Clifford (Geometric) Algebra for Practitioners

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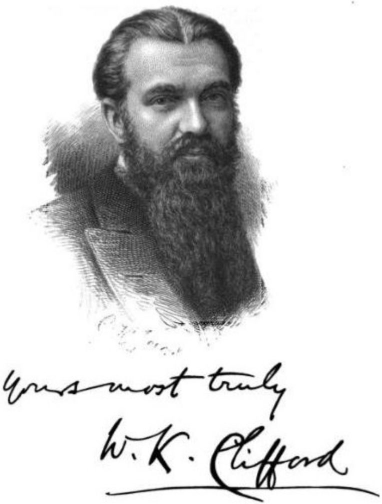
Overview



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Introduction to Maxima

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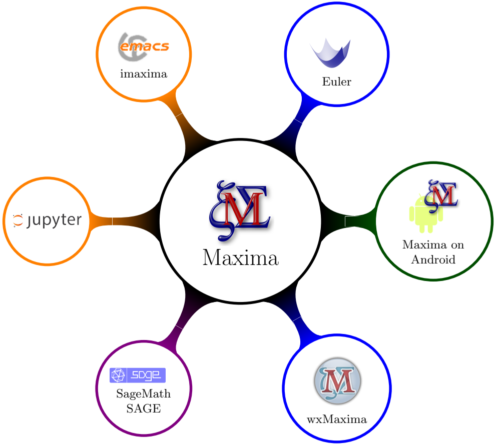
The Maxima ecosystem



- Maxima is widely used
- Maxima is mature: many tutorials, courses and textbooks use it
- Maxima is open source (GPL)
- open source allows for shorted development cycles

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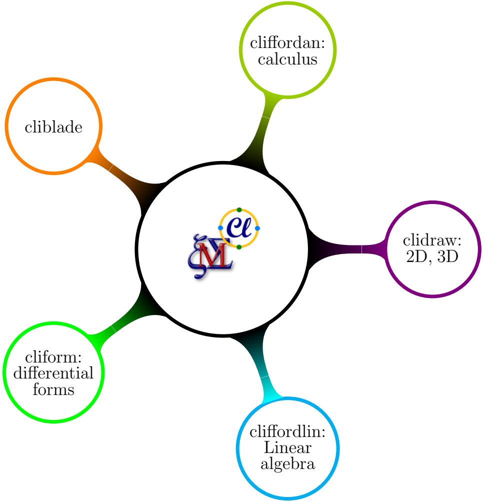
The `Clifford` package



- minimalist design
- mature: 2015 – 2026
- used in published math research
- open source: GPL license

Available from GitHub
`http://dprodanov.github.io/clifford/`

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Clifford algebras

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Geometric Algebra Fundamentals

The Euclidean Geometric Algebra $C\ell_{p,q}$

Generated by orthonormal basis vectors $\{e_1, e_2, \ldots, e_n\}$ with the associative geometric product:

$$e_i e_i = \delta_{ij},\ i \leq p; \quad e_i e_i = -\delta_{ij},\ i > p$$

$$e_i e_j = -e_j e_i \ \ (i \neq j)$$

- **Product Decomposition**: $ab = a \cdot b + a \wedge b$ (Inner symmetric contraction + Outer antisymmetric wedge).
- **Multivector Structure**: Mixes scalars (grade 0), vectors (grade 1), bivectors (grade 2), and pseudoscalars (grade 3).

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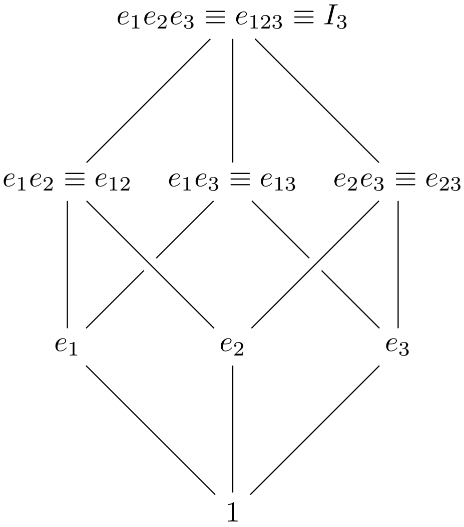
Example: Euclidean Geometric Algebra $\mathbb{G}^3$



- 1 scalar; 3 vectors; 3 bivectors; 1 pseudoscalar $I_3 = e_{123}$ (commutes with everything)
- bivectors encode rotations

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Geometric Algebra Products

- scalar product

$$A * B := \langle AB \rangle_0$$

commutative: $A * B = B * A$

- outer product

$$a \wedge A_k := \frac{1}{2}\left(aA_k + (-1)^k A_k a\right)$$

mixed: $a \wedge A_k = (-)^k A_k \wedge a$

- inner product (Hestenes convention)

$$a \cdot b = a * b$$

$$a \cdot A_k := \frac{1}{2}\left(aA_k - (-1)^k A_k a\right)$$

mixed: $a \cdot A_k = (-)^{k-1} A_k \wedge a$

$$(a_1 a_2 \ldots a_r) \cdot (b_1 b_2 \ldots b_s) := \begin{cases} ((a_1 a_2 \ldots a_r) \cdot b_1) \cdot (b_2 \ldots b_s), & r \geq s \\ (a_1 a_2 \ldots a_{r-1}) \cdot (a_r \cdot (b_1 b_2 \ldots b_s)), & r < s \end{cases}$$

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Main functions in `Clifford`

- GRADE FUNCTIONS

`grade` (expr)	grade decomposition of expression
`scalarpart` (expr)	$< expr >_0$
`vectorpart` (expr)	$< expr >_1$
`grpart` (expr,k)	$< expr >_k$
`mvectorpart` (expr)	$< expr >_{2+}$
`bdecompose` (expr)	blade decomposition of expression

Grade projection operator : $\langle \ \rangle_k$

Grade decomposition : $M = \sum_{k=0}^{n} \langle M \rangle_k$

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Product Simplification Demo

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Main functions in `Clifford`

- PRODUCT SIMPLIFICATION

`cliffsimpall` (expr)	full simplification of expressions
`dotsimp1` (ab)	canonic reordering of dot products
`dotsimpc` (ab)	simplification of dot products
`dotinvsimp` (ab)	simplification of inverses
`powsimp` (ab)	simplification of exponents

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Quaternions

```
load('clifford);
clifford(e,0,2);
mtable1([1, e[1],e[2], e[1] . e[2]]);
```

Quaternion multiplication table

$$\begin{pmatrix} 1 & e_1 & e_2 & e_1.e_2 \\ e_1 & -1 & e_1.e_2 & -e_2 \\ e_2 & -e_1.e_2 & -1 & e_1 \\ e_1.e_2 & e_2 & -e_1 & -1 \end{pmatrix}$$

(minimal manual formatting)

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Outer product in $\mathbb{G}^3$

```
clifford(e, 3);
mtable2o();
```

Outer product multiplication table

$$\begin{pmatrix} 1 & e_1 & e_2 & e_3 & e_1 \cdot e_2 & e_1 \cdot e_3 & e_2 \cdot e_3 & e_1 \cdot e_2 \cdot e_3 \\ e_1 & 0 & e_1 \cdot e_2 & e_1 \cdot e_3 & 0 & 0 & e_1 \cdot e_2 \cdot e_3 & 0 \\ e_2 & -e_1 \cdot e_2 & 0 & e_2 \cdot e_3 & 0 & -e_1 \cdot e_2 \cdot e_3 & 0 & 0 \\ e_3 & -e_1 \cdot e_3 & -e_2 \cdot e_3 & 0 & e_1 \cdot e_2 \cdot e_3 & 0 & 0 & 0 \\ e_1 \cdot e_2 & 0 & 0 & e_1 \cdot e_2 \cdot e_3 & 0 & 0 & 0 & 0 \\ e_1 \cdot e_3 & 0 & -e_1 \cdot e_2 \cdot e_3 & 0 & 0 & 0 & 0 & 0 \\ e_2 \cdot e_3 & e_1 \cdot e_2 \cdot e_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ e_1 \cdot e_2 \cdot e_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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Inner product(s) in $\mathbb{G}^3$

Inner product types: `lc` – left contraction, `rc` – right contraction , `sym` – for symmetric

```
inprotype:lc;
mtable2i();
```

Left contraction multiplication table

$$\begin{pmatrix}
1 & e_1 & e_2 & e_3 & e_1.e_2 & e_1.e_3 & e_2.e_3 & e_1.e_2.e_3 \\
0 & 1 & 0 & 0 & e_2 & e_3 & 0 & e_2.e_3 \\
0 & 0 & 1 & 0 & -e_1 & 0 & e_3 & -e_1.e_3 \\
0 & 0 & 0 & 1 & 0 & -e_1 & -e_2 & e_1.e_2 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & -e_3 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & e_2 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -e_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}$$

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Main functions in `Clifford`

- INVOLUTIONS

`dotreverse` (ab)	Clifford reverse
`cinvolve` (expr)	Clifford involution
`dotconjugate` (expr)	Clifford conjugation
`invautom` (expr)	Inverse Blade Automorphism (IBA)

Vectors
Grade reversal :

$$(abcd)^{\sim} = dcba$$

involution :

$$(abcd)^{t} = (-a)(-b)(-c)(-d)(-a)$$

IBA :

$$(abcd)^{\sharp} = d^{\sharp}c^{\sharp}b^{\sharp}a^{\sharp}, \quad a^{\sharp} = \frac{a}{a*a}$$

Clifford conjugation : $\bar{A} = (A^{t})^{\sim}$

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Inverse in STA $C\ell_{1,3}^{+}$

Spinor

$$\Psi = a_1\mathbb{I} + a_2\gamma^{12} + a_3\gamma^{23} + a_4\gamma^{31} + a_5\gamma^{01} + a_6\gamma^{02} + a_7\gamma^{03} + a_8\gamma^{5}$$

Inverse

$$\Psi^{-1} = \frac{\widetilde{\Psi}(\Psi\widetilde{\Psi})^{\sharp}}{\Delta}, \quad \text{where} \quad \Delta = \Psi\widetilde{\Psi}(\Psi\widetilde{\Psi})^{\sharp}$$

```
clifford(e,1,3);
A:celem(c)$
C:Ae.dotreverse(Ae),expand, dotsimpc;
C:C,clifact;
(C)    c[1]^2-c[6]^2-c[7]^2-c[8]^2+c[9]^2+c[10]^2+c[11]^2-c[16]^2+2*(-(c[8]*c
    [9])+c[7]*c[10]-c[6]*c[11]+c[1]*c[16])*(e[1] . e[2] . e[3] . e[4])
DD:C.invautom(C),expand, dotsimpc,ratsimp$
F: dotreverse(Ae).invautom(C),expand, dotsimpc$
A2: F/DD$
Ae.A2,expand, dotsimpc$
%,ratsimp;
(%o13) 1
```

Representation $C\ell^{+}_{1,3}$

$$PP = \begin{bmatrix} \psi_1 - i\psi_9 & \psi_{10} - i\psi_{11} & \psi_8 - i\psi_{16} & \psi_6 - i\psi_7 \\ -\psi_{10} - i\psi_{11} & \psi_1 + i\psi_9 & \psi_6 + i\psi_7 & -\psi_8 - i\psi_{16} \\ \psi_8 - i\psi_{16} & \psi_6 - i\psi_7 & \psi_1 - i\psi_9 & \psi_{10} - i\psi_{11} \\ \psi_6 + i\psi_7 & -\psi_8 - i\psi_{16} & -\psi_{10} - i\psi_{11} & \psi_1 + i\psi_9 \end{bmatrix}$$

Extracting a the bivector component γ_{23}

```
 P11:diff(PP, psi[11]);
(P11)     matrix(
[0, -%i,     0,   0],
[-%i,    0,  0,   0],
[0,  0,   0,  -%i],
[0,  0,  -%i,     0]
)
(%i82)   K1:P11.PP,expand$
(%i83)   mat_trace(K1)/mat_trace(P11.P11);
(%o83)   psi[11]
```

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Numerical Demo

Faddeev-Le Verrer-Souriau algorithm

Theorem (Reduced-grade FVS algorithm)

Suppose that $A \in C\ell_{p,q}$ *is a multivector of maximal grade* $r \leq s$ *and* $A \subseteq \mathrm{span}[e_1, \ldots, e_s]$. *The Clifford inverse, if it exists, can be computed in* $k = 2^{\lceil s/2 \rceil}$ *steps as*

$$
\begin{array}{ll|ll}
m_1 &= A & c_1 = -kA * 1, & t_1 := -c_1 \\
m_2 &= Am_1 - t_1 & c_2 = -\frac{k}{2} A * m_1, & t_2 := -c_2 \\
\ldots & & \ldots & \\
m_k &= Am_{k-1} - t_k & c_k = -A * m_{k-1}, & t_k := -c_k
\end{array}
$$

until the step where $m_k = 0$ *so that*

$$A^{-1} = -m_{k-1}/c_k. \tag{1}$$

The inverse does not exist if $c_k = -\det A = 0$.
The (reduced) characteristic polynomial of A *is*

$$p_A(\lambda) = \lambda^k + c_1 \lambda^{k-1} + \ldots c_{k-1}\lambda + c_k. \tag{2}$$

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Inverse in $C\ell_{2,5}$

$A = 1 - 2e_{15} + 5e_{134}$

- $\mathrm{span}[A] = \{e_1, e_3, e_4, e_5\}$
- Reduced grade algorithm will run in $k = 2^{\lceil 4/2 \rceil} = 4$ steps:

$$
\begin{aligned}
&t_1 = -4, && m_1 = 1 + 5C - 2B, \quad B := e_{15}, \quad C := e_{134};\\
&t_2 = 48, && m_2 = -24 - 10C + 4B;\\
&t_3 = -88, && m_3 = 66 + 110C - 44B;\\
&t_4 = 484, && m_4 = 0
\end{aligned}
$$

- The characteristic polynomial factorizes

$$p_A(v) = 484 - 88v + 48v^2 - 4v^3 + v^4 = (22 - 2v + v^2)^2$$

$$A^{-1} = (1 - 5C + 2B)/22 \tag{3}$$

- The optimal formula is

$$detA = AA^{\sim} = 22 \Longrightarrow A^{-1} = A^{\sim}/22$$

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t_4 &= 484, & m_4 &= 0
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Inverse in $C\ell_{5,2}$

$A = 1 - e_2 + I$
Reduced grade algorithm will run in $2^{\lceil 7/2 \rceil} = 2^4 = 16$ steps

$$
\begin{aligned}
t_1 &= -16, & m_1 &= 1 - e_2 + I; \\
t_2 &= 120, & m_2 &= -15 + 14e_2 - 14I + 2C, \quad C = e_{134567}; \\
t_3 &= -560, & m_3 &= 105 - 89e_2 + 93I - 26C; \\
t_4 &= 1836, & m_4 &= -459 + 340e_2 - 388I + 156C; \\
t_5 &= -4560, & m_5 &= 1425 - 881e_2 + 1145I - 572C; \\
t_6 &= 9064, & m_6 &= -3399 + 1682e_2 - 2562I + 1454C; \\
t_7 &= -14960, & m_7 &= 6545 - 2529e_2 + 4557I - 2790C; \\
t_8 &= 20886, & m_8 &= -10443 + 3096e_2 - 6648I + 4296C; \\
t_9 &= -24880, & m_9 &= 13995 - 3051e_2 + 8091I - 5448C; \\
t_{10} &= 25480, & m_{10} &= -15925 + 2386e_2 - 8242I + 5694C; \\
t_{11} &= -22416, & m_{11} &= 15411 - 1475e_2 + 7007I - 4934C; \\
t_{12} &= 16716, & m_{12} &= -12537 + 596e_2 - 4932I + 3548C; \\
t_{13} &= -10480, & m_{13} &= 8515 - 35e_2 + 2795I - 1980C; \\
t_{14} &= 5400, & m_{14} &= -4725 - 50e_2 - 1150I + 850C; \\
t_{15} &= -2000, & m_{15} &= 1875 + 125e_2 + 375I - 250C; \\
t_{16} &= 625, & m_{16} &= 0
\end{aligned}
$$

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Inverse in $C\ell_{5,2}$

$A = 1 - e_2 + I$

- Inverse

$$A^{-1} = (1 - e_2 - 3\,I + 2\,C)/5, \quad C = e_{134567} \tag{4}$$

- The characteristic polynomial factorizes:

$$p_A(v) = (5 - 4v + 6v^2 - 4v^3 + v^4)^4$$

- The optimal formula is

$$B = AA^t = 1 - 2I$$

$$\det A = BB^{\sim} = 5$$

$$A^{-1} = A^t (AA^t)^{\sim}/5$$

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Inverse in $C\ell_{5,2}$

$A = 1 - e_2 + I$

- Inverse
$$A^{-1} = (1 - e_2 - 3\,I + 2\,C)/5, \quad C = e_{134567} \tag{4}$$
- The characteristic polynomial factorizes:
$$p_A(v) = (5 - 4v + 6v^2 - 4v^3 + v^4)^4$$
- The optimal formula is
$$B = AA^t = 1 - 2I$$
$$\det A = BB^{\sim} = 5$$
$$A^{-1} = A^t (AA^t)^{\sim}/5$$

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Calculus Demo

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Main functions in `Clifford`an

`ctotdiff`(f, x)	total derivative w.r.t. multivector x
`ctotintdiff`(f, x)	inner total derivative w.r.t. x
`ctotextdiff`(f, x)	outer total derivative w.r.t. x
`vectdiff`(f, ee, k)	vector derivative of order k w.r.t. basis vector list *ee*
`mvectdiff`(f, x, k)	multivector derivative of order k w.r.t. multivector x
`parmvectdiff`(f, x, k)	partial multivector derivative of order k w.r.t. multivector x
`convderiv`(f, t, xx, [vs])	convective derivative w.r.t. multivector x
`coordsubst`(x, eqs)	substitutes coordinates in multivector x w.r.t. new variables in the list *eqs*
`clivolel`(x, eqs)	computes volume element of $Span\,\{x\}$ w.r.t. new variables in the list *eqs*

Generalized derivatives in Clifford algebras

Definition

Vector $r \in Span\{e_1, \dots, e_n\}$ and reciprocal frames $e^k = e_k^{-1}$ so that.

$$e_k * e^j = \delta_k^j$$

Directional derivative

projection along the unit norm (multi)vector v:

$$(v * \nabla)F(x) \equiv {}_v\nabla F(x) := \sum_j v_j \frac{\partial F}{\partial x_j}$$

Vector derivative

$$\nabla_r F(x) := \frac{dF}{dr} = \sum_J \underbrace{\frac{\partial x_j}{\partial r}}_{e^J} \frac{\partial F}{\partial x_j} = e^J \partial_J F(x)$$

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Potential problems in $\mathbb{G}^3$

Potential equation for the Green's function

$$r = e_a x^a = e^a x_a$$

$$\nabla_r G = \delta(r)$$

$$G(x, y, z) = \frac{e_1\, x + e_2\, y + e_3\, z}{\sqrt{(x^2 + y^2 + z^2)^3}}$$

Cartesian coordinates

```
/* Initialization */
load('clifford);
load ('cliffordan);

/* G(3) construction */
clifford(e,3);
r:cvect([x,y,z]);
G:r/sqrt(-cnorm(r))^3;
mvectdiff(G,r);


mvectdiff(-1/sqrt(-cnorm(r)),r);
mvectdiff(-1/sqrt(-cnorm(r)),r,2);
```

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Potential problems in $\mathbb{G}^3$

Cylindrical coordinates

```
cyl_eq:[x=rho*cos(phi), y=rho*sin(phi)];
  declare( [rho, phi], scalar);
  GG_c:coordsubst(G, cyl_eq),factor;
  rc:coordsubst(r, cyl_eq);
  mvectdiff(GG_c,rc);
  V:coordsubst(-1/sqrt(-cnorm(r)),cyl_eq);
  mvectdiff(V,rc);
```

$$GG_c = \frac{e_1 \rho \cos\phi + e_2 \rho \sin\phi + e_3 z}{\left(\rho^2 + z^2\right)^{\frac{3}{2}}}$$

$$V = \frac{e_1 \rho \cos\phi + e_2 \rho \sin\phi + e_3 z}{\left(\rho^2 + z^2\right)^{\frac{3}{2}}}$$

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Euler-Lagrange problems in $\mathbb{G}^3$

```
AA: celem(A,[t,x,y,z]);
dependsv(A,[t,x,y,z]);
r:cvect([x,y,z]);
F:mvectdiff(AA,t-r);
F:factorby(F, %elements);
L:lambda([x],1/2*scalarpart(cliffsimpall(x.x)))(F);
S:scalarpart(F);
V:vectorpart(F);
Q:grpart(F,2);
L-1/2*(S.S+V.V+Q.Q),cliffsimpall;
dA:mvectdiff(AA,r);
EuLagEq2(L, t+r,[AA,dA]);
```

D'Alembert's equation

$$\nabla_r \nabla_r \bullet A = 0$$

$$\left(-A_{t\,tt} + A_{t\,xx} + A_{t\,yy} + A_{t\,zz}\right) + e_1 \left(-A_{x\,tt} + A_{x\,xx} + A_{x\,yy} + A_{x\,zz}\right)$$
$$+ e_2 \left(-A_{y\,tt} + A_{y\,xx} + A_{y\,yy} + A_{y\,zz}\right) + e_3 \left(-A_{z\,tt} + A_{z\,xx} + A_{z\,yy} + A_{z\,zz}\right) = 0$$

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