Geometric Integration of Hamiltonian Systems by Solvable Structures

Saša Krešić-Jurić

Faculty of Science, University of Split

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Motivation

We prospose a new method for explicit integration of Hamiltoninan systems and the construction of action–angle variables by the method of solvable structures.

The Arnold–Liouville theorem
Assume we are given

- (M, ω) symplectic manifold, $\dim M = 2n$,
- $X_H \in \mathfrak{X}(M)$ Hamiltonian vector field induced by $H \in C^\infty(M)$.

If the Hamiltonian system (M, ω, H) possesses n independent first integrals $F_1, \dots, F_n \in C^\infty(M)$ in involution, and $c \in \mathbb{R}^n$ is a regular point of

$$F = (F_1, \dots, F_n) \colon M \to \mathbb{R}^n,$$

then

- $M_c = F^{-1}(c)$ is n–dimensional submanifold invariant under the flow of X_H,
- if M_c is compact and connected, then $M_c \simeq \mathbb{T}^n$ and there exists action–angle variables defined by a symplectic map $\phi(q, p) = (I, \theta)$ such that

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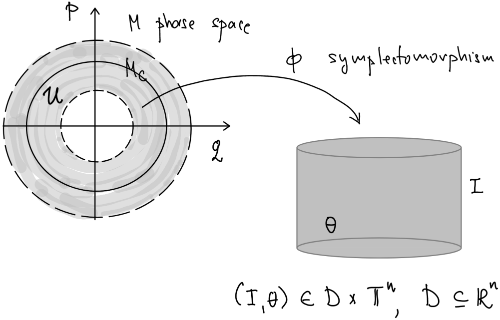
along the flow of X_H

$$\dot{I}_k = 0, \quad \dot{\theta}_k = \omega_k(I), \quad \omega_k(I) = \frac{\partial(\phi_* H)}{\partial I_k}.$$

I_k action variables

θ_k angle variables on the invariant torus $\mathbb{T}^n$



Limitations of the theorem

- The symplectomorphism $\phi\colon (q,p) \mapsto (I,\theta)$ is often difficult to find explicitly.
- The action variables $I_k = \frac{1}{2\pi}\oint_{\gamma_k} \sum_{i=1}^{n} p_i dq_i$ require identification of n independent cycles γ_k on the level sets which is difficult in high dimensions.

Standard explicit integration methods

- Hamilton–Jacobi equation
- Lax pair method
- Algebraic–geometric integration

The solvable-structure approach

A method for integrating involutive distributions which yields

- explicit solutions of Hamilton's equations,
- canonical variables of action–angle type.

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Solvable Structures

Let M be an n–dimensional smooth manifold, and let $\mathcal{D} \subset TM$ be a rank–r smooth distribution on M, $r < n$:

$$\mathcal{D} = \mathrm{span}\{X_1, \ldots, X_r\}.$$

$\mathcal{D}$ is integrable if there exists a submanifold $N \subset M$ such that $T_pN = \mathcal{D}_p$ for all $p \in N$.

By Frobenius theorem, $\mathcal{D}$ is integrable if and only if it is *involutive*, i.e.

$$[X_i, X_j] = 0, \quad i, j = 1, \ldots r.$$

- Solvable structures is a method for obtaining integral manifolds of a distribution $\mathcal{D} \subset TM$.
- It is based on the concept of *symmetry* of a distribution.
- Since solvable structures are constructed locally, we may assume that all vector fields and differential forms are defined on open subsets $U \subseteq \mathbb{R}^n$.

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Definition (Symmetry of a distribution)

Let $\mathcal{D} = \mathrm{span}\{X_1, \ldots, X_r\}$ be a rank–r smooth distribution on $U \subseteq \mathbb{R}^n$. A vector field $Y \in \mathfrak{X}(U)$ is called a symmetry of $\mathcal{D}$ if

1. $X_1, \ldots, X_r, Y$ are linearly independent on U,
2. $[Y, X_i] \in \mathcal{D}$ for $i = 1, \ldots, r$.

If Y is a symmetry of $\mathcal{D}$, then its flow ϕ_t preserves the distribution $\mathcal{D}$: $\phi_t^* X_i \in \mathcal{D}$.

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1. $X_1, \ldots, X_r, Y_1, \ldots, Y_{n-r}$ are linearly independent on U,
2. Y_1 is a symmetry of $\mathcal{D}$,
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The existence of a solvable structure for an involutive distribution $\mathcal{D}$ depends on the properties of the Pfaffian forms that define $\mathcal{D}$.

A rank–r distribution $\mathcal{D}$ on $U \subset \mathbb{R}^n$ can be represented by $n-r$ linearly independent one–forms $\omega_i \in \Omega(U)$ that annihilate $\mathcal{D}$:

$$\mathcal{D} = \ker \omega_1 \cap \ldots \cap \ker \omega_{n-r}.$$

If $\mathcal{D}$ is integrable, then the integral manifolds of $\mathcal{D}$ are locally given by the Pfaffian equations

$$\omega_1 = 0, \quad \omega_2 = 0, \quad \ldots \quad \omega_{n-r} = 0.$$

Theorem

Assume that $\mathcal{D} = \ker \omega_1 \cap \ldots \cap \omega_{n-r}$ is a rank–r involutive distribution on a convex domain $U \subseteq \mathbb{R}^n$. Then $\mathcal{D}$ admits a solvable structure if and only if

$$d\omega_k = \sum_{j=k+1}^{n-r} \theta_{kj} \wedge \omega_j, \quad k = 1, \ldots, n-r$$

for some one–forms θ_{kj} where $d\omega_{n-r} = 0$.

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Integration of involutive distributions by solvable structures

By using the solvable structure

- we can construct the Pfaffian forms that define $\mathcal{D}$,
- and integrate the Pfaffian equations to obtain the integral manifolds of $\mathcal{D}$.

Let $(X_1, \ldots, X_r, Y_1, \ldots, Y_{n-r})$ be a solvable structure for $\mathcal{D} = \operatorname{span}\{X_1, \ldots, X_r\}$. The Pfaffian forms that define $\mathcal{D}$ can be constructed as follows:

1. Define the smooth function

$$\lambda = Y_{n-r} \lrcorner \ldots \lrcorner Y_1 \lrcorner X_r \lrcorner \ldots \lrcorner X_1 \lrcorner \tau$$

   where τ is the volume form on $\mathbb{R}^n$.

2. Define the one–forms

$$\omega_k = \frac{1}{\lambda} Y_{n-r} \lrcorner \ldots \lrcorner \hat{Y}_k \lrcorner \ldots Y_1 \lrcorner X_r \lrcorner \ldots \lrcorner X_1 \lrcorner \tau, \quad k = 1, \ldots, n-r,$$

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The one–forms constructed by the above procedure have the following properties:

$$\mathcal{D} = \ker \omega_1 \cap \ldots \cap \omega_s, \quad s = n - r,$$

$$d\omega_s = 0, \quad d\omega_k = \sum_{j=k+1}^{s} \theta_{kj} \wedge \omega_j, \quad k = 1, \ldots, s-1.$$

The integral manifold of $\mathcal{D}$ is constructed by successive integration of the sequence $(\omega_1, \ldots, \omega_s)$ starting with ω_s:

$$\begin{aligned} d\omega_s &= 0 && \Rightarrow \quad \omega_s = dI_s, \\ d\omega_{s-1} &= \theta_{s-1,s} \wedge \omega_s && \Rightarrow \quad \omega_{s-1} = dI_{s-1} \text{ on } \{I_s = c_s\} \\ &\vdots \\ d\omega_1 &= \sum_{j=2}^{s} \theta_{1,j} \wedge \omega_j && \Rightarrow \quad \omega_1 = dI_1 \text{ on } \{I_2 = c_2, \ldots, I_s = c_s\} \end{aligned}$$

The integral manifold of $\mathcal{D}$ is given by level set

$$N = \big\{x \in U \mid I_1(x) = c_1, \ldots, I_s(x) = c_s\big\}, \quad c_i \in \mathbb{R}.$$

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Solvable Structures for Hamiltonian Systems

Consider a completely integrable Hamiltonian system on a symplectic manifold M,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad (q,p) \in M, \tag{1}$$

with n independent integrals $F_1, \dots, F_n$ in involution. Define the vector fields

$$A = \partial_t + X_H, \quad X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}$$

on the extended phase space $\mathbb{R} \times M$.

In the following, we

1. construct a solvable structure for $\mathcal{D} = \mathrm{span}\{A\}$,
2. compute the Pfaffian forms associated with $\mathcal{D}$,
3. integrate the Pfaffian forms to obtain explicit solutions and action–angle variables of the Hamiltonian system (1).

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Construction of the solvable structure

The sequence vector fields $(A, X_{F_1}, \ldots, X_{F_n})$ is completed by adding Hamiltonian vector fields $X_{G_1}, \ldots, X_{G_n}$ such that

1. X_{G_1} is a symmetry of $\mathcal{D}_0 = \mathrm{span}\{A, X_{F_1}, \ldots, X_{F_n}\}$,
2. X_{G_k} is a symmetry of $\mathcal{D}_k = \mathcal{D}_0 \oplus \mathrm{span}\{X_{G_1}, \ldots, X_{G_{k-1}}\}$.

One can prove that locally there exist functions $G_1, \ldots, G_n$ on M such that

$$[X_{G_i}, X_{G_j}] = 0,$$

$$[X_{G_i}, X_{F_j}] = \sum_{k=1}^{n} f_{ij}^k \, X_{F_k},$$

$$[X_{G_i}, A] = \sum_{k=1}^{n} h_{ik} \, X_{F_k}.$$

for some smooth functions f_{ij}^k and h_{ik} defined on an open subset of M.

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for some smooth functions f_{ij}^k and h_{ik} defined on an open subset of M.

The Arnold–Liouville theorem yields the *canonical solvable structure*.

Theorem 1

Let (M, ω) be a $2n$–dimensional symplectic manifold and let X_H be a completely integrable Hamiltonian vector field on M with functionally independent first integrals $F_1, \ldots, F_n \in C^\infty(M)$ in involution. If the common level set $M_c = F^{-1}(c)$ is compact and connected, then in a neighbourhood of M_c there exist locally defined functions $G_1, \ldots, G_n$ such that

$$\{G_i, G_j\} = 0, \quad \{G_i, F_j\} = \delta_{ij}.$$

Consequently, on the extended phase space $\mathbb{R} \times M$, the sequence of vector fields

$$A = \partial_t + X_H, X_{F_1}, \ldots, X_{F_n}, X_{G_1}, \ldots, X_{G_n}$$

forms a local solvable structure for $\mathcal{D} = \mathrm{span}\{A\}$ such that

$$[X_{G_i}, X_{G_j}] = 0, \quad [X_{G_i}, X_{F_j}] = 0, \quad [X_{G_i}, A] \in \mathrm{span}\{X_{F_1}, \ldots, X_{F_n}\}.$$

A more general class of solvable structure can be constructed as follows.

Theorem 2

Let (M,ω) be a $2n$–dimensional symplectic manifold, and let X_H be a completely integrable Hamiltonian vector field with first integrals $F_1,\ldots,F_n$ in involution. Let $U\subset M$ be an open subset on which $F_1,\ldots,F_n$ are functionally independent. Assume that on U there exist functions $G_1,\ldots,G_n$ satisfying $\{G_i,G_j\}=0$ and $\{G_i,F_j\}=\delta_{ij}$. Define

$$\tilde{G}_i=\sum_{j=1}^{n}K_{ij}(F)G_j,\quad i=1,\ldots,n,$$

where $K=\left[K_{ij}\right]$ is a nonsingular matrix of smooth functions $F=(F_1,\ldots,F_n)$ such that

$$\sum_{l=1}^{n}K_{il}\frac{\partial K_{jk}}{\partial F_l}-K_{jl}\frac{\partial K_{ik}}{\partial F_l}=0,\quad i,j,k=1,\ldots n. \tag{2}$$

Then $(A,X_{F_1},\ldots,X_{F_n},X_{\tilde{G}_1},\ldots,X_{\tilde{G}_n})$, $A=\partial_t+X_H$, is a local solvable structure for $\mathcal{D}=\operatorname{span}\{A\}$ such that

$$[X_{\tilde{G}_i},X_{\tilde{G}_j}]=0,\ \ [X_{\tilde{G}_i},X_{\tilde{G}_j}],\,[X_{\tilde{G}_i},A]\in\operatorname{span}\{X_{F_1},\ldots,X_{F_n}\}.$$

Remark

The compatibility conditions (2) is equivalent with $[Y_i, Y_j] = 0$ where

$$Y_i = \sum_{j=1}^{n} K_{ij} \frac{\partial}{\partial F_j}. \tag{3}$$

If $\det[K_{ij}] \neq 0$, the vector fields $Y_1, \ldots, Y_n$ are linearly independent, hence there exist a local coordinate system in which $[Y_i, Y_j] = 0$.

Theorem: Invariance under gauge transformations

Let $\tilde{G}_i$, $i = 1, \ldots, n$ be the functions satisfying the assumptions of Theorem 2. Then the Poisson bracket

$$\{\tilde{G}_i, \tilde{G}_j\} = 0, \quad \{\tilde{G}_i, F_j\} = K_{ij}(F). \tag{4}$$

is invariant under the gauge transformation

$$\tilde{G}_i \mapsto \tilde{G}_i + Y_i(\Phi)$$

where Y_i is the vector field (3) and $\Phi = \Phi(F)$ is a smooth function.

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Pfaffian Forms for the Solvable Structure

The Pfaffian forms associated with the solvable structure in Theorem 2 for a Hamiltonian vector field X_H on a symplectic manifold M provide

- explicit solutions of Hamilton's equations,
- action–angle variables of the system.

Let $(A, X_{F_1}, \ldots, X_{F_n}, X_{G_1}, \ldots, X_{G_n})$ be a solvable structure for $A = \partial_t + X_H$ constructed in Theorem 2.

Let τ be the volume form on the extended phase space $\mathbb{R} \times M$ and let

$$\lambda = X_{G_n} \lrcorner \ldots \lrcorner X_{G_1} \lrcorner X_{F_n} \lrcorner \ldots \lrcorner X_{F_1} \lrcorner A \lrcorner \tau.$$

For $k = 1, \ldots, n$, define

$$\omega_{n+k} = \frac{1}{\lambda} X_{G_n} \lrcorner \ldots \lrcorner \hat{X}_{G_k} \lrcorner \ldots \lrcorner X_{G_1} \lrcorner X_{F_n} \lrcorner \ldots \lrcorner X_{F_1} \lrcorner A \lrcorner \tau, \quad (X_{G_k} \text{ omitted}),$$

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Explicit compuation of the Pfaffian forms in the action–angle variables yields:

$$\omega_{n+k} = (-1)^{n+k+1} dP_k,$$

$$\omega_k = \frac{(-1)^n}{\lambda}\Big(f_k dt + \sum_{j=1}^{n}(-1)^j h_{kj}\, dQ_j\Big), \quad k = 1, \ldots, n,$$

$$f_k = \left|\frac{\partial(H, F_1, \ldots, \hat{F}_k, \ldots, F_n)}{\partial(P_1, P_2, \ldots, P_n)}\right|, \quad h_{kj} = \left|\frac{\partial(F_1, \ldots, \hat{F}_k, \ldots, F_n)}{\partial(P_1, \ldots, \hat{P}_j, \ldots, P_n)}\right|, \quad (F_k \text{ ommited}).$$

Since $\varphi(q,p) \to (Q,P)$ is a symplectic transformation, $\varphi^*\omega_k$ have the **same structure** as ω_k which yields $Q = Q(q,p)$ and $P = P(q,p)$.

Integration algorithm

1. The last n forms are exact differentials $\varphi^*\omega_{n+k} = dI_{n+k}$.
2. The first n forms are exact differentials $\varphi^*\omega_k = dI_k$ on the submanifolds $M_{k+1} = \{I_{k+1} = c_{k+1}, \ldots, I_{2n} = c_{2n}\}$.
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Connection with the Arnold–Liouville theorem

- Integration of the last n forms yields the action variables P_k.
- Integration of the first n forms yields the angle variables Q_k.
- The obtained variables are canonical:

$$\{Q_i, Q_j\} = 0,, \quad \{P_i, P_j\} = 0, \quad \{Q_i, P_j\} = \delta_{ij}.$$

Example: Rational Calogero–Moser System

The rational Calogero–Moser system describes N particles on a line with Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + g^2 \sum_{i<j} \frac{1}{(q_i - q_j)^2}$$

where $\{q_i, q_j\} = \{p_i, p_j\} = 0$, $\{q_i, p_j\} = \delta_{ij}$.

- The quantum version of the system was studied by Calogero in 1971.
- The classical version of the system was one of the first systems integrated by the Lax pair method (Moser 1975).

The system has a Lax representation $\dot{L} = [L, M]$ with $N \times N$ matrices

$$L_{ij} = p_i\,\delta_{ij} + \sqrt{-1}\,g\,\tfrac{1-\delta_{ij}}{q_i - g_j}$$

and

$$M_{ij} = \sqrt{-1}\,g\left(-\delta_{ij}\sum_{k\neq i}\tfrac{1}{(q_i-q_k)^2} + \tfrac{1-\delta_{ij}}{(q_i-q_j)^2}\right).$$

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Case: $N = 2$ particles

Hamiltonian vector field:

$$X_H = p_1 \frac{\partial}{\partial q_1} + p_2 \frac{\partial}{\partial q_2} + \frac{2g^2}{(q_1 - q_2)^3}\Big(\frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2}\Big).$$

First integrals $F_k = \mathrm{Tr}(L^k)$ in involution:

$$F_1 = p_1 + p_2, \quad F_2 = p_1^2 + p_2^2 + \frac{2g^2}{(q_1 - q_2)^2}.$$

A solvable structure for the distribution $D = \mathrm{span}\{A\}$, $A = \partial_t + X_H$, is given by $(A, X_{F_1}, X_{F_2}, X_{G_1}, X_{G_2})$ where

$$G_1 = q_1 + q_2, \quad G_2 = (q_1 - q_2)(p_1 - p_2)$$

and

$$\begin{aligned}
&[X_{G_1}, X_{G_2}] = 0, && \\
&[X_{G_1}, X_{F_1}] = 0, && [X_{G_1}, X_{F_2}] = -2X_{F_1}, \\
&[X_{G_2}, X_{F_1}] = 0, && [X_{G_2}, X_{F_2}] = 4(p_1 + p_2)X_{F_1} - 4X_{F_2}, \\
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Pfaffian forms associated with the solvable structure

$$\omega_4 = \frac{1}{\lambda} X_{G_1} \lrcorner X_{F_2} \lrcorner X_{F_1} \lrcorner A \lrcorner \tau = \frac{1}{2(-F_1^2 + 2F_2)} \big(F_1 dF_1 - dF_2 \big),$$

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Integration of the Pfaffian forms

- The forms ω_3 and ω_4 are exact:

$$\omega_4 = dI_4, \quad I_4 = -\frac{1}{4} \ln \big(-F_1^2 + 2F_2 \big),$$

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- The form ω_2 is exact on the submanifold $M_3 = \{I_3 = c_3,\ I_4 = c_4\}$:

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Since the Pfaffian forms vanish on the submanifold $M_1 = \{I_k = c_k,\ 1 \le k \le 4\}$, i.e.

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the functions F_k and G_k satisfy the system of equations

$$F_1 = 2c_3, \qquad G_1 = 2(c_3 t - c_1), \tag{5}$$

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Explicit solutions

By solving the system (5)–(6) explicitly for q and p, we find

$$q_1(t) = c_3 t - c_1 + \frac{1}{2}\left[e^{-4c_4}(t+2c_2)^2 + 4g^2 e^{4c_4}\right]^{\frac{1}{2}},$$

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Consistent with solutions found in Brzezinski et. et. (Phys. Lett. A, 1999).

Action variables

The last two forms ω_3 and ω_3 in the (Q, P) variables:

$$\omega_4 = -dP_2 \quad \Rightarrow \quad P_2 = \frac{1}{4}\ln\Big((p_1 - p_2)^2 + \frac{4g^2}{(q_1-q_2)^2}\Big),$$

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This implies

$$Q_1 = q_1 + q_2, \quad Q_2 = (q_1 - q_2)(p_1 - p_2).$$

- The transformation

$$\phi\colon (q,p) \mapsto \left(q_1 + q_2, (q_1 - q_2)(p_1 - p_2), \frac{1}{4}\ln\left((p_1 - p_2)^2 + \frac{4g^2}{(q_1 - q_2)^2}\right), \frac{1}{2}(p_1 + p_2)\right).$$

is symplectic.

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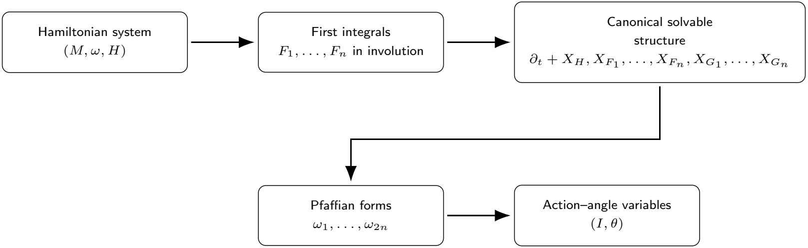
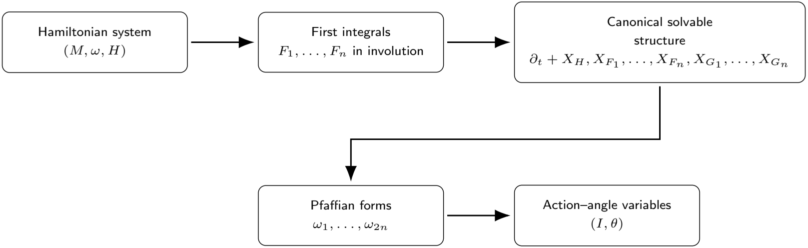
Solvable Structure Method: Summary



1. **Explicit local solutions** are obtained directly by integrating the Pfaffian forms associated with the solvable structure.
2. **Action–angle variables**:
   - the last n forms yield the action variables P_i,
   - the first n forms yield the angle variables Q_i.
3. The transformation $(q, p) \mapsto (Q, P)$ is **symplectic by construction**.

Solvable Structure Method: Summary



1. **Explicit local solutions** are obtained directly by integrating the Pfaffian forms associated with the solvable structure.
2. **Action–angle variables**:
   - the last n forms yield the action variables P_i,
   - the first n forms yield the angle variables Q_i.
3. The transformation $(q, p) \mapsto (Q, P)$ is **symplectic by construction**.

Thank you for your attention.

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