Space of Split Curvature

Patrik Peška

Department of Algebra and Geometry
Faculty of Science, Palacký University Olomouc

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Talk plan

1. Motivation and definition of split curvature
2. Why every space of split curvature is semisymmetric
3. Classification by the rank of the Ricci tensor
4. Geodesic mappings and the Sinyukov equation
5. A counterexample with non-constant scalar curvature
6. Equidistant spaces and concircular vector fields

Main message

Spaces of split curvature form a very restrictive curvature class, but they still admit rich geodesic mapping behaviour. In particular, nontrivial geodesic mappings do *not* force the scalar curvature to be constant.

Roadmap

Split curvature

$\Downarrow$

Semisymmetry

$\Downarrow$

Rank classification

$\Downarrow$

Geodesic mappings

$\Downarrow$

Counterexample

Background: curvature tensors

Let $(\mathbb{V}_n, g)$, $n > 2$, be a (pseudo-)Riemannian space with Levi-Civita connection ∇.

$$\Gamma^h_{ij} = g^{h\alpha}\Gamma_{ij\alpha},$$

$$\Gamma_{ijk} = \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right),$$

$$R^h{}_{ijk} = \partial_j \Gamma^h_{ik} + \Gamma^\alpha_{ik}\Gamma^h_{j\alpha} - \partial_k \Gamma^h_{ij} - \Gamma^\alpha_{ij}\Gamma^h_{k\alpha},$$

$$R_{ij} = R^\alpha{}_{i\alpha j}, \qquad \mathcal{R} = g^{\alpha\beta}R_{\alpha\beta}.$$

Convention

The sign convention is fixed explicitly, because curvature identities are sensitive to signs.

Definition: split curvature

Definition

A space $(\mathbb{V}_n, g)$, $n > 2$, is called a space of split curvature if its Riemann tensor has the form

$$R_{hijk} = \varepsilon\, S_{hi} S_{jk}, \qquad \varepsilon = \pm 1,$$

where S_{hi} is a non-vanishing tensor.

Equivalently,

$$R = \varepsilon\, S \otimes S.$$

- The tensor S is necessarily skew-symmetric.
- Dimension $n = 2$ is excluded, since the condition is automatically satisfied there.
- Allowing $\varepsilon = -1$ is important in the (pseudo-)Riemannian setting.

First central result

Theorem

Every space of split curvature is semisymmetric.

A (pseudo-)Riemannian space is semisymmetric if

$$R^h{}_{ijk,[lm]} = 0,$$

where the comma denotes covariant derivative and brackets denote alternation.
Equivalently, by the Ricci identity,

$$R \circ R = 0.$$

Why this matters

The result places spaces of split curvature inside the classical theory of semisymmetric spaces, studied by Shirokov, Cartan, Sinyukov, Szabó, Mikeš, and others.

Proof idea: from S-semisymmetry to semisymmetry

Starting from

$$R_{hijk} = \varepsilon\, S_{hi} S_{jk},$$

one obtains

$$S_{h\alpha} R^{\alpha}{}_{ijk} = \varepsilon\, S_{h\alpha} S_{\beta i} g^{\alpha\beta} S_{jk}.$$

Because S is skew-symmetric,

$$S_{h\alpha} R^{\alpha}{}_{ijk} + S_{\alpha i} R^{\alpha}{}_{hjk} = 0.$$

By the Ricci identity this is

$$S_{hi,[jk]} = 0.$$

Then

$$R_{hijk,[lm]} = \varepsilon \{S_{hi} S_{jk}\}_{,[lm]} = \varepsilon \left(S_{hi,[lm]} S_{jk} + S_{hi} S_{jk,[lm]} \right) = 0.$$

Bianchi identity and decomposability of S

The first Bianchi identity gives

$$R_{hijk} + R_{ijhk} + R_{jhik} = 0.$$

Using $R_{hijk} = \varepsilon S_{hi} S_{jk}$, this becomes

$$S_{hi} S_{jk} + S_{hj} S_{ki} + S_{hk} S_{ij} = 0.$$

Since $S \neq 0$ and S is skew-symmetric, there exist one-forms a, b such that

$$S_{jk} = a_j b_k - a_k b_j.$$

Thus

$$R_{hijk} = \varepsilon\,(a_h b_i - a_i b_h)(a_j b_k - a_k b_j).$$

Ricci tensor and scalar curvature

From the decomposable form of $S = a \wedge b$, the Ricci tensor is

$$R_{ij} = \varepsilon\Big\{(b^\alpha b_\alpha)a_i a_j + (a^\alpha a_\alpha)b_i b_j - (a^\alpha b_\alpha)(a_i b_j + a_j b_i)\Big\}.$$

The scalar curvature is

$$\mathcal{R} = 2\varepsilon\Big\{(a^\alpha a_\alpha)(b^\beta b_\beta) - (a^\alpha b_\alpha)^2\Big\}.$$

Interpretation

The scalar curvature is controlled by the Gram determinant of a and b.

Corrected classification

Classification by rank Ric

Spaces of split curvature divide naturally into three disjoint types:

Type	Ricci rank	Condition
A	$\operatorname{rank}\operatorname{Ric} = 2$	$(a^\alpha b_\alpha)^2 \neq (a^\alpha a_\alpha)(b^\beta b_\beta)$
B	$\operatorname{rank}\operatorname{Ric} = 1$	equality holds, but at least one of $a^\alpha a_\alpha$, $b^\alpha b_\alpha$ is nonzero
C	$\operatorname{rank}\operatorname{Ric} = 0$	$a^\alpha a_\alpha = b^\alpha b_\alpha = a^\alpha b_\alpha = 0$

Type A has nonzero scalar curvature; types B and C have $\mathcal{R} = 0$. Type C is Ricci-flat.

Why a previous classification fails

A previous theorem used an identity of the form

$$\frac{1}{\mathcal{R}^2} R_{ijkl} = R_{li} R_{kj} - R_{lj} R_{ki}.$$

But this expression is not meaningful when

$$(a^\alpha b_\alpha)^2 = (a^\alpha a_\alpha)(b^\beta b_\beta),$$

because then

$$\mathcal{R} = 0.$$

Correction

A classification based on rank Ric avoids division by the scalar curvature and covers all cases, including isotropic and (pseudo-)Riemannian possibilities.

Geodesic mappings

Let $\mathbb{A}_n$ and $\overline{\mathbb{A}}_n$ be spaces with affine connections. A diffeomorphism is a geodesic mapping if it maps geodesics onto geodesics.
In common coordinates, the Levi-Civita equations are

$$\bar{\Gamma}_{ij}^h = \Gamma_{ij}^h + \psi_i \delta_j^h + \psi_j \delta_i^h.$$

For (pseudo-)Riemannian spaces this is equivalent to

$$\bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik}.$$

If $\psi_i \not\equiv 0$, the mapping is nontrivial.

Linear form: the Sinyukov equation

The Levi-Civita equations are equivalent to the linear Sinyukov equation

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik},$$

where a_{ij} is a symmetric regular tensor and λ_i is a gradient vector.
The integrability conditions include

$$a_{i\alpha} R^{\alpha}{}_{jkl} + a_{j\alpha} R^{\alpha}{}_{ikl} = 0, \qquad \lambda_\alpha R^{\alpha}{}_{ijk} = 0.$$

For split curvature this yields

$$a_{i\alpha} a^{\alpha} = \tau a_i, \qquad a_{i\alpha} b^{\alpha} = \tau b_i, \qquad \lambda_\alpha a^{\alpha} = \lambda_\alpha b^{\alpha} = 0.$$

Scalar curvature under geodesic mappings

A previously stated theorem claimed:

> *Every space of split curvature admitting a nontrivial geodesic mapping has constant scalar curvature.*

This is true automatically in types B and C, because $\mathcal{R} = 0$.

But in general it is false

There exist spaces of split curvature that admit nontrivial geodesic mappings and have non-constant scalar curvature.

The proof is by an explicit 3-dimensional example.

The 3-dimensional example

Consider $\mathbb{V}_3$ with metric

$$g_{ij} = \mathrm{diag}(1, a, a), \qquad a = (x^1)^2 f(x^2, x^3) \neq 0.$$

Equivalently,

$$ds^2 = (dx^1)^2 + (x^1)^2 f(x^2, x^3)\big((dx^2)^2 + (dx^3)^2\big).$$

Let

$$\omega = f^2 - \frac{|\nabla f|^2}{2f} + \frac{\Delta f}{2},$$

where

$$|\nabla f|^2 = f_2^2 + f_3^2, \qquad \Delta f = f_{22} + f_{33}.$$

Curvature of the example

For the metric

$$ds^2 = (dx^1)^2 + (x^1)^2 f(x^2, x^3)((dx^2)^2 + (dx^3)^2),$$

there is one independent nonzero curvature component:

$$R_{2323} = (x^1)^2 \omega.$$

The Ricci tensor satisfies

$$R_{11} = 0, \qquad R_{22} = R_{33} = \frac{\omega}{f}.$$

The scalar curvature is

$$\mathcal{R} = \frac{2\omega}{(x^1)^2 f^2}.$$

If $\omega \neq 0$, the space is non-flat and is of split curvature of type A.

Non-constant scalar curvature

Since

$$\mathcal{R} = \frac{2\omega}{(x^1)^2 f^2},$$

and $\omega/f^2 \neq 0$ is independent of x^1, we have

$$\partial_1 \mathcal{R} \neq 0.$$

Therefore the scalar curvature is non-constant.

Consequence

Any non-flat space $\mathbb{V}_3$ with this metric is a type A space of split curvature with non-constant scalar curvature.

The nontrivial geodesic mapping

The metric

$$ds^2 = (dx^1)^2 + (x^1)^2 f(x^2, x^3)((dx^2)^2 + (dx^3)^2)$$

admits a geodesic mapping onto a space with metric

$$d\bar{s}^2 = \frac{p}{(1 + q(x^1)^2)^2}(dx^1)^2 + \frac{p(x^1)^2}{1 + q(x^1)^2} f(x^2, x^3)((dx^2)^2 + (dx^3)^2),$$

where

$$p \neq 0, \qquad q \neq 0, \qquad 1 + q(x^1)^2 \neq 0.$$

The corresponding function is

$$\psi = -\frac{1}{2} \ln |1 + q(x^1)^2|.$$

Counterexample: conclusion

We have constructed a space satisfying three properties:

1. It is a non-flat space of split curvature.
2. It admits a nontrivial geodesic mapping.
3. Its scalar curvature is non-constant.

Therefore

The claim that split curvature plus nontrivial geodesic mapping implies constant scalar curvature is false.

The corrected picture is more subtle: the scalar curvature is automatically zero only in types B and C, but type A allows non-constant scalar curvature.

Equidistant spaces and concircular vector fields

A (pseudo-)Riemannian space is equidistant if there exists a nonzero vector field φ_i such that

$$\varphi_{i,j} = \tau g_{ij}.$$

- In Yano's terminology, φ_i is a concircular vector field.
- If τ is constant, it is called convergent.
- The integrability condition is
$$\varphi_\alpha R^\alpha{}_{ijk} = 0.$$

For split curvature this implies

$$\varphi_\alpha a^\alpha = 0, \qquad \varphi_\alpha b^\alpha = 0.$$

The example is equidistant

For the metric

$$ds^2 = (dx^1)^2 + (x^1)^2 f(x^2, x^3)((dx^2)^2 + (dx^3)^2),$$

consider

$$\varphi_i = x^1 \delta_i^1.$$

A direct computation gives

$$\varphi_{1,1} = 1, \qquad \varphi_{2,2} = (x^1)^2 f, \qquad \varphi_{3,3} = (x^1)^2 f.$$

Thus

$$\varphi_{i,j} = g_{ij},$$

so $\tau = 1$ and φ_i is convergent.

Main conclusions

1. Split curvature spaces satisfy

$$R_{hijk} = \varepsilon S_{hi} S_{jk}, \qquad S = a \wedge b.$$

2. Every space of split curvature is semisymmetric.
3. The correct classification is obtained by $\operatorname{rank} \mathrm{Ric}$: types A, B, and C.
4. Type A may have non-constant scalar curvature.
5. There are explicit type A examples admitting nontrivial geodesic mappings.
6. Concircular vector fields clarify the relation with equidistant spaces.

Suggested discussion points

- How far can the classification by $\operatorname{rank} \mathrm{Ric}$ be refined?
- Which type A spaces admit nontrivial geodesic mappings?
- Can the equidistant condition be characterised purely from split curvature data?
- What changes in signatures where isotropic vectors play a central role?
- Are there broader classes of semisymmetric spaces with similar counterexamples?

Reference I

N. S. Sinyukov,
Geodesic Mappings of Riemannian Spaces,
Nauka, Moscow, 1979.

Z. I. Szabó,
Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R=0$,
J. Differential Geom. **17** (1982), 531–582.

J. Mikeš et al.,
Geodesic Mappings and Their Generalizations,
Palacký University, Olomouc, 1996.

Reference II

P. Peška and J. Mikeš,
Space of split curvature,
J. Geom. **117** (2026), Article 14.
DOI: 10.1007/s00022-026-00794-9.

Thank you for your attention.