
# Geometric Algebra for the Elastica: Frames, Lagrangians, and Conserved Quantities

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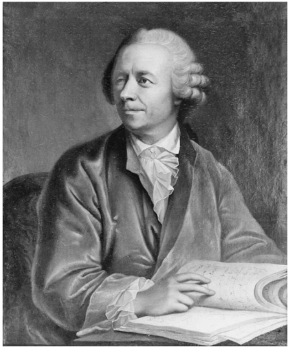
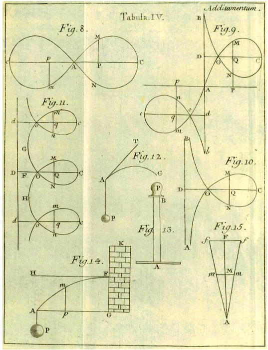
# The Euler Elastica



Leonard Euler




Minimizes elastic energy (1744) : $\int \frac{ds}{RR}$

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# Motivation

- **Core Topic**: Non-linear beam models are based on Euler's classical *elastica*.
- **Traditional Limits**: Coordinate-dependent descriptions and tricky differential systems.
- **Proposed Solution**: Coordinate-free formulation using 3D **Geometric Algebra (GA)**.
- **Advantage**: Calculations remain dimensionally co-variant and completely coordinate-free.

# Geometric Algebra Fundamentals

## The Euclidean Geometric Algebra $\mathbb{G}^3$

Generated by orthonormal basis vectors $\{e_1, e_2, e_3\}$ with the associative geometric product:

$$e_i e_i = 1, \quad e_i e_j = -e_j e_i \ \ (i \neq j)$$

- **Product Decomposition**: $ab = a \cdot b + a \wedge b$ (Inner symmetric contraction + Outer antisymmetric wedge).
- **Multivector Structure**: Mixes scalars (grade 0), vectors (grade 1), bivectors (grade 2), and pseudoscalars (grade 3).

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# Euclidean Geometric Algebra $\mathbb{G}^3$

$$
\begin{tikzcd}[column sep=small]
 & e_1e_2e_3 \equiv e_{123} \equiv I_3 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
e_1e_2 \equiv e_{12} \arrow[d, dash] \arrow[dr, dash] & e_1e_3 \equiv e_{13} \arrow[dl, dash] \arrow[dr, dash] & e_2e_3 \equiv e_{23} \arrow[dl, dash] \arrow[d, dash] \\
e_1 \arrow[dr, dash] & e_2 \arrow[d, dash] & e_3 \arrow[dl, dash] \\
 & 1 &
\end{tikzcd}
$$

- 1 scalar; 3 vectors; 3 bivectors; 1 pseudoscalar $I_3 = e_{123}$ (commutes with everything)
- bivectors encode rotations

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# Euler-Lagrange formulation

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# Variational Euler–Lagrange Formulation

- **Action Principle**: Modeled using an unextendable centerline beam configuration.
- **First-Order Constrained Lagrangian**:

$$\mathcal{L}(t, t') := \frac{1}{2}t'^2 + \frac{\lambda}{2}(t^2 - 1)$$

  where $\lambda$ is a scalar Lagrange multiplier, and $|t| = 1$.
- **Equations of Motion**: Varying independent fields leads directly to:

$$\frac{d}{ds}(\lambda t) - \frac{d^2}{ds^2}t' = 0$$

- **Dimensional Confinement**: Proves an elastica always natively resides within a maximum 3D subspace of $\mathbb{G}^n$.

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# Conservation Laws & Noether Currents

By studying translational, rotational, and scaling symmetries, the system inherits explicit conservation laws:

## Noetherian Currents

- **Linear Momentum (Vector)**: $J = \lambda t - t''$, where $J' = 0$.
- **Angular Momentum (Bivector)**: $\mathcal{M} = \gamma \wedge J + t \wedge t'$, where $\mathcal{M}' = 0$.
- **Energy / Hamiltonian (Scalar)**: $H = J \cdot t + \frac{1}{2}\kappa^2 = \text{const.}$

- **Quaternion Duality**: Total momentum $\mathcal{F} = H + \mathcal{M}$ behaves as a constant quaternion. (Curious observation or a deeper coincidence?)

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# The Curvature Equation

- **Torsion Coupling**: Pseudoscalar invariance isolates a geometric lock:

$$\kappa^2 \tau = c \quad \text{(constant Torsion coupling)}$$

- **GA Advantage**: Obviates solving independent systemic ODEs for torsion.
- **Lagrange Multiplier Value**: Evaluated as $\lambda = H - \frac{3}{2}\kappa^2$.

## Main Differential Equations

**First-Order Form**: $\left(H - \dfrac{\kappa^2}{2}\right)^2 + \kappa'^2 - j^2 = -\dfrac{c^2}{\kappa^2}$

**Second-Order Form**: $\kappa'' + \dfrac{1}{2}\kappa^3 - H\kappa = \dfrac{c^2}{\kappa^3}$

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# Classification of invariants

$$\begin{tikzcd}
& \|t\| = 1 & \|\sigma\| = \kappa & \|\omega\| = \kappa^2\tau & \\
\gamma(s) \arrow[r, "\prime"] & t \arrow[u] \arrow[d] \arrow[r, "\wedge t'"] & t \wedge t' = \sigma \arrow[u] \arrow[d] \arrow[r, "\wedge t''"] & t \wedge t' \wedge t'' = \omega \arrow[u] \arrow[d] \arrow[r, "\wedge t'''"] & 0 \\
& t' \arrow[d] \arrow[ur, dashed, no head, "\sigma = 0", "\text{line}"'] & \sigma' \arrow[d] \arrow[ur, dashed, no head, "\omega = 0", "\text{2D elastica}"'] & \omega' \arrow[d] & \\
& t' = 0 & \sigma' = 0 \arrow[r, dashed, no head, "\text{helix}"'] & \omega' = 0 & \\
& \text{line} & \text{circle} & \text{3D elastica} &
\end{tikzcd}$$

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# Integrability

# Differential Geometry & Mobile Frames

Frame variations are driven compactly by a frame bivector $\Omega$ via $t_i' = \Omega \cdot t_i$.

| **Frame Type** | **Darboux Bivector $\Omega$** | **Physical Context** |
|---|---|---|
| **Frenet** | $\Omega_f = t' \wedge t + b' \wedge b$ | Standard curvature $\kappa$ and torsion $\tau$. |
| **Gradient** | $\Omega_g = t' \wedge t + n' \wedge n$ | Optimal for medium physical interaction. |
| **Bishop** | $\Omega_b = t' \wedge t$ | Parallel transport; tracking without singularities. |

Table: Summary of frame expressions in $\mathbb{G}^3$

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# The Bishop (Inertial) Frame

The Bishop (parallel transport) frame $E_b := (t, \eta, \beta)^T$ provides an alternative tracking system that remains globally well-defined even when the curve acceleration vanishes.

## Mathematical Formulation & Frame Rotation

- **Acceleration Split**: $t' = k_1\eta + k_2\beta$, where $k_1 = \kappa\cos\phi(s)$ and $k_2 = \kappa\sin\phi(s)$
- **Phase Optimization**: The parameter $\phi' = -\tau$ tracks cumulative structural torsion.
- **Minimalist Darboux Bivector**: Reduces to an exceptionally compact form:

$$\Omega_b = t' \wedge t$$

- **Singularity Removal**: Avoids the geometric breakdowns standard Frenet frames face at vanishing curvature zones ($\kappa = 0$).
- **Elastica Integrability**: Acts as the foundational reference system to reconstruct the entire 3D curve layout by simple quadrature.

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# Geometric Interpretation: Cartan Connections

The frame bivector $\Omega$ acts as a flat **Cartan connection**—specifically, a Lie-algebra-valued connection form representing an infinitesimal rotation along the curve.

## The Maurer-Cartan Structural Equation

The structural consistency of any moving frame $E$ driven by a connection 1-form $\Omega$ requires satisfying the classical **Maurer-Cartan equation**:

$$d\Omega + \frac{1}{2}[\Omega, \Omega] = 0 \implies d\Omega + \Omega \wedge \Omega = 0$$

- **1D Flatness Condition**: Because the parameter space is a one-dimensional arc length $s$, the curvature 2-form identically vanishes, rendering the connection inherently *flat*.
- **Frame Compatibility**: Ensures that consecutive mobile frame operations obey the compatibility sequence:

$$\frac{dE}{ds} = \Omega E \implies \frac{d\Omega_b}{ds} = j\,(t \wedge e_3)$$

- **Integrability Baseline**: Satisfying this constraint provides the structural foundation that makes the 3D elastica curve completely integrable via the Bishop frame.

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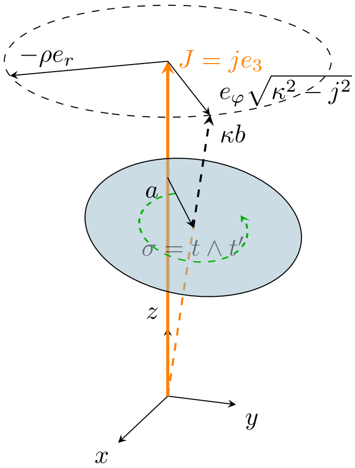
# Integration of the system




Preferred cylindrical coordinate system

Theorem: Quadrature Reconstructions
The overall coordinate-free elastica geometry integrates cleanly as:

$$j\gamma(s) = \left( Hs - \frac{1}{2}\int_0^s \kappa^2\, ds \right) e_3 + (\mathcal{M} + \Omega_b) \cdot e_3$$

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# 3D Spatial Solutions: Preferred Cylindrical Coordinates

## Cylindrical Coordinate Transformations

The spatial curve configuration $r(s) = \rho(s)e_r(s) + z(s)e_3$ where

$$\kappa^2(s) - \kappa^2(0) = -4q\, m\, \mathrm{sn}^2(\sqrt{q}s|m) \tag{1}$$

$$j\rho(s) = \kappa(s)\sin\theta \tag{2}$$

$$jz(s) = \frac{H}{3}s - \frac{2\,m}{3}(q-2)s - 2\sqrt{q}\,\mathcal{E}(\sqrt{q}s|m) \tag{3}$$

$$\varphi(s) = -\frac{cs}{2j} + \frac{c}{2j}\frac{(2Hj^2 - c^2)}{\sqrt{q}\,(j^2p - c^2)}\,\Pi\left(\frac{4\,m\,q\,j^2}{j^2p - c^2},\, \mathrm{am}\,(\sqrt{q}\,s|m)\,\middle|\,m\right) \tag{4}$$

- **Radial Profile**: The bounding cylinder radius $\rho(s)$ is a closed algebraic function of local curvature.
- **Vertical Displacement**: The axial track $z(s)$ remains independent of torsion and uncouples cleanly.
- **Angular Phase**: The phase rotation $\varphi(s)$ resolves explicitly through the incomplete elliptic integral of the third kind $\Pi$, mapping structural spatial twist.

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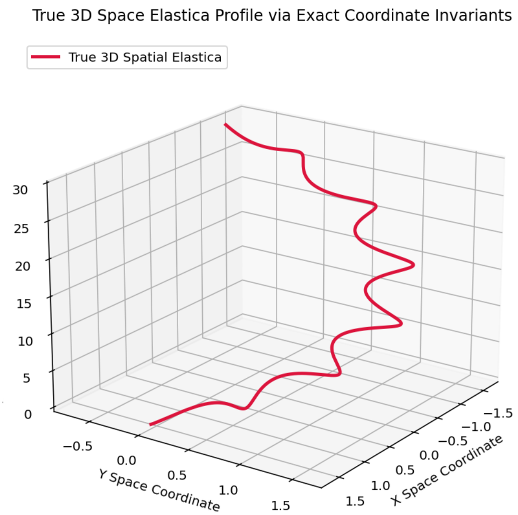
# Numerical Implementation

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# Exploration of the 3D Solution Space

The global topology of the 3D elastica is governed entirely by the modular parameter $m$, the torsion coupling constant $c$, and the linear momentum amplitude $j$.

## Parameter Boundaries & Curve Topology

- **Planar vs. Spatial ($c = 0$ vs. $c \neq 0$):** When $c = 0$, the curve collapses entirely to a flat plane. For $c \neq 0$, the non-vanishing torsion locks spatial twisting via $\kappa^2\tau = c$.
- **The Linear Regime ($m \to 0$):** Elliptic functions degenerate into standard trigonometric waves, generating stable circular helices or uniform cylindrical configurations.
- **The Undulating Regime ($0 < m < 1$):** Curvature $\kappa(s)$ oscillates periodically without sign changes, forming regular non-inflectional spatial loops on a cylinder.
- **The Soliton Boundary ($m \to 1$):** Jacobi functions map to localized hyperbolic profiles (sech, $\tanh$), yielding non-periodic asymptotically straight rods or solitary spatial knots.

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# Exploration of the 3D Solution Space

## Phase Parameter Transformations ($m > 0$ vs. $m < 0$)

The characteristic parameter $n$ of the elliptic phase integral $\Pi(n, \phi|m)$ is driven by the sign of the modular parameter $m$:

$$n = \frac{4\,m\,q\,j^2}{j^2 p - c^2} \quad \Longrightarrow \quad \begin{cases} n > 0 & \text{for } m > 0 \quad \text{(Smooth Undulating Tracks)} \\ n < 0 & \text{for } m < 0 \quad \text{(Inflectional Reversals)} \end{cases}$$

- **Branch Configuration**: Shifting to $m < 0$ forces a negative characteristic ($n \to -n$), requiring a Riemann surface branch shift to preserve physical continuity.
- **Modulus Realignment**: Analytical continuation forces a real parameter mapping under the standard modular transformation rule:

$$m \longrightarrow \bar{m} = \frac{-m}{1-m} > 0$$

- **Geometric Result**: If $m > 0$, the phase angle $\varphi(s)$ advances strictly monotonically. If $m < 0$, the sign flip allows directional reversals, triggering spatial loops or physical knots on the cylinder wall.

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# Planar Solutions: Case $m > 0$

An analytical solution for the positive modular parameter case $m > 0$ expressed natively via Jacobi elliptic functions.

## Example

The spatial curve tracks according to the system:

$$\kappa(s) = \sqrt{2}\sqrt{j+H}\,\mathrm{dn}\left(\frac{\sqrt{j+H}s}{\sqrt{2}}\middle|\frac{2j}{j+H}\right) \tag{5}$$

$$x(s) = \frac{\kappa(0)-\kappa(s)}{j} \tag{6}$$

$$z(s) = \frac{Hs}{j} - \frac{\sqrt{2}\sqrt{j+H}}{j}\,\mathcal{E}\left(\frac{\sqrt{j+H}s}{\sqrt{2}}\middle|\frac{2j}{j+H}\right) \tag{7}$$

- **Curvature Tracking**: Evaluated via the Jacobi delta amplitude (dn) function.
- **Coordinate** $x(s)$: Dictated entirely by total local curvature drop.
- **Coordinate** $z(s)$: Integrated cleanly using the elliptic integral of the second kind $\mathcal{E}$.

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# Planar Solutions: Case $m < 0$

An analytical formulation for negative modular parameter domains $m < 0$, requiring complex domain mapping.

## Example

The system shifts using inverted parameter footprints:

$$\kappa(s) = \sqrt{2}\sqrt{-j+H}\,\mathrm{dn}\left(\frac{\sqrt{-j+H}s}{\sqrt{2}}\middle|\frac{2j}{j-H}\right) \tag{8}$$

$$x(s) = \frac{\kappa(0)-\kappa(s)}{j} \tag{9}$$

$$z(s) = \frac{Hs}{j} - \frac{\sqrt{2}\sqrt{-j+H}}{j}\,\mathcal{E}\left(\frac{\sqrt{j+H}s}{\sqrt{2}}\middle|\frac{2j}{j-H}\right) \tag{10}$$

- **Sign Flipping**: Linear momentum parameter $j$ changes orientation within radicals.
- **Evaluation Guard**: Demands the use of explicit **analytical continuation identities**.
- **Numeric Safety**: Prevents computational breakdowns inside traditional float solvers.

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# Physical Applications & Exact Integration

- **Planar Elastica ($c = 0$)**: Curvature simplifies to $\kappa = 2q\,\mathrm{dn}(qs|m)$, resolved via Jacobi delta amplitude functions.
- **Helix Verification**: Validates matching physical balances for circular helix paths ($H = \frac{k^2}{2} - T^2$).
- **Preferred Cylindrical Coordinates**: Choosing $e_3 = J/j$ establishes global cylindrical symmetry for clear tracking layouts.

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# Conclusions & Engineering Applications

- **Main Achievement**: Developed a completely integrable formulation of 3D elastica without complex non-Euclidean space mapping.
- **Algorithmic Generalization**: Easily updates to higher-dimensional continuum mechanics due to GA invariance.
- **Numerical Methods Impact**: Directly supports current Geometric Algebra implementations in:
  - Finite Element Modeling (**FEM**)
  - Boundary Element Methods (**BEM**)
  - Computer Vision and Image Segmentation workflows

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THANK YOU FOR THE ATTENTION!

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D. Prodanov, *Elastic Curves and Euler–Bernoulli Constrained Beams from the Perspective of Geometric Algebra*, Mathematics **13**(16), 2555 (2025).