# Non-Commutative (NC) Integrable Systems (Part 2)

↖ NC spaces

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9 June 2026 in Varna

Sponsored by Asahipen Hikari Foundation

↖ paint company

# Plan of Talk

§1 Introduction (5 min)

§2 NC integrable systems (8 min)

§3 ADHM Construction of instantons (15 min)

Yesterday

Today

§4 NC KdV eq. & Quasideterminants (15 min) 20 ~

- Integrability ($\infty$ conserved quantities, Quasidet. solutions)

§5 ASDYM revisited (5~10 min)

~ Towards Unification of Integrable Systems

- Codim 1 (KP-type) Solitons in ASDYM

# §4 NC KdV eqs. & Quasideterminants

## NC Korteweg-de Vries (KdV) eq.

$$4u_t + u_{xxx} + 3(u \star u_x + u_x \star u) = 0$$

**Moyal products**

$$f(x) \star g(x) := f(x)\, e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}}\, g(x)$$

c-number

$$= f(x)\cdot g(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f \cdot \partial_\nu g + \mathcal{O}(\theta^2)$$

$$[x^\mu, x^\nu]_\star := x^\mu \star x^\nu - x^\nu \star x^\mu = x^\mu \cdot x^\nu + \frac{i}{2}\theta^{\mu\nu} - \left(x^\nu \cdot x^\mu + \frac{i}{2}\theta^{\nu\mu}\right) = i\theta^{\mu\nu}$$

# NC KdV eq. has infinite conserved densities:

$$4u_t + u_{xxx} + 3(u u_x + u_x u) = 0 \quad \begin{pmatrix}\text{products are}\\ \text{all Moyal}\end{pmatrix}$$

$[t, x] = i\theta$

$$\sigma_n = \mathrm{coef}_{-1}\, \tilde{L}^n - 3\theta\Big( (\mathrm{coef}_{-1}\tilde{L}^n) \diamond u_3' + (\mathrm{coef}_{-2}\tilde{L}^n) \diamond u_2' \Big)$$

Strachan's product (comm. nonassoc.)

$$\text{w/ } \tilde{L} = \sqrt{\partial_x^2 + u} = \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + \cdots$$

$$\mathrm{coef}_{-k}\, \tilde{L}^n = \big(\cdots + ⊘\, \partial^{-k} + \cdots = \tilde{L}^n\big)$$

this one

[H. hep-th/0311206]

NC KP/GD hierarchies

- suggests infinite symmetry
- Significance is not clarified

# Lax representation

spectral parameter

$$\begin{cases} Lf = 0 \\ Mf = 0 \end{cases} \cdots \text{①} \quad \text{w/} \quad \begin{cases} L := \partial_x^2 + u - \zeta \\ M := \partial_t + 4\partial_x^3 + 6u\,\partial_x + 3u_x \end{cases}$$

· Compatible condition of ① $[L, M] = 0 \iff$ KdV eq.

# Darboux transformation

$\zeta = \lambda$ (fix)

Prepare a special sol. of ① : $\begin{cases} L\theta = 0 \\ M\theta = 0 \end{cases}$ $\quad \theta = f(\lambda)$

The following trf. leaves ① as it is :

$$(\mathrm{D}) \quad \begin{cases} L \mapsto \tilde{L} = G_\theta L G_\theta^{-1} \\ M \mapsto \tilde{M} = G_\theta M G_\theta^{-1} \\ f \mapsto \tilde{f} = G_\theta f \end{cases} \qquad \begin{aligned} G_\theta &:= \theta\,\partial_x\,\theta^{-1} \cdots \text{②} \\ &= \partial_x - \theta_x\,\theta^{-1} \end{aligned}$$

The Darboux trf (D) induces $\tilde{u} = u + 2(\theta_x \theta^{-1})_x$

n-iterations of (D) from a trivial seed sol.

$$\begin{cases} (\partial_x^2 - \lambda_k)\theta_k = 0 \\ (\partial_t + 4\partial_x^3)\theta_k = 0 \end{cases} \quad k \in \{1, \cdots, n\}$$

$$u_n = -2\partial_x \begin{vmatrix} \theta_1 & \cdots & \theta_n & 0 \\ \theta_1' & \cdots & \theta_n' & 0 \\ \vdots & & \vdots & \vdots \\ & & & 0 \\ \theta_1^{(n-1)} & \cdots & \theta_n^{(n-1)} & 1 \\ \theta_1^{(n)} & \cdots & \theta_n^{(n)} & \boxed{0} \end{vmatrix}$$

derivative

Quasideterminant

squares

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} := d - CA^{-1}B \xrightarrow[d:1\times 1]{\text{commutative limit}} \frac{\begin{vmatrix} A & B \\ C & d \end{vmatrix}}{|A|}$$

(Schur complement)

The Darboux trf (D) induces $\tilde{u} = u + 2(\theta_x \theta^{-1})_x$

n-iterations of (D) from a trivial seed sol.

$$\begin{cases} (\partial_x^2 - \lambda_k)\theta_k = 0 \\ (\partial_t + 4\partial_x^3)\theta_k = 0 \end{cases}$$

$k \in \{1, \cdots, n\}$

$$u_n = -2\partial_x \begin{vmatrix} \theta_1 & \cdots & \theta_n & 0 \\ \theta_1' & \cdots & \theta_n' & 0 \\ \vdots & & \vdots & \vdots \\ & & & 0 \\ \theta_1^{(n-1)} & \cdots & \theta_n^{(n-1)} & 1 \\ \theta_1^{(n)} & \cdots & \theta_n^{(n)} & \boxed{0} \end{vmatrix}$$

$$\xrightarrow[\text{limit}]{\text{com.}} 2\partial_x \frac{\partial_x Wr(\theta_1, \cdots, \theta_n)}{Wr(\theta_1, \cdots, \theta_n)} = 2\partial_x^2 \log \overbrace{Wr(\theta_1, \cdots, \theta_n)}^{\tau\text{-fcn}}$$

det

Hirota trf.

# Quasideterminant

- Not just a NC generalization
- But rather relates to inverse matrix

**Def** [Gelfand-Retakh, 1991]

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}, \quad Y = X^{-1} = \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{n1} & \cdots & y_{nn} \end{pmatrix}$$

$$\Rightarrow \quad |X|_{ij} := y_{ji}^{-1} \qquad \left( \underset{\text{limit}}{\overset{\text{com.}}{=}} \frac{1}{\det X^{ij}} \cdot \det X \right)$$

$(i,j)$-th quasideterminant

## Examples (recursively obtained)

$n=1)$ $\quad |X|_{11} = x_{11}$

**Recall**: $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow X^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & (C-DBA)^{-1} \\ (B-AC^{-1}D)^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$

$\left[ \text{New symbol} : |X|_{ij} \equiv \begin{pmatrix} & \vdots & \\ \cdots & \boxed{x_{ij}} & \cdots \\ & \vdots & \end{pmatrix} \right]$

$n=2)$ $\quad \begin{pmatrix} \boxed{x_{11}} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = x_{11} - x_{12}\, x_{22}^{-1}\, x_{21}\,,$

$\begin{pmatrix} x_{11} & \boxed{x_{12}} \\ x_{21} & x_{22} \end{pmatrix} = x_{12} - x_{11}\, x_{21}^{-1}\, x_{22}\,, \quad \cdots$

$$\left[\text{New symbol: } |X|_{ij} \equiv \begin{pmatrix} & \vdots & \\ \cdots & \boxed{x_{ij}} & \cdots \\ & \vdots & \end{pmatrix}\right]$$

**n = 1)** $|X|_{11} = x_{11}$

**Recall:** $X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \Rightarrow X^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & (C-DBA)^{-1} \\ (B-AC^{-1}D)^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$

**n = 2)**

$$\begin{pmatrix} \boxed{x_{11}} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = x_{11} - x_{12}\, x_{22}^{-1}\, x_{21} ,$$

$$\begin{pmatrix} x_{11} & \boxed{x_{12}} \\ x_{21} & x_{22} \end{pmatrix} = x_{12} - x_{11}\, x_{21}^{-1}\, x_{22} , \quad \cdots$$

**n = 3)**

$$\begin{array}{ccc} A & & B \\ & \left(\begin{array}{c|cc} \boxed{x_{11}} & x_{12} & x_{13} \\ \hline x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{31} \end{array}\right) & \\ C & & D \end{array} = x_{11} - (x_{12}\ x_{13}) \underbrace{\begin{pmatrix} x_{22} & x_{23} \\ x_{32} & x_{31} \end{pmatrix}}_{\text{can be written by 2x2 results}}{}^{-1} \begin{pmatrix} x_{21} \\ x_{31} \end{pmatrix}$$

# Some formula

**Prop**

$k=1,\cdots,n-1$

$$\begin{vmatrix} a_{11} & \cdots & a_{k1}\lambda & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{1n} & \cdots & a_{kn}\lambda & \cdots & \boxed{a_{nn}} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{k1} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{1n} & \cdots & a_{kn} & \cdots & \boxed{a_{nn}} \end{vmatrix}$$

**NC Jacobi id.**

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} = \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix}$$

Very powerful!

**Homological rel.**

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & \boxed{h} & i \end{vmatrix} = \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} \begin{vmatrix} A & B & C \\ D & f & g \\ 0 & \boxed{0} & 1 \end{vmatrix}$$

# n-soliton solution:

$$u_n = -2\partial_x \begin{vmatrix} \theta_1 & \cdots & \theta_n & 0 \\ \theta_1' & \cdots & \theta_n' & 0 \\ \vdots & & \vdots & \vdots \\ & & & 0 \\ \theta_1^{(n-1)} & \cdots & \theta_n^{(n-1)} & 1 \\ \theta_1^{(n)} & \cdots & \theta_n^{(n)} & \boxed{0} \end{vmatrix}$$

$$\begin{cases} (\partial_x^2 - \lambda_k)\,\theta_k = 0 \\ (\partial_t + 4\partial_x^3)\,\theta_k = 0 \end{cases}$$

$$k \in \{1, \cdots, n\}$$

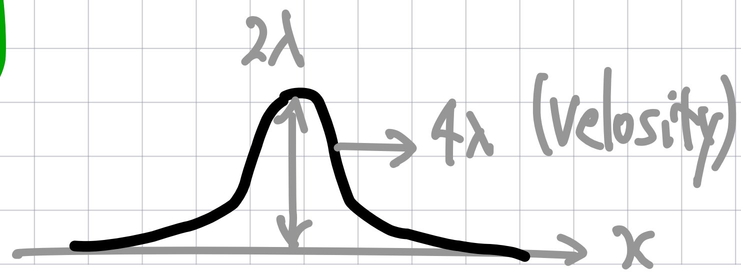
$$\theta_k = \exp\left(\sqrt{\lambda_k}\,(x - 4\lambda_k t)\right) + \exp\left(-\sqrt{\lambda_k}\,(x - 4\lambda_k t)\right)$$

**(n = 1)**

$\operatorname{sech} x := \dfrac{1}{\cosh x}$

$$u(t,x) = 2\lambda \operatorname{sech}^2\left(\sqrt{\lambda}\,x - 4\sqrt{\lambda}^3\, t\right)$$

height of water surface



# Soliton Scattering

[H, hep-th/0610006.]
[H, arXiv: 1806.05188]

<u>Bad News</u> when $f(x)$ and $g(x)$ are real, $f(x) \star g(x)$ is not in general

The solutions could be real in some cases:

(i) 1-soliton solutions are the same as commutative ones because of

$$f(x-vt) \star g(x-vt) = f(x-vt) \cdot g(x-vt) \quad \forall v$$

$[t, x] = i\theta$

(ii) N-soliton solutions in the asymptotic region

# Ride on a comoving frame with I-th soliton:

$$\tilde{x} := x - 4\lambda_I t \; : \; \text{finite}$$

$$x - 4\lambda_k t = \underbrace{x - 4\lambda_I t}_{\tilde{x}} + 4(\lambda_I - \lambda_k)t$$

$$\downarrow t \to \pm\infty$$

$$\pm\infty$$

Assume $0 < \lambda_1 < \cdots < \lambda_I < \cdots < \lambda_n$

$$\theta_k \xrightarrow[t\to\infty]{} \begin{cases} e^{-\sqrt{\lambda_k}(x-4\lambda_k)t} & k<I \\ e^{+\sqrt{\lambda_k}(x-4\lambda_k)t} + e^{-\sqrt{\lambda_k}(x-4\lambda_k)t} & k=I \\ e^{+\sqrt{\lambda_k}(x-4\lambda_k)t} & k>I \end{cases}$$

$$u \xrightarrow{t\to\infty} \begin{vmatrix} 1 & \cdots & 1 & e^{\sqrt{\lambda_I}(x-\lambda_I t)}+e^{-\sqrt{\lambda_I}(x-\lambda_I t)} & \cdots & 1 & 0 \\ (-\lambda_1^{\frac{1}{2}}) & & (-\lambda_{I-1}^{\frac{1}{2}}) & \sqrt{\lambda_I}\left(e^{\sqrt{\lambda_I}(x-\lambda_I t)}-e^{-\sqrt{\lambda_I}(x-\lambda_I t)}\right) & & (\lambda_n^{\frac{1}{2}}) & 0 \\ (-\lambda_1^{\frac{1}{2}})^2 & & (-\lambda_{I-1}^{\frac{1}{2}})^2 & \vdots & & (\lambda_n^{\frac{1}{2}})^2 & \vdots \\ \vdots & & \vdots & & & \vdots & 0 \\ & & & & & & 1 \\ (-\lambda_1^{\frac{1}{2}})^n & \cdots & (-\lambda_{I-1}^{\frac{1}{2}})^n & \underbrace{\sqrt{\lambda_I}^{\,n}\left( \qquad\qquad \right)}_{} & & (\lambda_n^{\frac{1}{2}})^n & \boxed{0} \end{vmatrix}$$

← I forgot "4"...

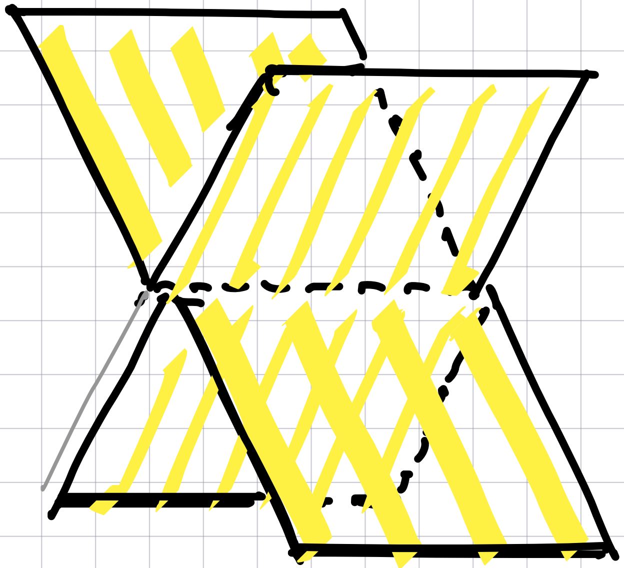
x-dependence is here only "$f(x-\lambda_I t)$"

⇒ In $t\to\pm\infty$ (asymptotic region), Moyal products reduce to ordinary products (coincides with commutative ones)

NC n-soliton solution (asymptotically)

= "non-linear superposition" of n 1-solitons

(with the same phase shifts as commutative case)



# §5 ASDYM revisited

We present soliton solutions of ASDYM whose codim is one (not 4 : instantons)

TMP 122, 284
[Nimmo, Gilson, Ohta, '00]

These are described in terms of

quasi-Wronskian generated by Darboux trf.

or

quasi-Grammian " binary Darboux trf.

In KP theory, $\tau$-functions are Wronskians (= Grammians)

Here we show the Grammian solutions.

ASDYM eq. $F_{zw}=0,\ F_{\tilde{z}\tilde{w}}=0,\ F_{z\tilde{z}}-F_{w\tilde{w}}=0$

has two potential forms:

- J-matrix form (Yang's eq.)

$$\partial_{\tilde{z}}\left((\partial_z J)J^{-1}\right)-\partial_{\tilde{w}}\left((\partial_w J)J^{-1}\right)=0 \quad \cdots ①$$

- K-matrix form (Newman's eq.)

$$\partial_z\partial_{\tilde{z}}K-\partial_w\partial_{\tilde{w}}K+[\partial_w K,\partial_z K]=0 \quad \cdots ②$$

- There exist actions whose EoM are ① or ②
- Choose: $z=x^1+x^3,\ w=x^2+x^4,\ \cdots\in\mathbb{R} \Rightarrow (++--)$

# 4-dim WZW model (J-matrix)

[Donaldson '85]

**Action:** $S_{4dWZW} = S_\sigma + S_{WZ}$ $\quad J(x) \in GL(N,\mathbb{C})$

$$S_\sigma = \underline{\frac{i}{4\pi}} \int_{\mathbb{U}} \underbrace{\omega \wedge \mathrm{Tr}\left[(\partial J)J^{-1} \wedge (\tilde{\partial} J)J^{-1}\right]}_{\mathcal{L}_\sigma}$$

$$S_{WZ} = -\underline{\frac{i}{12\pi}} \int_{\mathbb{U}} \underbrace{A \wedge \mathrm{Tr}\left[(dJ)J^{-1}\right]^3}_{\mathcal{L}_{WZ}}$$

$(z, w, \tilde{z}, \tilde{w})$: local coords of $\mathbb{U}$

w/ $\omega = dA$ : Kähler form of $M_4$

$\mathbb{U}$ : $z = x^1 + x^3,\ w = x^2 + x^4, \cdots$ $\qquad \omega = \frac{i}{2}(dz \wedge d\tilde{z} - dw \wedge d\tilde{w})$

$$d = \partial + \tilde{\partial},\quad \partial = dw\,\partial_w + dz\,\partial_z,\quad \tilde{\partial} = d\tilde{w}\,\partial_{\tilde{w}} + d\tilde{z}\,\partial_{\tilde{z}}$$

**EoM:** $\partial_{\tilde{z}}\left((\partial_z J)J^{-1}\right) - \partial_{\tilde{w}}\left((\partial_w J)J^{-1}\right) = 0$ Yang eq. (= ASDYM)

# n-soliton sols. for $G = U(2)$:

PRD 111, 086023 [Li-H.-Huang-Zhang '25]

$$\underset{2\times 2}{J} = \begin{vmatrix} \Omega(\theta,\theta) & \Lambda^{-1}\theta^{\dagger} \\ \theta & \boxed{1_2} \end{vmatrix}$$

← quasi-Grammian type (One binary Darboux trf.)

$$\underset{2\times n}{\theta} = \begin{pmatrix} a_1^2 e^{\xi_1} & a_2^2 e^{\xi_2} & \cdots & a_n^2 e^{\xi_n} \\ b_1^2 e^{-\xi_1} & b_2^2 e^{-\xi_2} & & b_n^2 e^{-\xi_n} \end{pmatrix}, \quad \underset{n\times n}{\Lambda} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

$$\xi_k = \underbrace{\lambda_k \alpha_k z + \beta_k \tilde{z} + \lambda_k \beta_k w + \alpha_k \tilde{w}}_{\text{linear in spacial coords}}$$

sols. of $\theta_z = \theta_{\tilde{w}}\Lambda$, $\theta_w = \theta_{\tilde{z}}\Lambda$

($\Omega$: sol. of $\Lambda^{\dagger}\Omega - \Omega\Lambda = \theta^{\dagger}\theta$)

## ※ Quasideterminant (Schur complement)

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} := d - CA^{-1}B \qquad (d, A: \text{square matrices})$$

# Action Density $\mathcal{L}_{\mathrm{4dWZW}} = \mathcal{L}_\sigma + \mathcal{L}_{\mathrm{WZ}}$

$\operatorname{sech} x = \dfrac{1}{\cosh x}$

## One soliton (on $\mathbb{U}$)

$\propto (\lambda - \bar{\lambda})^3$

$$\mathcal{L}_\sigma = \frac{1}{8\pi} d_{11} \operatorname{sech}^2 X$$

$$\mathcal{L}_{\mathrm{WZ}} \equiv 0 \quad \text{(identically)}$$

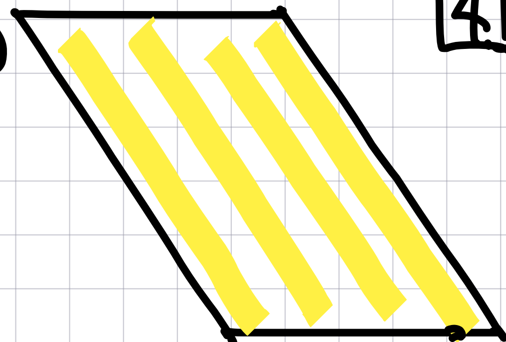
実

$X := \xi + \bar{\xi}$ : linear in $x^\mu$

peak

$X = 0$

4D



3-dim hyperplane (codim 1)

not instanton!

Similar!

## cf. KP soliton

$$u = 2\partial_x^2 \log\left(e^X + e^{-X}\right) \propto \operatorname{sech}^2 X$$

linear in t, x, y

# Two Soliton (on III)

$X_k = \xi_k + \bar{\xi}_k$, $\Theta_{12} = \Theta_1 - \Theta_2$

$i\Theta_k = \xi_k - \bar{\xi}_k$

$$\mathcal{L}_a = \frac{\left[ A\cosh^2 X_1 + B\cosh^2 X_2 + C_\pm \cosh^2\left(\frac{X_1 + X_2 \pm i\Theta_{12}}{2}\right) + D_\pm \cosh^2\left(\frac{X_1 - X_2 \pm i\Theta_{12}}{2}\right)\right]}{2\pi\left(a\cosh(X_1+X_2) + b\cosh(X_1-X_2) + c\cos\Theta_{12}\right)^2}$$

$$\xrightarrow[X_1:\text{const}]{r\to\infty} \propto \operatorname{sech}^2(X_1 \pm \delta)$$

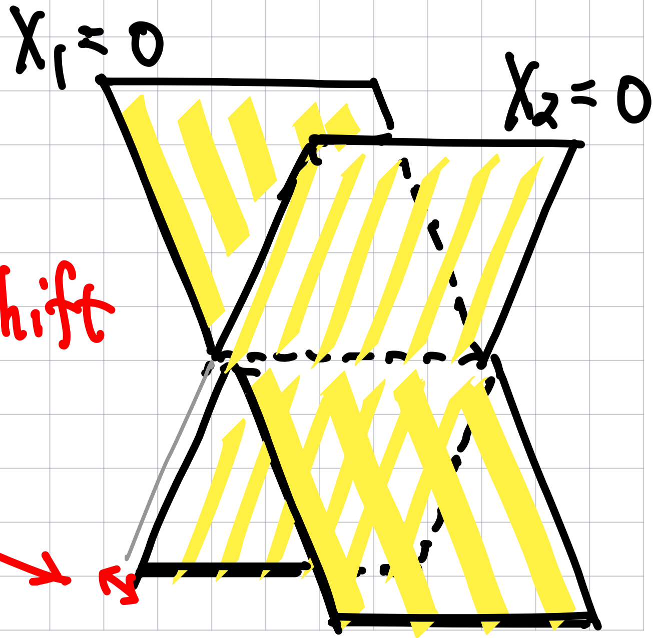
non-singular

$$\xrightarrow[X_2:\text{const}]{r\to\infty} \propto \operatorname{sech}^2(X_2 \pm \delta)$$

$$\xrightarrow[\text{otherwise}]{r\to\infty} 0$$

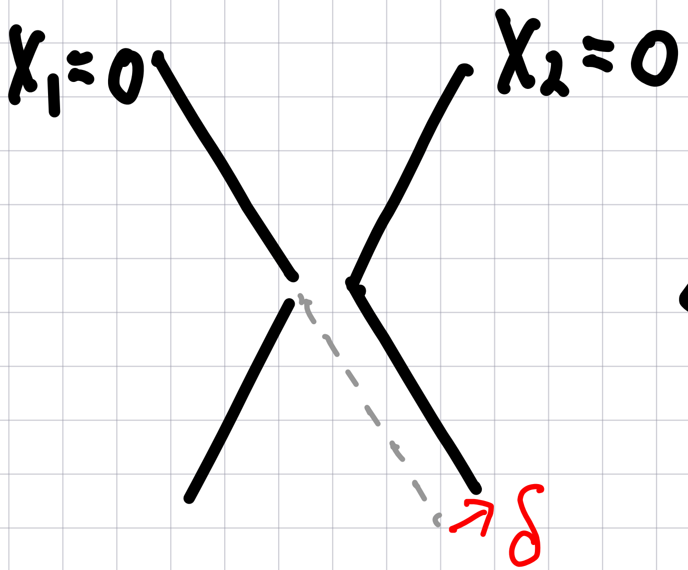
phase shift

(non-linear effect)

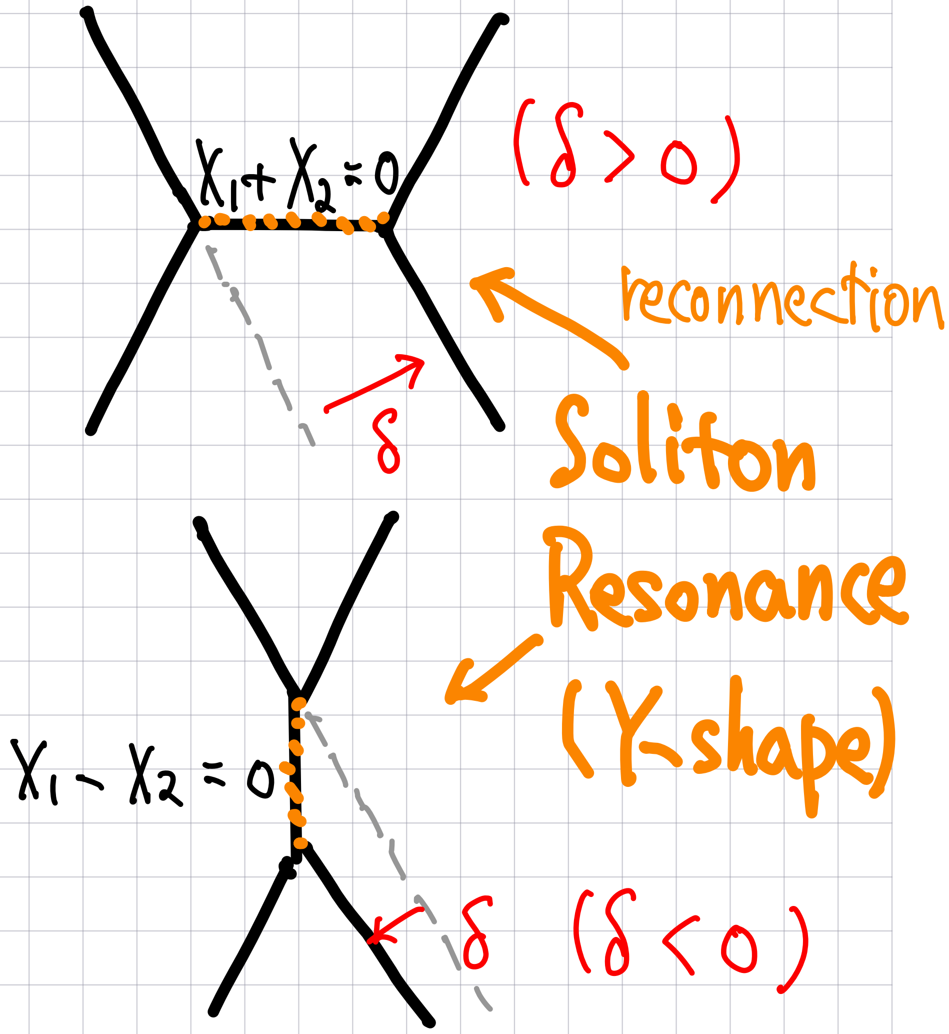



$\mathcal{L}_{WZ} = \begin{pmatrix} \text{very} \\ \text{long} \end{pmatrix}$

# ※Origin of Phase shift (断層) (KP case)




$\Leftrightarrow$

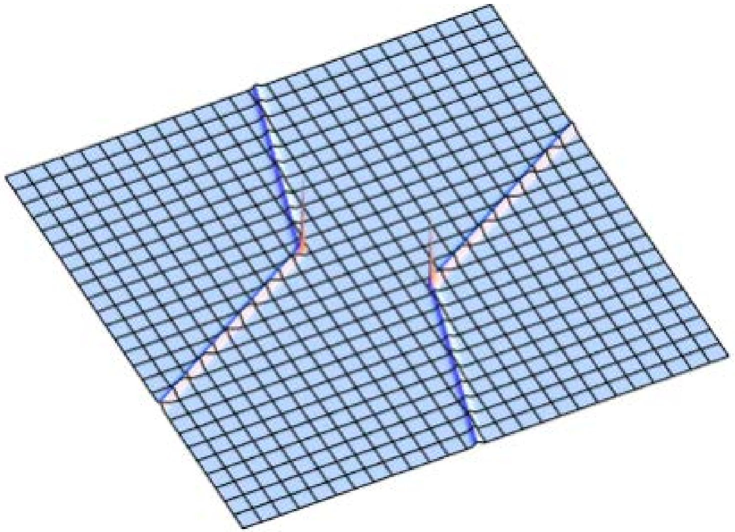



"Intermediate States" appear in large $\delta$

(Seen at shallow water waves)

[Book of Yuji Kodama]

(i) large $\delta$ ($>0$)



$$\delta = \log \frac{|\lambda_1 - \lambda_2|}{|\lambda_1 - \bar{\lambda}_2|}$$

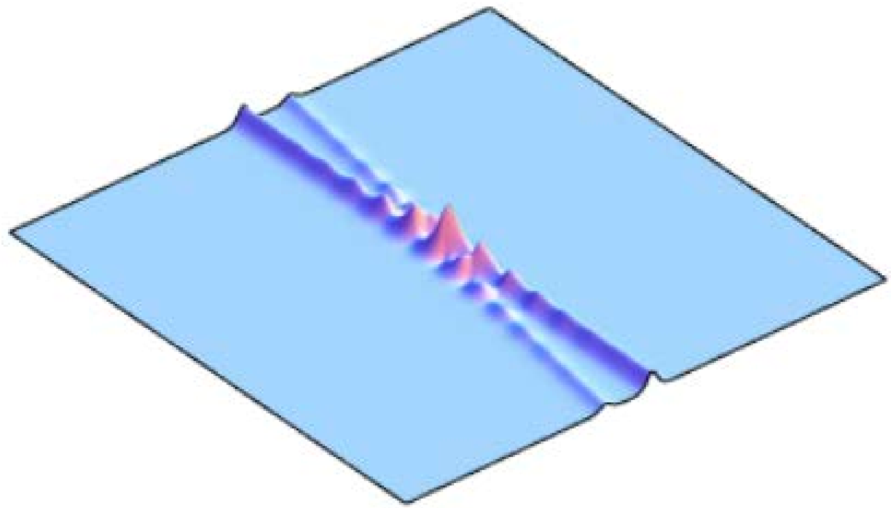
Not 2 Y-shaped soliton

But 2 V-shaped soliton! (new)

(Exists in 2+1 dim CBS eq) [Yu-Toda-Sasa-Fukuyama '98]

## (ii) large $\delta$ $(<0)$

with $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ $\left(\begin{array}{c}\Leftrightarrow \\ \xi_1 \to \xi_2\end{array}\right)$



Double pole solution (resonance)

※ Can be constructed directly from

$$\Theta = \begin{pmatrix} \xi e^{\xi} & e^{\xi} \\ -\xi e^{-\xi} & e^{-\xi} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$$

Jordan form

# Towards Unification of Integrable Systems

4d ASDYM $\underset{(++--)}{\mathbb{U}}$

$\downarrow$

various integrable eqs.

(KdV NLS, Toda, $\cdots$)

[Ward, Mason-Woodhouse ⋯]

# Towards Unification of Integrable Systems

4d Chern-Simons (CS)

↓

various

solvable models

(spin chains, PCM, ⋯ )

[ Costello - Yamazaki - Witten, ⋯ ]
(2017)

4d ASDYM $\underset{(++--)}{\text{on}}$

↓

various

integrable eqs.

( KdV NLS, Toda, ⋯ )

# Towards Unification of Integrable Systems

6d CS [Bittleston–Skinner, 2023] JHEP 02, 227

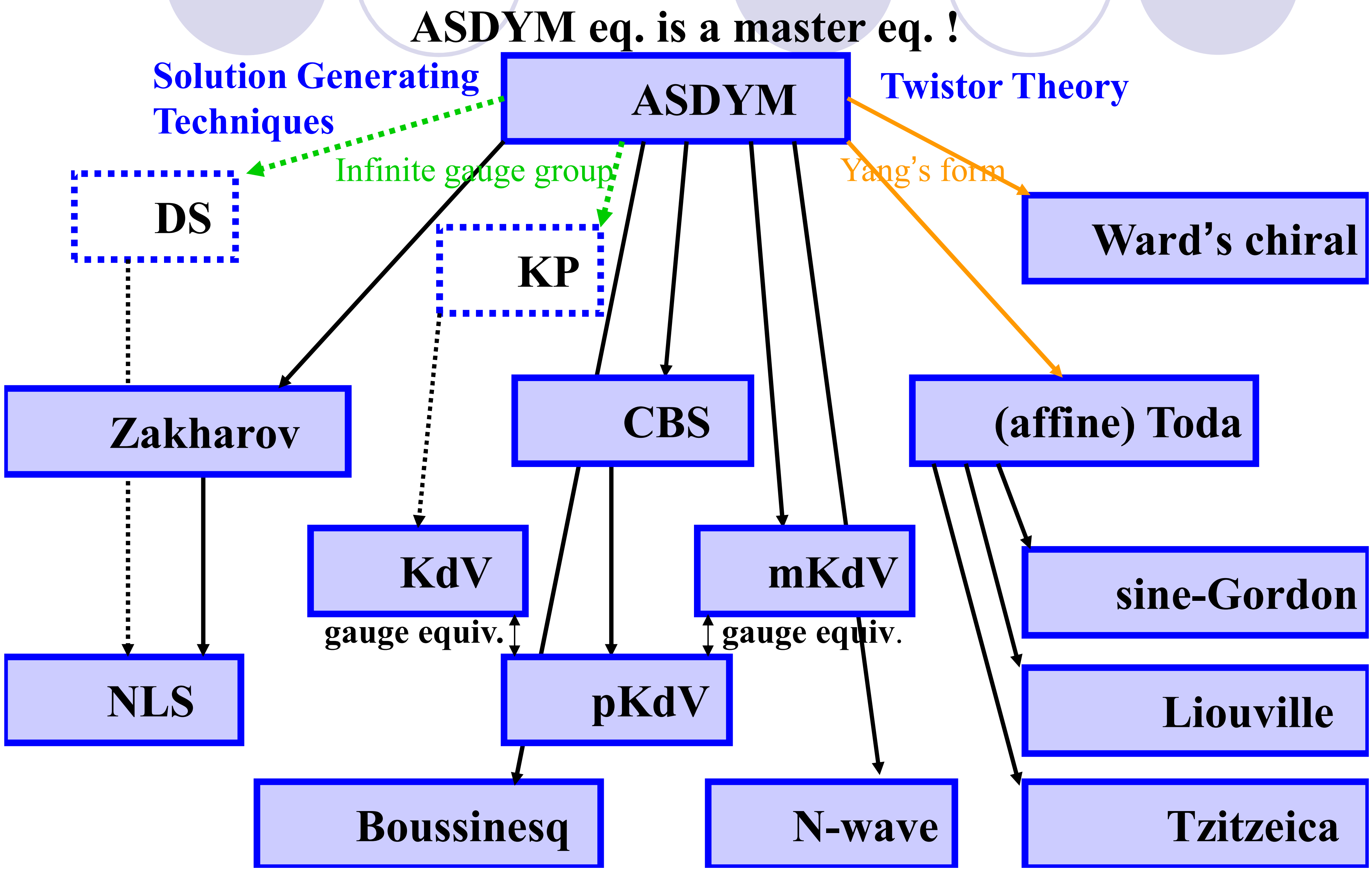
↙ ↘

4d CS ← duality relation → 4d ASDYM (+ + − −)

↓ ↓

various solvable models (spin chains, PCM, ⋯) | various integrable eqs. (KdV, NLS, Toda, ⋯)

**Ward's conjecture: Many (perhaps all?) integrable equations are reductions of the ASDYM eqs.**



# Towards Unification of Integrable Systems

6d CS

$\swarrow$ $\searrow$

4d CS

$\downarrow$

various solvable models

(spin chains, PCM, ... )

4d ASDYM

U (++−−)

hier.

$\downarrow$

various integrable eqs.

(KdV NLS, Toda, ...)

KP hier.

[Li-Maruno-Zhang, arXiv:2603.23060]

my postdoc. →

# Towards Unification of Integrable Systems

6d CS

NC 5d CS (Bittleston-Heuveline-Raghavendran-Skinner] [arXiv:2509.20643]

4d CS

various solvable models

(spin chains, PCM, ⋯)

KP eq!

4d ASDYM ∪ (++−−)

hier.

various integrable eqs.

(KdV NLS, Toda, ⋯)

KP hier.

[Li-Maruno-Zhang, arXiv:2603.23060]

my postdoc.

# Towards Unification of Integrable Systems

NC 6d CS

↙ NC 5d CS

↘ NC 4d ASDYM (++−−) hier.

NC 5d CS ↙ NC 4d CS

NC 5d CS ↘ KP eq !

NC 4d CS ↓ various NC solvable models

NC(spin chains, PCM, ⋯)

4d ASDYM ↓ various NC integrable eqs.

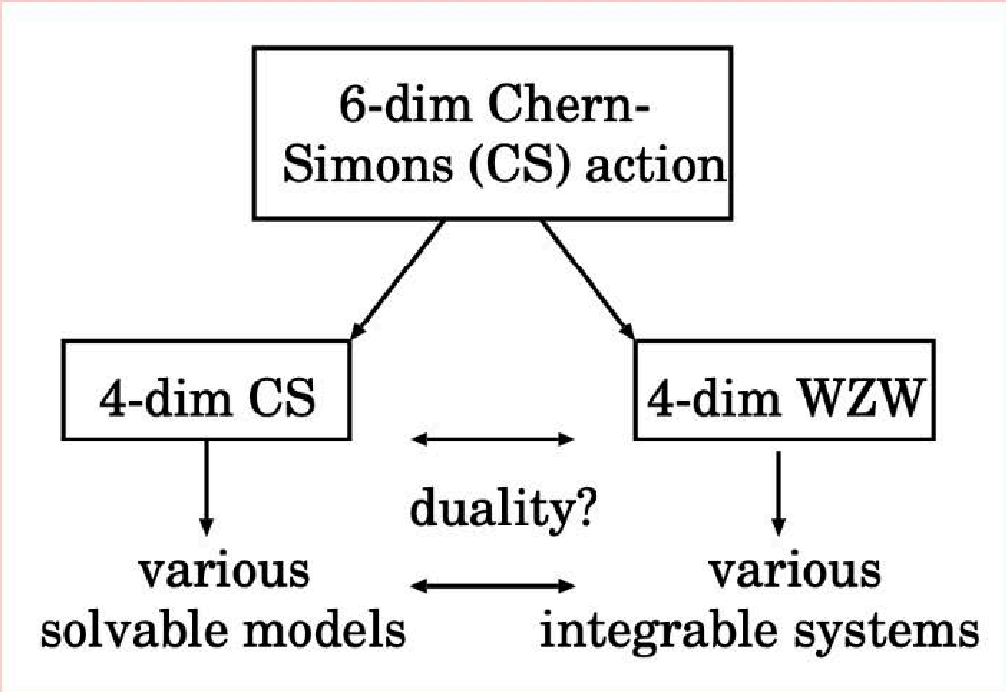
NC(KdV NLS, Toda, ⋯)

NC KP hier.

Workshop on the Unification (2025.10 Nagoya)

# Unification of Integrable Systems

**27～29 October, 2025 at Nagoya University and Online**

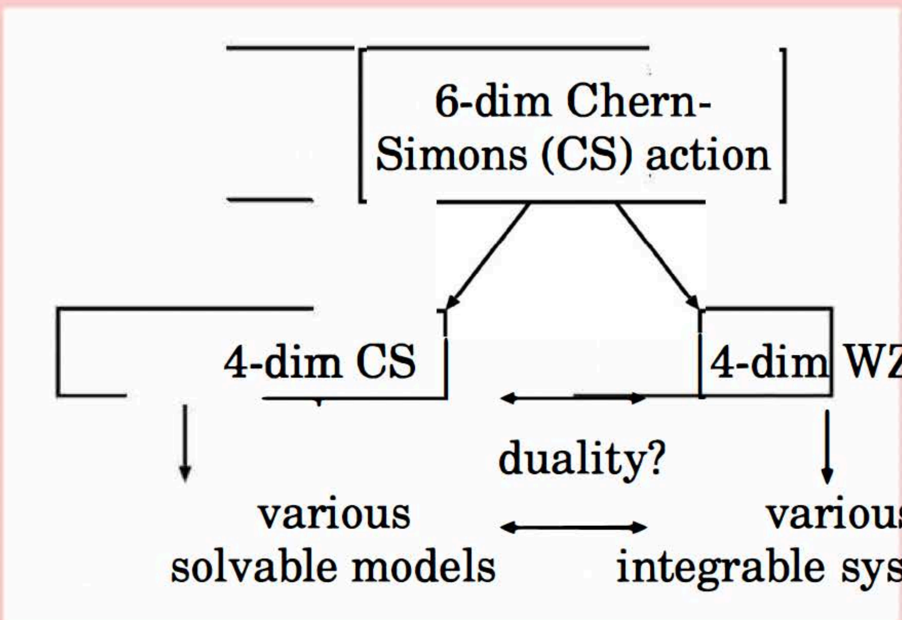


# Workshop on the Unification 2 (2026. Fall Osaka)

Daiwa Angro-Japanese Foundation

## Unification of Integrable Systems 2

Autumn, 2026 in Osaka and Online



# Workshop on the Unification 3 (2027 Summer York, UK)

# ?

# Welcome to Join!

Thank You

Very Much

# References (Reviews)


- NC integrable system : [H, arXiv:1012.6043]
- NC instantons : [H-Nakatsu, arXiv:1311.5227]
- Quasideterminants : [GGRW, math/0208146]


Useful Slides & Videos

※ Workshop on NC integrable systems

& on Unification of Integrable Systems

Nagoya Math-Phys Seminars