Geodesic Mappings and Their Generalizations

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et al.

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Dedicated to Professor Vladimir Berezovskii (1962 – 2023)

Sts. Konstantine & Helene, Varna (Koral)

Geodesic, holomorphically projective mappings and their generalizations play an important role in the differential geometry of Riemannian and more general manifolds.

As is known, the gnomonic projection of sphere, which is an example of geodesic mapping, has been known since the time of Thales of Miletus, and stereographic projection, which is an example of conformal mapping, has been known since the time of Apollonius of Perga.
From a mathematical point of view, these problems were studied by Mercator and later by Lagrange.
Questions of conformal mappings were also studied by Euler within the framework of the theory of complex functions. In studying geodesic mappings of spaces of constant curvature, Beltrami in 1865 established the validity of the non-Euclidean geometry of

Lobachevsky, Bolyai and Gauss.

Karl F. Gauss, Nikolai I. Lobachevsky, János Bolyai
1777-1855 1792-1856 1802-1860



An appendix showing the absolutely true science of space (26 pages)

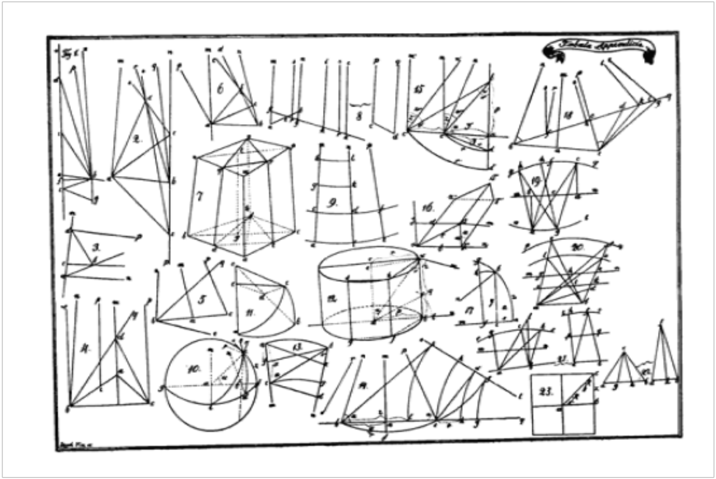
APPENDIX.

SCIENTIAM SPATII *absolute veram* exhibens:

a veritate aut falsitate Axiomatis XI *Euclidei*
(a priori haud unquam decidenda) in-
dependentem: adjecta ad casum fal-
sitatis, quadratura circuli
geometrica.

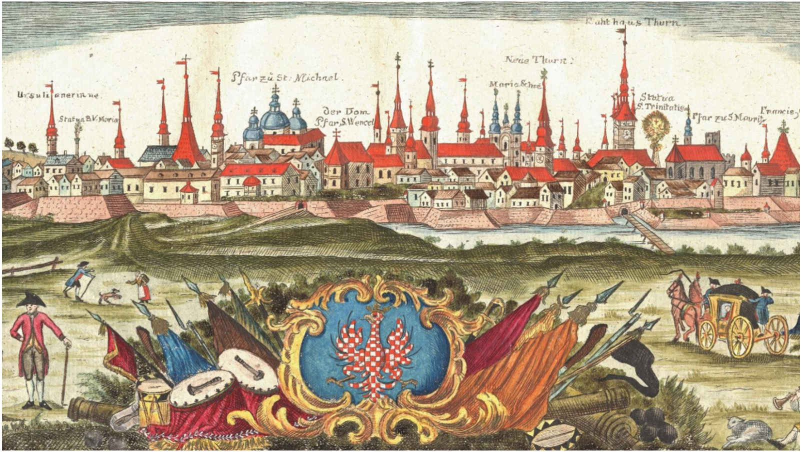
Auctore JOHANNE BOLYAI de eadem, Geometrarum
in Exercitu Caesareo Regio Austriaco Ca-
strensium Capitaneo

Attachment of the Appendix



János Bolyai (1802–1860), in Olomouc from 1832 to 1833

Olomouc, the metropolis of Moravia



Later, Levi-Civita (1896) found the basic equations of geodesic mappings of Riemannian spaces, and also showed their application to modeling mechanical processes.

In the twenties, Weyl and other authors found new results for more general types of spaces, especially spaces with affine connection. In addition, they pointed out the connection between these questions and the then new theory of relativity founded by Einstein.

Eisenhart's monographs present the basic facts of Riemannian and pseudo-Riemannian geometry:

L.P. Eisenhart, Riemannian geometry. Princeton University Press, 1926.
Reprint: Oxford University Press. VII, 306p. (1967).
L.P. Eisenhart, Non-Riemannian geometry. Princeton University Press, 1927.
Reprint: NY: Dover Publications, viii, 184 p. (2005).
L.P. Eisenhart, Continuous groups of transformations. Princeton University Press, 1933.
Unabridged and corrected republ.: NY: Inc. IX, 301 p. (1961).

A.P. Norden presents many facts of the theory of spaces with affine connection:

A. Norden, *Spaces with affine connection*. Moscow, 1950.
Reprint: 2nd ed., Moscow, Nauka, 1976.

Later, many new results concerning geodesics and other types of mappings of Riemannian spaces and their generalizations were presented in monographs:

K. Yano, S. Bochner, Curvature and Betti numbers. Princeton Univ. Press IX, 190 p. (1953).
S. Kobayashi, K. Nomizu, Foundations of differential geometry. Vol. I, II. (1963, 1969).
A.Z. Petrov, New methods in the theory of relativity. Moscow, Nauka, (1966).
A.Z. Petrov, Einstein spaces. Pergamon Press. XIII, 411 p. (1969).
N.S. Sinyukov, *Geodesic mappings of Riemannian spaces*. M.: Nauka, (1979).

Here are the results obtained by many authors up to 1979, and the theory of geodesic, almost geodesic and holomorphically projective mappings is presented in detail.

In Sinyukov's monograph, about a quarter of the new results belong to me. Especially they are the theory of geodesic mappings of semisymmetric and generalized recurrent spaces, as well as the entire Chapter 5, it is devoted to holomorphic-projective mappings.

My research and books present in detail, with proofs, facts and new results of geodesic, holomorphic-projective, conformal and other mappings and transformations, and also present the foundations of the theory of Riemannian spaces and their generalizations:

J. Mikeš, Geodesic mappings of affine-connected and Riemannian spaces. J. Math. Sci., New York 78, No. 3, 311-333 (1996).

J. Mikeš, Holomorphically projective mappings and their generalizations. J. Math. Sci., New York 89, No. 3, 1334-1353 (1998);

J. Mikeš, M.L. Gavril'chenko, Zh. Radulovich, Geodesic mappings and deformations of Riemannian spaces. Podgorica: Izdatel'stvo CID. Odessa: Izd. OGU, (1997).

J. Mikeš, A. Vanžurová, I. Hinterleitner, Geodesic mappings and some generalizations. Palacky Univ. Press, Olomouc, (2009).

J. Mikeš et al., Differential Geometry of Special Mappings, Palacky Univ. Press, Olomouc, (2015, 2019).

In this lecture, we present new results concerning geodesic and holomorphically projective mappings under the assumptions of compactness and completeness of geodesics and their images. These conditions have a rigidity character, where the system of geodesics forms a “skeleton" that guarantees it, see

J. Mikeš, I. Hinterleitner, N.I. Guseva, Geodesic mappings “in the large” of Ricci-flat spaces with n complete geodesic lines, *Math. Notes* **108**(1–2) (2020), 292–296.
J. Mikeš, I. Hinterleitner, P. Peška, L. Vítková, Rigidity of holomorphically projective mappings of Kähler and hyperbolic Kähler spaces with finite complete geodesics, *Geometry* **2**(1) (2025), No. 3.

In paper

P. Peška, J. Mikeš, L. Rýparová, O. Chepurna, On general solutions of equidistant vector fields on two-dimensional (pseudo-) Riemannian spaces *Filomat* **37**(25) (2022), 8569–8574.

we obtained a general solution for geodesic mappings of equidistant spaces, for main conditions of differentiability metrics.

In following papers

Mikeš, J.; Guseva, N. I.; Peška, P.; Rýparová, L. Rotary mappings and projections of a sphere. Math. Notes 110, No. 1, 152-155 (2021).

Mikeš, J.; Guseva, N. I.; Peška, P.; Rýparová, L., Almost geodesic mappings and projections of a sphere, *Math. Notes* **111** (2022), 498–502.

we constructed global rotary and almost geodesic mappings for compact spaces.

Geodesic Mappings and Their Generalizations

Topics

1. Geodesic mappings
2. Holomorphically projective mappings
3. F-planar mappings

1. Geodesic mapping theory for $V_n \to \bar{V}_n$ of class C^1

Assume the (pseudo-) Riemannian manifolds $V_n = (M, g)$ and $\bar{V}_n = (\bar{M}, \bar{g})$. Here V_n, $\bar{V}_n \in C^1$, i.e. metrics $g, \bar{g} \in C^1$ which means that their components g_{ij}, $\bar{g}_{ij} \in C^1$.

Definition

A diffeomorphism $f\colon V_n \to \bar{V}_n$ is called a *geodesic mapping* of V_n onto $\bar{V}_n$ if f maps any geodesic in V_n onto a geodesic in $\bar{V}_n$.

A manifold V_n admits a geodesic mapping onto $\bar{V}_n$ if and only if the *Levi-Civita equations*

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X \tag{1}$$

hold for any tangent fields X, Y and where ψ is a linear differential form, ∇ and $\bar{\nabla}$ are the Levi-Civita connections.

If $\psi \equiv 0$ than f is *affine* or *trivially geodesic*.

In local form: $\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi_i(x)\delta^h_j + \psi_j(x)\delta^h_i$,

where $\Gamma^h_{ij}(\bar{\Gamma}^h_{ij})$ are the Christoffel symbols of V_n and $\bar{V}_n$.

The Levi-Civita equations (1) are equivalent to the following equations

$$(2) \qquad \bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi \bar{g}_{ik}$$

where "," denotes the covariant derivative on V_n. It is known that

$$\psi_i = \partial_i \Psi, \quad \Psi = \frac{1}{2(n+1)} \ln \left| \frac{\det \bar{g}}{\det g} \right|, \quad \partial_i = \partial / \partial x^i.$$

Sinyukov proved that the Levi-Civita equations (1) are equivalent to

$$(3) \qquad a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik},$$

where $a_{ij} = \mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}; \quad \lambda_i = -\mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\beta i} \psi_\alpha.$
From (3) follows $\lambda_i = \partial_i \lambda = \partial_i(\frac{1}{2}\, a_{\alpha\beta} g^{\alpha\beta})$. On the other hand

$$(4) \qquad \bar{g}_{ij} = \mathrm{e}^{2\Psi} \tilde{g}_{ij}, \quad \Psi = \frac{1}{2} \ln \left| \frac{\det \tilde{g}}{\det g} \right|, \quad \|\tilde{g}_{ij}\| = \|g^{i\alpha} g^{j\beta} a_{\alpha\beta}\|^{-1}.$$

The above formulas are the criterion for geodesic mappings $V_n \to \bar{V}_n$ globally as well as locally.

Mikeš and Berezovski (1962-2023) for the case geodesic mappings from equiaffine space A_n onto (pseudo-) Riemannian space $\bar{V}_n$ proved that the Levi-Civita equations (1) are equivalent to

$$a^{ij}{}_{,k} = \lambda^i \delta^j_k + \lambda^j \delta^i_k, \tag{5}$$

where

$$a^{ij} = \mathrm{e}^{2\Psi} \bar{g}^{ij} \quad \text{and} \quad \lambda^i = -a^{i\alpha}\psi_\alpha.$$

The above formulas (5) are the criterion for geodesic mappings $A_n \to \bar{V}_n$ globally as well as locally.

See: J. Mikeš, V. Berezovski, *Geodesic mappings of affine-connected spaces onto Riemannian spaces*. DGA (Eger, 1989), 491–494, Colloq. Math. Soc. J. Bolyai, 56, North-Holland, Amsterdam, 1992. MR1211677, Zbl 0792.53012.

J. Mikeš at al. *Differential Geometry of Special Mappings*, Olomouc Univ. Press, 2015.

Eastwood and Matveev obtained analogically results for geodesic mappings of space with affine connection A_n onto (pseudo-) Riemannian space $\bar{V}_n$:

M. Eastwood, V. Matveev. *Metric connections in projective differential geometry*. Symmetries and overdetermined systems of partial differential equations, 339–350, IMA Vol. Math. Appl., 144, Springer, New York, 2008. MR2384718.

They did not realise that

any space with affine connection
is projective equivalent of equiaffine space!

See locally:

É. Cartan, *Sur les variété à connexion projective*. S. M. F. Bull. 52, 205-241, 1924.

J.M. Thomas, *Asymmetric displacement of a vector*. Trans. AMS 28:4, 658-670, 1926.

L.P. Eisenhart, *Non-Riemannian geometry*. Princeton Univ. Press. 1926. AMS Colloq. Publ. 8, 2000, in p. 105.

A. Norden, *Spaces with affine connection*. Moscow, 1950, 2nd ed., Nauka, 1976.

globally:

I. Hinterleitner, J. Mikeš, *Projective equivalence and spaces with equi-affine connection*. J. Math. Sci. 177, 546-550, 2011 (Fundam. Prikl. Mat. 16, 47-54, 2010).

J. Mikeš at al. *DG of Special Mappings*, Olomouc Univ. Press, 2015; pp. 264–265.

2. Geodesic bifurcations

L. Rýparová and J. Mikeš studied a question about the existence of geodesic in detail. It is proved that there exist geodesic bifurcations, that are described as situations when two or more geodesics pass through the given point in the given direction.

See: Rýparová L., Mikeš J. On geodesic bifurcations, Geometry, Integrability and Quantization 18, 2017, 217–224.

A different definition of bifurcations can be found in

Thielhelm H., Vais A., Wolter F.-E. Geodesic bifurcation on smooth surfaces. The Visual Computer 31:2, 2015, 187–204.

Here, geodesics connect two points but do not have the same direction at any of them. Bifurcation theory is presented in

Arnold V. I. et al. Bifurcation theory and catastrophe theory. Springer, Berlin, 1999.

Besides other examples, the following theorem is proved.

Let $\mathcal{S}_2$ be a surface of revolution given by the equations:

$$x = r(u)\cos v,\ \ y = r(u)\sin v,\ \ z = z(u),$$

where v is a parameter from $(-\pi, \pi)$ and $u \in I \subset (-1, 1) \subset \mathbb{R}$, $I = \langle u_1, u_2\rangle$, $r(u) = 1/\sqrt{1 - u^{2\alpha}}$, and $z'^2(u) + r'^2(u) = 1$.

Theorem

Geodesic bifurcations exist on the surface of revolution $\mathcal{S}_2$ for $\alpha \in (\frac{1}{2}, 1)$.

The statement can be proved by the existence of geodesics given by the equations:

$$I.\ u = 0,\ \ v = s; \quad \text{(gorge circle)}$$

$$II.\ u = ((1-\alpha)s)^{\frac{1}{1-\alpha}},\quad v = s - \frac{((1-\alpha)s)^{\frac{1+\alpha}{1-\alpha}}}{1+\alpha}.$$

It is straightforward to verify that the curves given by the above equations are geodesics. These two geodesics pass through the same point (0;0) and have the same tangent vector (0;1).

See: Rýparová L., Mikeš J. On geodesic bifurcations, Geometry, Integrability and Quantization 18, 2017, 217–224.

These properties are usually omitted in the standard textbooks of differential geometry.

The above result is used to construct the surface of revolution, where bifurcations of closed geodesic exist.

See: Rýparová L., Mikeš J. Bifurcation of closed geodesic, Geometry, Integrability and Quantizations 19, 2018, 188–192.

Next, there are constructed Riemannian and Kähler product spaces, where these bifurcations also exist. The last example shows that it is also possible to construct the pseudo-Riemannian spaces with bifurcations of isotropic geodesics, i.e., geodesics that have vanishing length.

See: Rýparová L., Mikeš J., Sabykanov A.: On geodesic bifurcations of product spaces, J. Math. Sci. 239:1, 2019, 86–91.

The metric of the above surface of revolution $\mathcal{S}_2$ has a form

$$ds^2 = du^2 + f(u)\,dv^2.$$

In our case $f(u) = r(u)^2$. As is well known, this surface admits geodesic mapping onto a surfaces of revolution $\bar{\mathcal{S}}_2$ with metrics

$$d\bar{s}^2 = \frac{p\,du^2}{(1+q\,f)^2} + \frac{p\,f(u)\,dv^2}{1+q\,f}, \qquad p, q - \mathrm{const.}$$

See: Mikeš J. On existence of nontrivial global geodesic mappings on n-dimensional compact surfaces of revolution.
DGA Conf., Brno, Czechosl., 1989. World Sci. 129–137, 1990.

Rýparová L., Mikeš J.: On rotary mappings of surfaces of revolution,
Proc. of 4th Conf. MITAV, 2017, Brno, 208–216.

Metrics of $\mathcal{S}_2$ and $\bar{\mathcal{S}}_2$ have a class of differentiability C^1,

but they are not of class C^2.

Therefore, the theories about them will not be realized for them.

Let V_n and $\bar{V}_n \in C^2$, then for geodesic mappings $V_n \to \bar{V}_n$ the Riemann and the Ricci tensors transform in this way

$$\bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik} \quad \text{and} \quad \bar{R}_{ij} = R_{ij} - (n-1)\psi_{ij},$$

where $\psi_{ij} = \psi_{i,j} - \psi_i \psi_j$,
and the Weyl tensor of projective curvature, which is defined in the following form

$$W^h_{ijk} = R^h_{ijk} + \frac{1}{n-1}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right),$$

is invariant.

The integrability conditions of the Sinyukov equations (5)

have the following form

(6) $$a_{i\alpha}R^{\alpha}_{jkl} + a_{j\alpha}R^{\alpha}_{ikl} = g_{ik}\lambda_{j,l} + g_{jk}\lambda_{i,l} - g_{il}\lambda_{j,k} - g_{jl}\lambda_{i,k}.$$

After contraction with g^{jk} we get

(7) $$n\lambda_{i,l} = \mu g_{il} - a_{i\alpha}R^{\alpha}_{l} + a_{\alpha\beta}R^{\alpha}{}_{il}{}^{\beta}$$

where $R^{\alpha}{}_{il}{}^{\beta} = g^{\beta k}R^{\alpha}{}_{ilk}$; $R^{\alpha}_{l} = g^{\alpha j}R_{jl}$ and $\mu = \lambda_{\alpha,\beta}g^{\alpha\beta}$.

From the analysis of the integrability conditions of fundamental equations of geodesic mappings Sinyukov obtained the following

Theorem

(Pseudo-) Riemannian spaces V_n admit geodesic mapping onto (pseudo-) Riemannian spaces $\bar{V}_n$ then and only then if in V_n exists Cauchy type closed linear system of partial differential equations respective unknown functions $a_{ij}(x)$, $\lambda_i(x)$ and $\mu(x)$:

$$\begin{aligned} a_{ij,k} &= \lambda_i g_{jk} + \lambda_j g_{ik}, \\ n\,\lambda_{i,k} &= \mu g_{ik} - a_{i\alpha} R^\alpha_k - a_{\alpha\beta} R^\alpha{}_{ik}{}^\beta, \\ (n-1)\,\mu_{,k} &= 2(n+1)\lambda_\alpha R^\alpha_k + a_{\alpha\beta}(2R^\alpha{}_{k,}{}^\beta - R^{\alpha\beta}{}_{,k}). \end{aligned} \tag{8}$$

See: N.S. Sinyukov, *Geodesic mappings of Riemannian spaces*. M.: Nauka, 1979.

From Sinyukov system (8) elementary follows that

If $V_n \in C^r$ $(r > 2)$ admits geodesic mappings onto $\bar{V}_n \in C^3$, then $\bar{V}_n \in C^r$.

Theorem

If $V_n \in C^r$ $(r > 2)$ admits geodesic mappings onto $\bar{V}_n \in C^3$,

then in neighbourhood of point x_0 which Riemannian tensor satisfy the following inequality

$$R_{hijk}(x_0) \not\equiv c \cdot (g_{hj}g_{ik} - g_{hk}g_{ij})|_{x_0}$$

system (8) is reduced in the following form

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} \tag{9}$$

where vector λ_i is linearly depend of tensor a_{ij} with coefficients which constructed of metric tensor g of V_n.

5. Geodesic mapping between $V_n \in C^r$ $(r > 2)$ and $\bar{V}_n \in C^1$

Theorem 2

If $V_n \in C^r$ $(r > 2)$ admits geodesic mappings onto $\bar{V}_n \in C^1$,

$$\text{then } \bar{V}_n \in C^r.$$

This means that the differentiability class of the metric is preserved under a geodesic mapping.

See:

I. Hinterleitner, J. Mikeš, *Geodesic mappings onto Riemannian manifolds and differentiability*. Geometry, integrability and quantization XVIII, 183–190, Bulgar. Acad. Sci., Sofia, 2017.

J. Mikeš at al. *DG of special mappings*, Olomouc Univ. Press, 2015. PP. 283–285.

Geodesic mapping theory for $A_n \to \bar{V}_n$

From analysis integrability conditions of fundamental equations of geodesic mappings $A_n \to \bar{V}_n$ we with Berezovskii obtained the following

Theorem

Equiaffine spaces A_n admit geodesic mapping onto (pseudo-) Riemannian spaces $\bar{V}_n$ then and only then if in A_n exists Cauchy type closed linear system of partial differential equations respective unknown functions $a^{ij}(x)$, $\lambda^i(x)$ and $\mu(x)$:

$$(10)\qquad \begin{aligned} a^{ij}{}_{,k} &= \lambda^i \delta^j_k + \lambda^j \delta^i_k, \\ n\,\lambda^i{}_{,k} &= \mu\delta^i_k - a^{i\alpha} R_{\alpha k} - a^{\alpha\beta} R^i_{\alpha\beta k}, \\ (n-1)\,\mu_{,k} &= 2(n+1)\lambda_\alpha R^\alpha_k + a_{\alpha\beta}(2R^\alpha{}_{k,}{}^\beta - R^{\alpha\beta}{}_{,k}). \end{aligned}$$

See: J. Mikeš, V. Berezovski, *Geodesic mappings of affine-connected spaces onto Riemannian spaces*. DGA (Eger, 1989), 491–494, Colloq. Math. Soc. J. Bolyai, 56, North-Holland, Amsterdam, 1992. MR1211677, Zbl 0792.53012.

J. Mikeš at al. *DG of special mappings*, Olomouc Univ. Press, 2015. PP. 281–283.

Geodesic mapping between $A_n \in C^{r-1}(r>2)$ and $\bar{V}_n \in C^1$

Theorem 2

If $A_n \in C^{r-1}$ $(r > 2)$ admits geodesic mappings onto $\bar{V}_n \in C^1$,

$$\text{then } \bar{V}_n \in C^r.$$

See:

I. Hinterleitner, J. Mikeš, *Geodesic mappings onto Riemannian manifolds and differentiability*. Geometry, integrability and quantization XVIII, 183–190,
Bulgar. Acad. Sci., Sofia, 2017.

J. Mikeš at al. *DG of special mappings*, Olomouc Univ. Press, 2015. PP. 283–285.

4. On geodesic mappings of Einstein spaces

Einstein spaces V_n are characterized by the condition

$$Ric = \mathrm{const} \cdot g,$$

so $V_n \in C^2$ would be sufficient.

We remark that spaces of constant curvature are Einstein spaces and three-dimensional Einstein spaces always have constant curvature. Therefore many properties of Einstein spaces appear when

$$V_n \in C^3 \quad \text{and} \quad n > 3.$$

It is known that Riemannian spaces of constant curvature form a closed class with respect to geodesic mappings (Beltrami theorem).

We continue with geodesic mappings of Einstein spaces $V_n \in C^3$. On basis of Theorem 1 it is natural to suppose that $\bar{V}_n \in C^3$.

4. On geodesic mappings of Einstein spaces

In 1978 (see PhD thesis – 1979, Math. Notes – 1980, J. Math. Sci. 1996) Mikeš proved that under these conditions the following theorem holds:

Theorem 3

If the Einstein space V_n admits a nontrivial geodesic mapping onto a (pseudo-) Riemannian space $\bar{V}_n$, then $\bar{V}_n$ is an Einstein space.

From Theorem 2 this Theorem holds for $V_n \in C^r$, $r \geq 3$, and $\bar{V}_n \in C^1$.
It is known (*) that an Einstein space V_n belongs to C^ω, i.e.
for all points of V_n, there exists local coordinate system x
for which $g_{ij}(x) \in C^\omega$ (analytic coordinate system).

(*) D.M. DeTurck, J.L. Kazdan, *Some regularity theorems in Riemannian geometry.* Ann. Sci. Éc. Norm. Supér. (4) 14:3, 249-260, 1981.

See: A. Besse, *Einstein spaces.* Springer, 1987.

From this result by DeTurck and Kazhdan the Theorem 3 holds GLOBALLY and exists common coordinate system in which

$$V_n \in C^\omega \text{ and } \bar{V}_n \in C^\omega.$$

On geodesic mappings of four-dimensional Einstein spaces

In 1982, based on these above results, (see [*]) Mikeš and Kiosak proved that the following theorem holds (locally):

Theorem 4

If the four-dimensional Einstein space V_4 admits a nontrivial geodesic mapping onto a (pseudo-) Riemannian space $\bar{V}_4$,
then V_4 and $\bar{V}_4$ are spaces with constant curvature.

From Theorem 2 and above results by Kazhdan and DeTurck this Theorem holds GLOBALLY for $V_4 \in C^r$, $r \geq 3$, and $\bar{V}_4 \in C^1$.

[*] J. Mikeš, V.A. Kiosak, *On geodesic maps of four dimensional Einstein spaces*. Odessk. Univ. Moscow: Archives at VINITI, 9.4.82, No. 1678-82, 19p. 1982.

See: J. Mikeš at al. *DG of special mappings*, Olomouc Univ. Press, 2015. PP. 326–327.

The article Kiosak V.A., Matveev V.S. Complete Einstein metrics are geodesically rigid. Comm. Math. Phys. 289:1, 383–400, 2009. deliberately claims that the proof is cumbersome, but it can be seen that this is not the case.

On projective transformations of Einstein spaces

Finally we notes that in

J. Mikeš, On geodesic mappings of Einstein spaces. Math. Notes 28, 922-923, 1980; transl. from Mat. Zametki 28, 935-938, 1980.

is proved the following

Theorem 5

An Einstein space with nowhere-vanishing scalar curvature admits a nontrivial geodesic mapping if and only if it admits a nontrivial projective transformation.
If an Einsteinian space with zero scalar curvature admits a nontrivial geodesic mapping, then it admits an affine transformation.

We with S. Formella obtained metrics of all Einstein spaces which admit geodesic mappings.

S. Formella, J. Mikeš, *Geodesic mappings of Einstein spaces.* Szczecińske rocz. naukove, Ann. Sci. Stetinenses. **9** I. (1994) 31-40.

Spaces not admitting nontrivial geodesic mappings and projective transformations

A very important problem is the distinguishing of spaces which do not admit nontrivial geodesic mappings (NGM) and nontrivial projective transformations (NPT). These spaces are defined (modulo affine mappings) by the position of geodesic lines, and from the viewpoint of possible application to physics — by the position of model particle trajectories. This is the RIGIDITY problem respective geodesic mappings.
As we have already mentioned, the spaces where NGM do not exist admit neither NPT nor null-geodesic transformations, nor covariantly nonconstant concircular vector fields; for brevity we shall not stress this again.
The problem mentioned was considered in two aspects:

local and *global*.

It is natural that if a space does not admit NGM everywhere,
it does not admit global NGM. On the other hand, a space where global NGM does not exist can admit NGM locally in some regions.

Spaces not admitting nontrivial geodesic mappings locally

In 1953, H. Takeno and M. Ikeda (Theory of spherically symmetric space-times. VII. Space-times with corresponding geodesics. J. Sci. Hiroshima Univ. 1953, A17. 1 75–81) considered GM from spherically symmetric spaces V_4,

and in 1954, N.S. Sinyukov (On geodesic mappings of Riemannian spaces onto symmetric Riemannian spaces. Dokl. Akad. Nauk SSSR, n. Ser. 98, 21-23(1954)) proved that the symmetric and recurrent Riemannian spaces V_n with nonconstant curvature do not admit NGM.

Let us recall that the *symmetric spaces* S_n^1 and the *recurrent spaces* K_n^1 are characterized by the conditions

$$R^h_{ijk,l} = 0 \text{ and } R^h_{ijk,l} = \varphi_l R^h_{ijk},\ \varphi_l \not\equiv 0, \text{ respectively.}$$

The spaces S_n^1 were first investigated by P.A. Shirokov and E. Cartan; the spaces K_n^1 were introduced by Ruse.

In [1] projectively Euclidean spaces S_n^1 and K_n^1 were studied.

[1] P.A. Shirokov, Selected Investigations on Geometry, Kazan', 1966;
J. Mikeš, Dj. Moldobayev, and A. Sabykanov On recurrent equiaffine projectively Euclidean spaces. Issled. po Topol. i Geometrii, Bishkek, 1991;
A. Sabykanov, J. Mikeš, P. Peška, Symmetric, semisymmetric, and recurrent projectively Euclidean spaces. J. Math. Sci., New York 276, No. 3, 410-416 (2023).

The geodesic mappings from
projectively symmetric ($WS^1_n : W^h_{ijk,l} = 0$) and
it projectively recurrent ($WK^1_n : W^h_{ijk,l} = \varphi_l W^h_{ijk}; \varphi_l \not\equiv 0$)
(pseudo-) Riemannian spaces were studied in many papers.

But it turned out that $WS^1_n \equiv S^1_n$ and $WK^1_n \equiv K^1_n \cup SCC$, see W. Glodek A note on riemannian spaces with recurrent projective curvature. Pr. Nauk. Inst. Mat. Fiz. Teor. Wr. Ser. Stud. Mater. 1970, 1, 9–12.

It is shown for affine-connected spaces that $WS^1_n \equiv S^1_n$.
But there exist spaces $WS^1_n \not\equiv S^1_n$ and $WK^1_n \not\equiv K^1_n$,
see J. Mikeš Projectively symmetric and projectively recurrent spaces of an affine connection Tr. Geometr. Sem. Kazan', 1981, 13, 61–62.

V.R. Kaygorodov (in 1984) introduced into consideration the *generally recurrent spaces D_n^m*, defined by the conditions

$$R^h_{ijk,l_1 l_2 \cdots l_m} = \sum_{s=1}^{m} \overset{s}{\Omega}_{l_s l_{s+1} \cdots l_m} \, R^h_{ijk,l_1 l_2 \cdots l_{s-1}},$$

where $\overset{s}{\Omega}$ are some tensors.

The spaces where $R^h_{ijk,l_1 l_2 \ldots l_m} = 0$ are called *m-symmetric S_n^m*, and the spaces where $R^h_{ijk,l_1 l_2 \ldots l_m} = \Omega_{l_1 l_2 \ldots l_m} R^h_{ijk}$, $\Omega \not\equiv 0$, are called *m-recurrent K_n^m*.

Note that many spaces D_n^m are semisymmetric spaces Ps_n. In particular, S_n^1, S_n^2, $K_n^m \subset Ps_n$.

In 1968 W. Roter showed that K_n^2 do not admit NPT, and in 1976 J. Mikeš proved that the spaces considered below with nonconstant curvature do not admit NGM: K_n^m; S_n^2; D_n^2; D_n^m, where $\overset{2}{\Omega} \not\equiv 0$.

In 1991 V.S. Sobchuk added to this list the semisymmetric spaces S_n^m.

Neither do S_n^3, $n > 4$ (see [Mikeš, Sobchuk 1991]), S_n^4, $n > 4$ (see [Sobchuk 1991]), and S_n^m, $2n > m + 3$ (see [Mikeš 1992]) of nonconstant curvature, which cannot be semisymmetric, admit NGM.

As we have mentioned, four-dimensional Einsteinian spaces with nonconstant curvature do not admit NGM

J. Mikeš, V.A. Kiosak On geodesic mappings of four-dimensional Einsteinian spaces. Odessk. Univ. Deposited at VINITI, 9.04.82, No. 1678-82 (1982).

This is true also for non-Einsteinian Ricci-symmetric ($R_{ij,k} = 0$), Ricci-2-symmetric ($R_{ij,kl} = 0$), see J. Mikeš, On geodesic mappings of 2-Ricci symmetric Riemannian spaces. Math. Notes 28, 622-624, 1981,
Ricci-3-symmetric ($R_{ij,klm} = 0$, $n > 4$, see [Mikeš, Sobchuk 1991]),
Ricci-4-symmetric ($R_{ij,klmp} = 0$, $n > 4$, see [Sobchuk 1991]) and
Ricci-m-symmetric ($R_{ij,l_1 l_2 \ldots l_m} = 0$, $2n > m + 2$, see [Mikeš 1992]) spaces.
Supposing in above formulas, instead of the Riemannian tensor, the Weyl tensor of the projective curvature W^h_{ijk}, we shall define the *projective generally recurrent spaces* WD^m_n, WS^m_n, and WK^m_n. For Riemannian spaces, $WS^m_n \equiv S^m_n$ and $WK^2_n \equiv K^2_n \cup SCC$.
The situations where $WK^m_n \equiv K^m_n$ are considered.
In Mikeš J., Tolobaev O.S. Symmetric and projectively symmetric affinely connected spaces. Collect. Sci. Works, Frunze/Kyrgyzstan, 58-63, 1988. examples of equiaffine spaces $WD^m_n \not\equiv D^m_n$, $WS^m_n \not\equiv S^m_n$, and $WK^m_n \not\equiv K^m_n$ are given.

Similar problems were considered for *conformally symmetric* CS_n^1 and for *conformally recurrent* CK_n^1, as well as for spaces CS_n^2 and CK_n^2. In this case, the Riemannian tensor in above formula is replaced by the Weyl tensor of conformal curvature C^h_{ijk}.
It is obvious that $CS_n^1,\ CK_n^1,\ CS_n^2,\ CK_n^2 \subset CPs_n \subset CPs_n(B)$.
The conformally generally semisymmetric spaces $CPs_n(B) \not\equiv Ps_n(B)$ with $C^h_{ijk} \not\equiv\ 0$ do not admit NGM.
Spaces not admitting NGM have been specified under more complicated recurrent conditions:

$$\text{(a)} \quad T^{(\alpha\beta)}_{ijkl,m} = T^{(\alpha\beta)}_{\gamma\delta\varepsilon\nu} S^{\gamma\delta\varepsilon\nu}_{ijklm}, \qquad \text{(b)} \quad T^{(\alpha\beta)}_{ij,k} = R^{(\alpha}_{\gamma} S^{\beta)\gamma}_{ijk} - R^{(\alpha}_{\gamma} S^{|\gamma|\beta)}_{ijk},$$

where $T^{\alpha\beta}_{ijkl} \equiv \delta^{\alpha}_{(i} R^{\beta}_{j)kl} - g_{k(i} M^{\alpha\beta}_{j)l} + g_{l(i} M^{\alpha\beta}_{j)k};\ \ nM^{\alpha\beta}_{jl} \equiv \delta^{\alpha}_{j} R^{\beta}_{l} - R^{\alpha\ \beta}_{.jl.}$,

$T^{\alpha\beta}_{ij} \equiv \delta^{\alpha}_{[i} A^{\beta}_{j]};\ A^i_j \equiv \sum\limits_{s=0}^{m} R^{s|\,i}_{j} a^s;\ R^{s+1|\,i}_{j} \equiv R^{s|\,i}_{\alpha} R^{\alpha}_{j};\ R^{0|i}_{j} \equiv \delta^i_j;$
$Rg\|A^i_j - \rho\delta^i_j\| > 1,\ W^h_{ijk} \not\equiv 0.$ and a^s are some functions.

The conditions (a) were obtained in [Sinyukov 1979]. The partial cases of (b) are given in [Mikeš 1988]; these conditions, in particular, are satisfied by Ricci-recurrent spaces; see [Mikeš 1988].
We can prove that there are no spaces $V_n(0)$ in which one of the following conditions holds:

(a) $R_{hijk,l} + R_{hi\alpha\beta,\gamma} S^{\alpha\beta\gamma}_{jk} S_l = R_{hi\alpha\beta} S^{\alpha\beta}_{jkl}$

(b) $R_{hijk,lm} = R_{hi\alpha\beta} S^{\alpha\beta}_{jklm} + R_{hijk,\alpha} S^{\alpha}_{lm}$;

(c) $R_{hijk,lmp} = R_{hi\alpha\beta} S^{\alpha\beta}_{jklmp} + R_{hijk,\alpha} S^{\alpha\beta}_{lmp} + R_{hijk,(lm} S_{p)}$; $R_{hijk,[lm]} = 0$;

(d) $R_{hijk,l_1 l_2 \cdots l_m} = R_{hi\alpha\beta} S^{\alpha\beta}_{jkl_1 l_2 \cdots l_m}$;

(e) $(R_{ij,k} - R_{ik,j})_{,l_1 l_2 \cdots l_m} = R_{jk\alpha\beta} S^{\alpha\beta}_{il_1 l_2 \cdots l_m}$,

where S is some tensor, and $R^h_{ijk} \not\equiv 0$.
On the basis of Theorem of semisymmetric ($\not\equiv SCC$), Ricci-semisymmetric ($\not\equiv ES$), Ricci-flat, Kählerian, and reducible spaces, in which these conditions hold, do not admit NGM.
For special cases, this result was obtained in [Mikeš 1988, 1991, Sobchuk 1988, 1991].

Riemannian spaces $\overline{V}_n$ $(n > 4)$ with nonconstant curvature do not admit NGM onto the spaces of an affine connection A_n, where the following conditions hold:

$$R^h_{ijk,l} = \varphi_l R^h_{ijk} + \nu_j R^h_{ikl} - \nu_k R^h_{ijl} + \delta^h_i A_{jkl} + \delta^h_j B_{ikl} - \delta^h_k B_{ijl} + \delta^h_l C_{ijk},$$

where φ, ν, A, B, C are some tensors. If A_n is equiaffine and $A_{jkl} = 0$, then the theorem is also valid for $n = 3, 4$, see [Mikeš 1988].

For equiaffine symmetric spaces S^1_n this theorem was proved in

Sinyukov, N. S. On geodesic mappings of Riemannian spaces onto symmetric Riemannian spaces. Dokl. Akad. Nauk SSSR, Ser. 98, 21-23 (1954).

For infinite-dimensional spaces this result was obtained by V.E. Fomin.

Fomin, V. E. A pair of infinite-dimensional Levi-Civita spaces can have no common geodesics. Tr. Geom. Semin. 17, 79-83 (1986).

In conclusion, we note that the above results were also for the Weyl space:

Hinterleitner I., Mikeš J. Geodesic mappings onto Weyl manifolds. J. Appl. Math. 2:1, 125-133, 2009.

Spaces not admitting nontrivial global geodesic mappings

G. Vranceanu "Proprietati globale ale spatiilor bui Riemann cu conexiune abina constanta," Stud. Cerc. Mat. Acad. RPR, 14, No. 1, 7-22 (1963). found compact spaces *without a boundary* which do not admit global NGM;
the work C. Simonescu, "Varietati Riemann in corepondenta geodezica definite pe un suport compact," In: Lucr. Sti. Inst. Politech. Brasov. Fac. Mec., Vol. 5 (1961), pp. 15-19. was devoted to the same problems.

Afterwards, the investigations of global GM of compact spaces without a boundary were continued by P. Venzi, E.N. Sinuykova, and J. Mikeš, see J. Mikeš, Geodesic mappings of affine-connected and Riemannian spaces. J. Math. Sci. (New York) 78:3, 311-333, 1996.

In paper J. Mikeš, On concircular global vector fields on compact Riemarmian spaces, Odessk. Univ. (1988); Deposited at Ukr. NIINTI, Kiev, 02.03.88, No. 615-Uk88 (1988). is devoted the following results:

It is correct that non-Einsteinian compact Ricci-semisymmetric Riemannian spaces $RicPs_n \in C^4$ do not admit global NGM onto Riemannian spaces $\overline{V}_n \in C^3$.

Compact Kählerian ($\in C^2$), Ricci-flat ($\in C^3$), and reducible ($\in C^2$) spaces V_n do not admit global NGM onto $\overline{V}_n$ of the same class. For properly Riemannian spaces this result is given in

E.N. Sinyukova, On geodesic mappings of some special Riemannian spaces, Mat. Zametki, 30, No. 6, 889-894 (1981).

In Sinyukova's papers, some spaces where global NGM do not exist were specified. Here also it is proved that compact properly Riemannian Einsteinian spaces $V_n \in C^2$ with nonnegative curvature do not admit NGM. This theorem generalizes and reinforces the results of R. Couty, "Transformations projectives sur un espace d'Einstein complect," C.R., Acad. Sci., 252, No. 8, 1096-1097 (1961).

We assume that the (pseudo-)Riemannian space V_n under consideration is connected and either it has no boundary or its boundary ∂V_n is (almost) Lipschitz. The following theorem is valid.

J. Mikeš, I. Hinterleitner, N.I. Guseva, Geodesic maps "in the large" of Ricci-flat spaces with n Complete geodesic lines, Math. Notes, 108(1), 292-296

Theorem 1

Suppose that a (pseudo-) Riemannian space $V_n \in C^1$ with metric g admits a geodesic mapping f onto a Ricci-flat (pseudo-)Riemannian space $\bar{V}_n$. If the scalar curvature R vanishes at a point $x_0 \in V_n$ and, throught his point, complete geodesic lines γ with complete images $f(\gamma)$ pass in n linearly independent directions, then this mapping is affine.

Theorem 2

Suppose that a (pseudo-)Riemannian space $V_n \in C^1$ with metric g admits a geodesic mapping f onto a Ricci-flat (pseudo-)Riemannian space $\bar{V}_n$.
If the sectional curvatures are not all equal at a point $x_0 \in V_n$ and, through this point, complete geodesic lines γ with complete images $f(\gamma)$ for which $R(x_0) \cdot g(\dot{\gamma}, \dot{\gamma}) \leq 0$ pass in n linearly independent directions, then this mapping is affine.

Note that the light line is an isotropic geodesic line and has tangent vector of length zero, i.e., $g(\dot{\gamma}, \dot{\gamma}) = 0$. Since the length of a vector is invariant under translation along a geodesic line, it follows that the quantity $g(\dot{\gamma}, \dot{\gamma})$ is invariant along the entire line γ.
A similar result was obtained by these authors for conformal mappings of Einstein spaces in

J. Mikeš, I. Hinterleitner, N. Guseva, there are no conformal Einstein rescalings of pseudo-Riemannian Einstein spaces with n complete light-like geodesics Mathematics 2019, 7(9), 801; https://doi.org/10.3390/math7090801

Using the methods of A. Švec (Afwat M., Švec A. Global differential geometry of hypersurfaces. Rozpr. ČSAV, 88:7, 75p. 1978),
J. Mikeš (Mikeš J. Global geodesic mappings and their generalizations for compact Riemannian spaces. Math. Publ. Opava. 1, 143-149, 1993)
obtained the following results for affine and geodesic mappings of compact orientable properly Riemannian spaces $V_n \in C^2$ $(n \geq 2)$ *with a boundary* ∂V.

Let V_n $(n \geq 2)$ admit an affine mapping onto $\overline{V}_n \in C^2$. If, at any point $m \in V_n$, the sectional curvature is positive (or nonnegative) for each two-dimensional direction and for each point $m \in \partial V$: $\overline{g}(X, Y) = f\, g(X, Y)$, then this mapping will be homothetic.

Let V_n admit GM onto $\overline{V}_n \in C^2$. If, at any point $m \in V_n$, the sectional curvature is nonpositive for each two-dimensional direction and for each point $m \in \partial V$: $2\overline{g}(X, Y) = f\, g(X, Y)$, then this mapping will be affine.

5. Holomorphically projective mapping theory for $K_n \to \bar{K}_n$ of class C^1

Assume the (elliptic, hyperbolic or parabolic) Kähler manifolds

$$K_n = (M, g, F) \text{ and } \bar{K}_n = (\bar{M}, \bar{g}, F).$$

Here K_n, $\bar{K}_n \in C^1$, i.e. $g, \bar{g} \in C^1$ which means that their components g_{ij}, $\bar{g}_{ij} \in C^1$ and F is complex or product structure for which

$$F^2 = \varepsilon\, I, \qquad g(X, FX) = \bar{g}(X, FX) = 0 \text{ for } \forall X, \qquad \nabla F = \bar{\nabla} F = 0,$$

where ∇ and $\bar{\nabla}$ are Levi-Civita connection of K_n and $\bar{K}_n$, $\varepsilon = \pm 1, 0$.

Definition

A diffeomorphism $f\colon K_n \to \bar{K}_n$ is called a *holomorphically projective mapping* of K_n onto $\bar{K}_n$ if f maps any analytically planar curve in $K_n$onto an analytically planar curve in $\bar{K}_n$.

Otsuki, Tashiro, Prvanović, ...

A manifold K_n admits a holomorphically projective mapping onto $\bar{K}_n$ if and only if the equations (for $\varepsilon = \pm 1$):

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X - \varepsilon\psi(FX)FY - \varepsilon\psi(FY)FX, \tag{11}$$

hold for any tangent fields X, Y and where ψ is a differential form. If $\psi \equiv 0$ than f is *affine* or *trivially holomorphically projective*.

In local form:

$$\bar{\Gamma}_{ij}^h = \Gamma_{ij}^h + \psi_i\delta_j^h + \psi_j\delta_i^h + \varepsilon\psi_\alpha F_i^\alpha F_j^h + \varepsilon\psi_\alpha F_j^\alpha F_i^h,$$

where Γ_{ij}^h and $\bar{\Gamma}_{ij}^h$ are the Christoffel symbols of K_n and $\bar{K}_n$, ψ_i, F_i^h are components of ψ, F, and δ_i^h is the Kronecker delta.

5. HP mapping theory for $K_n \to \bar{K}_n$ of class C^3

From analysis integrability conditions of fundamental equations of holomorphically projective mappings we obtained the following

Theorem

(Pseudo-) Kähler spaces K_n admit geodesic mapping onto (pseudo-) Kähler spaces $\bar{K}_n$ then and only then if in K_n exists Cauchy type closed linear system of partial differential equations respective unknown functions $a_{ij}(x)$, $\lambda_i(x)$ and $\mu(x)$:

$$
\begin{array}{rcl}
a_{ij,k} &=& \lambda_i g_{jk} + \lambda_j g_{ik} + \lambda_{\bar{i}} g_{\bar{j}\,k} + \lambda_{\bar{j}} g_{\bar{i}\,k}, \\
n\,\lambda_{i,k} &=& \mu g_{ik} - a_{i\alpha} R^\alpha_k - a_{\alpha\beta} R^{\alpha}{}_{ik}{}^{\beta}, \\
\mu_{,k} &=& -2\lambda_\alpha R^\alpha_k
\end{array}
\qquad (12)
$$

See:

J. Mikeš, *On holomorphically projective mappings of Kählerian spaces*. Ukr. Geom. Sb. 23, 90-98, 1980.

In: N.S. Sinyukov, *Geodesic mappings of Riemannian spaces*. M.: Nauka, 1979.

6. Holomorphically projective mapping between $K_n \in C^r$ $(r \geq 2)$ and $\bar{K}_n \in C^2$

In paper I. Hinterleitner, *Holomorphically projective mappings of (pseudo-) Kähler manifolds preserve the class of differentiability*. Filomat 30 (2016), no. 11, 3115–3122.
it is proved the above results for $K_n \in C^r$ $(r > 2)$ and $\bar{K}_n \in C^r$.

We obtained the following (J. Mikeš, I. Hinterleitner, P. Peška, and L. Vítková. Rigidity of holomorphically projective mappings of Kähler and hyperbolic Kähler spaces with finite complete geodesics. Geometry 2025, 2, 3, https://doi.org/10.3390/geometry2010003)

Theorem

If $K_n \in C^r$ $(r \geq 2)$ admits holomorphically projective mappings onto $\bar{K}_n \in C^2$, then $\bar{K}_n \in C^r$.

For parabolical Kähler spaces this result is proved in the paper
P. Peška, J. Mikeš, H. Chudá, M. Shiha, M. *On holomorphically projective mappings of parabolic Kähler manifolds.* Miskolc Math. Notes 17 (2016), no. 2, 1011–1019.

7. F-planar mapping theory

Assume the manifolds with affine connection with structures

$$A_n = (M, \nabla, F) \text{ and } \bar{A}_n = (\bar{M}, \bar{\nabla}, \bar{F}).$$

A curve ℓ is caled *F-planar* if its tangent vector λ satisfies

$$\nabla_t \lambda = a(t) \cdot \lambda + b(t) \cdot F(\lambda).$$

If $b(t) = 0$ then ℓ is a geodesic.

Definition

A diffeomorphism $f\colon A_n \to \bar{A}_n$ is called an F-planar mapping of A_n onto $\bar{A}_n$ if f maps any F-planar curve in A_n onto a F-planar curve in $\bar{A}_n$.

J. Mikeš J., N.S. Sinyukov, *On quasiplanar mappings of spaces of affine connection*. Sov. Math. 27:1, 63-70, 1983; transl. from Izv. Vyssh. Uchebn. Zaved., Mat. 248:1, 55-61, 1983.

For $n > 3$ Mikeš and Sinyukov proved the following

Theorem

A manifold A_n admits an F-planar mapping onto $\bar{A}_n$ if and only if the equations:

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X + \varphi(X)FY + \varphi(Y)FX, \tag{13}$$

hold for any tangent fields X, Y and where ψ and φ is a differential form. Moreover, it holds that structures are preserved: $\bar{F} = \alpha F + \beta Id$.
In local form: $\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_i \delta^h_j + \psi_j \delta^h_i + \varphi_i F^h_j + \varphi_j F^h_i$, $\bar{F}^h_i = \alpha F^h_i + \beta \delta^h_i$.
where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are the Christoffel symbols of K_n and $\bar{K}_n$,
ψ_i, φ_i, F^h_i are components of ψ, φ, F, and α, β are functions on A_n.

For $n \geq 3$, a more correct proof of these statements is given in the article

I. Hinterleitner, J. Mikeš, *On F-planar mappings of spaces with affine connections.* Note Mat. 27, No. 1, 111-118 (2007).

Finally, we remark, that in paper

I. Hinterleitner, J. Mikeš, and P. Peška, *Fundamental equations of F-planar mappings.* Lobachevskii J. Math. 2017, Vol. 38, No. 4, pp. 653–659.

there are proved fundamental equations of geodesic, holomorphically-projective and F-planar mappings from several different points of view. Among other things using operator forms.

In paper

J. Mikeš, *Special F-planar mappings of affinely connected spaces onto Riemannian spaces.* Mosc. Univ. Math. Bull. 49:3, 15-21, 1994.

there were found fundamental equations of F-planar mappings from A_n onto (pseudo-) Riemannian spaces $\bar{V}_n$ in form of linear system Cauchy type differential equations in covariant derivative.

Thank you very much for you attention!

Olomouc, the metropolis of Moravia