IJ = 0 \end{cases}$$

BPS condition = ASDYM

$2 = N$ D4-branes

$1 = k$ D0-branes



open strings

BPS condition = ADHM eq

[Douglas, hep-th/9512077]
[Witten, hep-th/9511030]

D-brane interpretation = k D0-N D4 BPS branes

[Nekrasov-Schwarz CMP 198 (98) 689]

$G = U(2)$ 1-instanton

$$*F_{\mu\nu} = -F_{\mu\nu}$$

$\longleftrightarrow$ 1:1 (trivial!?)

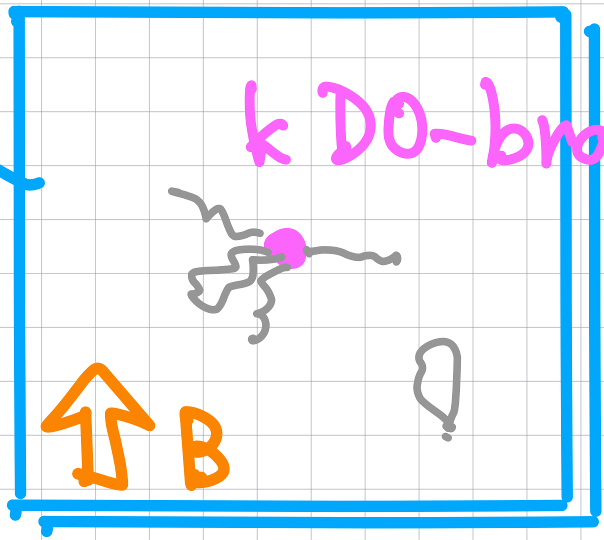
"$G = U(1)$" ADHM data

$B_{1,2}$ = arbitrary

$$\begin{cases} II^\dagger - J^\dagger J = \zeta \\ IJ = 0 \end{cases}$$

ζ: FI param.

BPS condition = ASDYM



k D0-branes

N D4-branes

BPS condition = ADHM eq

turn on bkg B-field

D-brane interpretation = k D0-N D4 BPS branes

$G=U(2)$ 1-instanton

$$*F_{\mu\nu} = -F_{\mu\nu}$$

$\longleftrightarrow$

1:1

(trivial!?)

"$G=U(1)$" ADHM data

$B_{1,2}$ = arbitrary

$$\begin{cases} II^\dagger - J^\dagger J = \zeta \\ IJ = 0 \end{cases}$$

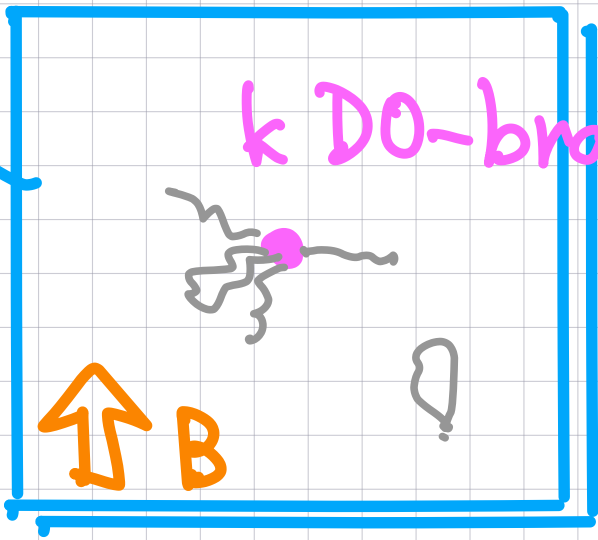
$\uparrow$ FI param.

BPS condition

‖

ASDYM

N D4-branes



BPS condition

‖

ADHM eq

turn on bkg B-field

§4 Noncommutative KdV eqs.

(products are all Moyal)

NC Korteweg-de Vries (KdV) eq.

$$u_t + u_{xxx} + 3(u u_x + u_x u) = 0$$

has infinite conserved densities :

Strachan's product

$$\sigma_n = \text{coef}_{-1} \tilde{L}^n - 3\theta\left((\text{coef}_{-1} \tilde{L}^n) \diamond u_3' + (\text{coef}_{-2} \tilde{L}^n) \diamond u_2'\right)$$

$$\text{w/ } \tilde{L} = \sqrt{\partial_x^2 + u} = \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + \cdots$$

$$\text{coef}_{-k} \tilde{L}^n = \left(\cdots + \underbrace{\text{ⅢⅢ}}_{\text{this one}} \partial^{-k} + \cdots = \tilde{L}^n\right)$$

[H. hep-th/0311206]