Problems of Extrinsic Geometry of Foliations

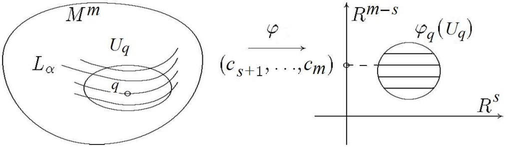
Vladimir Rovenski

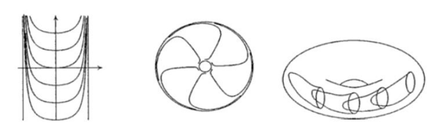
Department of Mathematics, University of Haifa

XXVI-th International Conference on Geometry, Integrability and Quantization

June 4–11, 2026 Varna, Bulgaria

Foliations are partitions of a manifold M into sets of submanifolds (leaves) of the same dimension. The leaves of a foliation are tangent to the involutive (Frobenius integrable) **distribution**. **Distribution** of M is a subbundle of the tangent bundle TM.

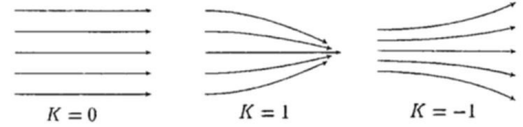




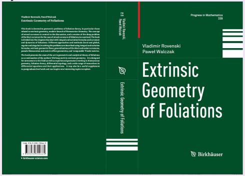
A distribution $\mathcal{D} \subset TM$ (or a foliation) on a Riemannian manifold (M, g) is **totally geodesic** $\Leftrightarrow$ its 2nd fundamental form vanishes, that is, *any geodesic of M that is tangent to $\mathcal{D}$ at one point is tangent to $\mathcal{D}$ at all its points.*

Example. Reeb foliation is not geodesic for any Riemannian metric.

Extrinsic Geometry (**EG**) of foliations is a field of Riemannian geometry, which studies the properties expressed by the second fundamental form, as being totally geodesic/umbilical/harmonic. The Riemann tensor (the 2nd derivative of metric) of foliations belongs to the intrinsic geometry, but its part called **mixed** or **mutual curvature** (the 1st derivative of metric) is a part of **EG**.



Behavior of geodesics and Jacobi fields for $K_{mix} = \mathrm{const}$



[1] V. Rovenski, P. Walczak, Extrinsic geometry of foliations. Progress in Math. 339, Birkhäuser, 2021

The following tools are important for **EG of foliations**:

- **IF** as obstacles for specifying extrinsic geometric properties,
- **Variation formulas** for extrema of functionals are also used in
- **EG flows of metrics** and their self-similar solutions (solitons).

The **Levi-Civita connection** ∇ is given by

$$2\,g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\ + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X).$$

$R_{X,Y} = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ – the **curvature tensor** of ∇,
$\mathrm{Ric}(X,Y) = \mathrm{Tr}(Z \to R_{Z,X}\,Y)$ – the **Ricci tensor**.
$\mathrm{S} = \mathrm{Tr}\,\mathrm{Ric} = \sum_i \mathrm{Ric}(e_i, e_i)$ – the **scalar curvature**.
$K(X,Y)$ – the **sectional curvature** of the plane $\sigma = X \wedge Y$.

Let $\mathcal{D}_i$ be $k \ge 2$ mutually orthogonal n_i-dim. non-degenerate distributions on a pseudo- Riemannian manifold $(M, g = \langle\cdot,\cdot\rangle)$. If $\sum n_i = \dim M$, then $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ is a **Riemannian almost product manifold**. A distribution $\mathcal{D}_i$ is **non-degenerate**, if g_x is a non-degenerate bilinear form on $\mathcal{D}_i(x) \subset T_x M$ for every $x \in M$.

This is applicable to studies of **multiply twisted** and **multiply warped products** as well as the **webs of foliations**.

Let ${}^{\top_i} : TM \to \mathcal{D}_i$ and ${}^{\perp_i} : TM \to \mathcal{D}_i^\perp$ be orthoprojectors.

The **second fundamental form** $h_i : \mathcal{D}_i \times \mathcal{D}_i \to \mathcal{D}_i^\perp$ and **integrability tensor** $T_i : \mathcal{D}_i \times \mathcal{D}_i \to \mathcal{D}_i^\perp$ of $\mathcal{D}_i$ of $\mathcal{D}_i$ are defined by

$$h_i(X,Y) = \tfrac{1}{2}\,(\nabla_X Y + \nabla_Y X)^{\perp_i},\ \ T_i(X,Y) = \tfrac{1}{2}\,[X,Y]^{\perp_i}\ \ (X, Y \in \mathcal{D}_i).$$

$H_i = \operatorname{Tr}_g\ h_i$ – the **mean curvature vector field** of $\mathcal{D}_i$.

The **shape operators** $(A_i)_Z$ of $\mathcal{D}_i$ with $Z \in \mathcal{D}_i^\perp$ and the operators $(T_i^\sharp)_Z$ are adjoint to the (1,2)-tensors h_i and T_i:

$$g((A_i)_Z(X), Y) = g(h_i(X,Y), Z), \quad g((T_i^\sharp)_Z(X), Y) = g(T_i(X,Y), Z).$$

$\mathcal{D}_i$ is **totally umbilical**, **harmonic**, or **totally geodesic**, if $h_i = (H_i/n_i)\, g$, $\ H_i = 0$, or, $\ h_i = 0$, respectively.

Similarly, we define $h_i^\perp$, $H_i^\perp = \operatorname{Tr}_g h_i^\perp$, $T_i^\perp$ for $\mathcal{D}_i^\perp$;
and h_{ij}, $H_{ij} = \operatorname{Tr}_g h_{ij}$, T_{ij} are defined for $\mathcal{D}_{ij} := \mathcal{D}_i \oplus \mathcal{D}_j$, etc.

De Rham Decomposition Theorem

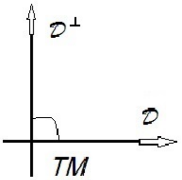
We say that $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ **splits** if all distributions $\mathcal{D}_i$ are integrable, M is locally the direct product $M_1 \times \ldots \times M_k$ with canonical foliations tangent to $\mathcal{D}_i$ and $g = g_1 \oplus \ldots \oplus g_k$.

When $(M, g, \mathcal{D}_1, \ldots, \mathcal{D}_k)$ is decomposed (or splits locally) into the product of k manifolds? The best known result in this direction is the **Decomposition theorem of de Rham**:

If each $\mathcal{D}_i$ is parallel with respect to the Levi-Civita connection of M, then any point $p \in M$ has a neighborhood U isometric to a product $M_1 \times \ldots \times M_k$ of Riemannian manifolds such that the submanifolds, which are parallel to the factor M_i, correspond to integral manifolds of the distribution $D_i|_U$. In the case that M is simply connected and complete the assertion is true with $U = M$.

The **mixed curvature of orthogonal distributions** $(\mathcal{D}, \mathcal{D}^\perp)$ on a (pseudo-)Riemannian manifold (M, g) is defined as an averaged mixed sectional curvature $K(E_a, \mathcal{E}_b) = \varepsilon_a\, \varepsilon_b\, g(R_{E_a, \mathcal{E}_b}\mathcal{E}_b, E_a)$:

$$\mathrm{S}_{\mathcal{D},\mathcal{D}^\perp} = \sum_{E_a \in \mathcal{D},\, \mathcal{E}_b \in \mathcal{D}^\perp} K(E_a, \mathcal{E}_b),$$



$E_a \in \mathcal{D}$, $\mathcal{E}_b \in \mathcal{D}^\perp$ – an adapted orthonormal frame.

If one of distributions is spanned by a unit vector field N, then

$$\mathrm{S}_{\mathcal{D},\mathcal{D}^\perp} = \mathrm{Ric}_{N,N},$$

$\mathrm{Ric}_{N,N} = \sum_{E_b \perp N} K(N, E_b)$ – **Ricci curvature** in the N-direction.

To simplify the presentation, we will sometimes assume that $g > 0$.

Definition 1

Let $TM = \mathcal{D}_1 \oplus \ldots \oplus \mathcal{D}_k$, and $\{E_1, \ldots, E_n\}$ – a local **adapted** orthonormal frame on M^n such that $\{E_1, \ldots, E_{n_1}\} \subset \mathcal{D}_1$, $\{E_{n_{i-1}+1}, \ldots, E_{n_i}\} \subset \mathcal{D}_i$ for $2 \le i \le k$. The **mixed scalar curvature** of $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ is $\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} = \sum_{i<j} \mathrm{S}_{\mathcal{D}_i,\mathcal{D}_j}$.

The mixed scalar curvature is involved in such research topics as 1) **IF** and splitting of foliations, 2) the mixed Einstein–Hilbert action, 3) prescribing curvature of foliations.

Proposition 1

We express the scalar curvature of (M, g)*:*

$$\mathrm{S} = 2\,\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} + \sum\nolimits_{i=1}^{k} \mathrm{S}_{\mathcal{D}_i},$$

where $\mathrm{S}_{\mathcal{D}_i}$ *– the scalar curvature of* $\mathcal{D}_i$ *(function on* M*) and*

$$2\,\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} = \sum\nolimits_{i=1}^{k} \mathrm{S}_{\mathcal{D}_i,\mathcal{D}_i^\perp}.$$

Multiply twisted/warped products

Definition 2

A **multiply twisted product** $F_0 \times_{u_1} F_1 \times \ldots \times_{u_k} F_k$ of Riemannian manifolds $(F_0, g_{F_0}), \ldots, (F_k, g_{F_k})$ is the product $M = F_0 \times \ldots \times F_k$ with the metric $g = g_{F_0} \oplus u_1^2\, g_{F_1} \oplus \ldots \oplus u_k^2\, g_{F_k}$, where $u_i : F_0 \times F_i \to (0, \infty)$ for $1 \le i \le k$ are smooth functions.

Multiply warped products, i.e., $u_i : F_0 \to (0, \infty)$, are special cases of multiply twisted products.

For a multiply twisted product: the **fibers**, i.e., the leaves of the F_0-foliation are totally geodesic; the **leaves** of the F_i-foliation, $i \ge 1$, are totally umbilical with the **mean curvature vectors** $H_i = -n_i P_0 \nabla(\log u_i)$ tangent to fibers.

The mixed scalar curvature in terms of the functions u_i is

$$\mathrm{S}_{\mathcal{D}_1, \ldots, \mathcal{D}_k} = \sum\nolimits_{i \ge 1} n_i\,(\Delta_{F_0}\, u_i)/u_i\,.$$

IF for $(M, g; \mathcal{D}, \mathcal{D}^\perp)$

IF are obtained by using the Divergence Theorem to appropriate vector fields, and can be viewed as "conservation laws" of quantities when the metric changes. **IF** for foliations relate **EG** of the leaves and curvature, and also create obstacles to the existence of foliations (or compact leaves) with given geometric properties.

$\mathrm{S}_{\mathcal{D},\mathcal{D}^\perp}$ belongs to **Extrinsic Geometry** of $M(g, \mathcal{D}, \mathcal{D}^\perp)$, since:

$$\mathrm{S}_{\mathcal{D},\mathcal{D}^\perp} = Q_{\mathcal{D},g} + \mathrm{div}(H + H^\perp), \tag{1}$$

$$Q_{\mathcal{D},g} = \langle H^\perp, H^\perp\rangle + \langle H, H\rangle - \langle h, h\rangle - \langle h^\perp, h^\perp\rangle + \langle T, T\rangle + \langle T^\perp, T^\perp\rangle. \tag{2}$$

Applying the Divergence Theorem on a closed M, gives the **IF**:

$$\int_M \big(\mathrm{S}_{\mathcal{D},\mathcal{D}^\perp} - Q_{\mathcal{D},g}\big)\, d\,\mathrm{vol}_g = 0. \tag{3}$$

For codimension-1 foliations (which will be discussed later), (3) is

$$\int_M \big(\sigma_2(A) - \tfrac{1}{2}\,\mathrm{Ric}_{N,N}\big)\,\mathrm{d}\,\mathrm{vol}_g = 0,$$

where $\sigma_2 = \sum_{i<j} k_i k_j$ is the **second elementary symmetric function** of the principal curvatures k_i of the leaves.

IF for $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$

The following result with **IF** generalizes (1) and (3) for $k > 2$.

Theorem 3

For $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ *we obtain*

$$\operatorname{div} \sum\nolimits_i (H_i + H_i^\perp) = 2\,\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - \sum\nolimits_i Q_{\mathcal{D}_i,g},$$

where $Q_{\mathcal{D}_i,g}$ *are given in* (2) *with* $\mathcal{D} = \mathcal{D}_i$ *.*

For a closed manifold M*, by the Divergence Theorem, we get* **IF***:*

$$\int_M \big(\,2\,\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - \sum\nolimits_i Q_{\mathcal{D}_i,g}\,\big)\,\mathrm{d}\,\mathrm{vol}_g = 0\,. \tag{4}$$

Example: **IF** for n codimension-one foliations

Let (M^n, g) admits n pairwise orthogonal codimension-one foliations $\mathcal{F}_i$, and N_i be unit vector fields orthogonal to $\mathcal{F}_i$. Summing (7) with N_i for $i = 1, \ldots, n$, and using the equality $\mathrm{S} = \sum_i \mathrm{Ric}_{N_i, N_i}$, we obtain the following **IF**:

$$\int_M \big(2 \sum_i \sigma_2(\mathcal{F}_i) - \mathrm{S} \big)\, \mathrm{d}\,\mathrm{vol}_g = 0. \qquad (5)$$

Two immediately consequences of (5):

- if $\mathrm{S} < 0$ then each foliation $\mathcal{F}_i$ cannot be totally umbilical;
- if $\mathrm{S} > 0$ then each foliation $\mathcal{F}_i$ cannot be harmonic.

Applications of **IF** to splitting of manifolds

Definition 4

A pair $(\mathcal{D}_i, \mathcal{D}_j)$ $(i \neq j)$ of distributions on $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ is

- **mixed totally geodesic**, if $h_{ij}(X, Y) = 0$ $(X \in \mathcal{D}_i, Y \in \mathcal{D}_j)$.
- **mixed integrable**, if $T_{ij}(X, Y) = 0$ $(X \in \mathcal{D}_i,\ Y \in \mathcal{D}_j)$.

Theorem 5

$(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ splits if any of the following is valid:

(i) $\mathcal{D}_1, \ldots, \mathcal{D}_k$ are integrable harmonic, each pair $(\mathcal{D}_i, \mathcal{D}_j)$ is mixed integrable, and $\underline{\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} \geq 0}$.

(ii) $\mathcal{D}_1, \ldots, \mathcal{D}_k$ are totally umbilical, each $(\mathcal{D}_i, \mathcal{D}_j)$ is mixed totally geodesic, $\langle H_i, H_j \rangle = 0$ $(i \neq j)$, M is closed, $\underline{\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} \leq 0}$.

Corollary 6

Let a <u>multiply twisted product</u> (M, g) satisfy $\langle H_i, H_j \rangle = 0$ $(i \neq j)$. If M is closed and $\underline{\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} \leq 0}$, then (M, g) splits.

Let M be a hypersurface in a Riemannian manifold $\bar{M}(c)$ of constant curvature c. M is called a **Dupin hypersurface** if

- the multiplicity of each principal curvature is constant on M;
- each principal curvature is constant along its principal directions.

It is known that **a compact Dupin hypersurface** M **in** $\bar{M}(c)$ $(c \geq 0)$ **has 1, 2, 3, 4 or 6 distinct principal curvatures**, that is $(M, g, \mathcal{D}_1, \dots, \mathcal{D}_k)$ with $k = 2, 3, 4, 6$.

$A : TM \to TM$ the shape operator, X_i its principal directions on TM, and $AX_i = \mu_i X_i$. The sectional curvature of $M \subset \bar{M}(c)$ is

$$K(X_i, X_j) = c + \mu_i\, \mu_j \qquad (i \neq j).$$

IF for Dupin hypersurfaces with $k = 2, 3$

If M has two distinct principal curvatures $\mu_1 < \mu_2$, then using $\mathrm{S}_{\mathcal{D}_1,\mathcal{D}_2} = n_1 n_2(c + \mu_1\mu_2)$, **IF** (3) can be rewritten as

$$\int_M \Big\{ n_1 n_2(c+\mu_1\mu_2) + \frac{n_1(1-n_1)\|\nabla\mu_1\|^2 + n_2(1-n_2)\|\nabla\mu_2\|^2}{(\mu_2-\mu_1)^2} \Big\}\, d\,\mathrm{vol} = 0.$$

If M has three principal curvatures $\mu_1 < \mu_2 < \mu_3$, then using

$$\mathrm{S}_{\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3} = n_1 n_2(c + \mu_1\mu_2) + n_1 n_3(c + \mu_1\mu_3) + n_2 n_3(c + \mu_2\mu_3).$$

IF (4) with $k = 3$ can be rewritten as

$$\int_M \Big\{ \sum\nolimits_{i<j} n_i\, n_j(c + \mu_i\mu_j) + \sum\nolimits_{i=1}^{3} 2\, n_i(1-n_i)\Big(\frac{\|P_j\nabla\mu_i\|^2 + \|P_l\nabla\mu_i\|^2}{(\mu_i-\mu_j)^2}\Big)\Big\}\, d\,\mathrm{vol}_g = 0,$$

where $(j, l) \in \{1, 2, 3\} \setminus \{i\}$ and $j < l$.

IF for codimension-1 foliations

Integral Formula (**IF**) for a **codimension-1 foliation** $\mathcal{F}$ with a unit normal N of a closed Riemannian manifold (M^{n+1}, g) means the vanishing of the integral over M, where integrand depends on Weingarten operator A of $\mathcal{F}$ and the curvature tensor R.

$\sigma_i(A)$ $(1 \le i \le n)$ – symmetric functions of eigenvalues of A. For few initial values of i, $i = 1, 2$:

$$\int_M \sigma_1(A)\, d\,\mathrm{vol}_g = 0, \text{ [G. Reeb, 1950]}, \tag{6}$$

$$\int_M \big(\sigma_2(A) - \tfrac{1}{2}\,\mathrm{Ric}_{N,N}\big)\, d\,\mathrm{vol}_g = 0, \text{ [T. Nora, 1983]}. \tag{7}$$

The proof of (6), (7) is based on the Divergence Theorem and calculation of $\mathrm{div}\, N$ and $\mathrm{div}(\sigma_1(A)\, N + \nabla_N N)$, respectively.

Define **symmetric functions** $\sigma_r(A)$: $\sum_{r=0}^{n}\sigma_r(A)t^r=\det(I_n+t\,A)$, tensor $R(X): Z\to(R_{Z,X}N)^\top$ and **Newton transformations** of A: $T_0(A)=I$ and $T_r(A)=\sigma_r(A)\,I-A\,T_{r-1}(A)$ $(1\le r\le n)$. We get

$$\operatorname{div}(T_r(A)Z+\sigma_{r+1}(A)N)=\sum_{j\le r}\operatorname{Tr}(T_{r-j}(A)R((-A)^{j-1}Z))\\ -(r+2)\sigma_{r+2}(A)+\operatorname{Tr}(T_r(A)R_N).$$

A series of **IF** with $1\le r\le\dim M-2$ for closed manifolds:

$$\int_M\Big((r+2)\sigma_{r+2}(A)-\operatorname{Tr}(T_r(A)R_N)\\ -\sum\nolimits_{j=1}^{r}\operatorname{Tr}(T_{r-j}(A)R((-A)^{j-1}Z))\Big)\mathrm{d}\operatorname{vol}_g=0. \qquad (8)$$

It can be generalized to singular foliations, under suitable integrability conditions.

Example. For $r=1$, (8) reduces to

$$\int_M\big(3\,\sigma_3(A)-\sigma_1(A)\operatorname{Ric}_{N,N}+\operatorname{Tr}(AR_N)-\operatorname{Ric}_{N,Z}\big)\mathrm{d}\operatorname{vol}_g=0.$$

Application: **IF** for foliated space-forms

- The **IF** (8) might be the main obstructions for recovering metrics by higher mean curvatures of codimension-one foliations.

The total σ_r's of a codimension-one foliation $\mathcal{F}$ on a compact space form $M^{n+1}(c)$ do not depend on $\mathcal{F}$: they depend on n, r, c and volume of M only,

$$\int_M \sigma_r(A)\,\mathrm{d}\,\mathrm{vol}_g = \begin{cases} c^{r/2}\binom{n/2}{r/2}\mathrm{Vol}(M,g), & n \text{ and } r \text{ even},\\ 0, & n \text{ or } r \text{ odd}.\end{cases} \tag{9}$$

Example. An amazing consequence of (9) for codimension-1 foliations of the round unit sphere S^3. By S. Novikov's theorem, any such foliation contains a leaf diffeomorphic to a torus. By Gauss-Bonnet theorem, there is a point with zero **Gaussian leaf curvature** $K_\mathcal{F}$. By (9) for $n = r = 2$ and $c = 1$, the average of $\sigma_2(A)$ is 1. Since $K_\mathcal{F} = 1 + \sigma_2(A)$, there is a point with $K_\mathcal{F} > 1+1 = 2$. Hence, *the set of values of the function* $K_\mathcal{F} : S^3 \to \mathbb{R}$ *contains the interval* $[0, 2+\varepsilon]$ *for some* $\varepsilon > 0$.

Compact manifolds with $\chi(M) \neq 0$ do not admit regular (codimension-1) distributions, while all of them admit such distributions defined outside some "set of singularities".

Definition 7 (A very simple notion of singular foliations)

A foliation $\mathcal{F}$ of the complement $M \setminus \Sigma$ of a union Σ of finitely many closed codimension ≥ 2 submanifolds of M is called a **singular foliation** of M; Σ itself is called the **singular locus** of $\mathcal{F}$.

Given Σ as in Definition 7 on (M, g), the hypersurfaces Σ_r (with small enough $r > 0$) consisting of all points in distance r from Σ form a singular foliation of a tubular neighborhood $\mathcal{N}$ of Σ.

Given a submanifold Σ of an oriented manifold M^n and an n-form Ω on $M \setminus \Sigma$, one can define the *improper integral* of Ω by

$$\int_M \Omega = \lim_{U \to \Sigma} \int_{M \setminus U} \Omega, \tag{10}$$

whenever the limit in (10) exists; here, U ranges over the family of all open neighborhoods of Σ.

Consider tubular neighborhoods $\mathcal{N}_r$ of radius $r > 0$.
Let $f \geq 0$ be a function defined on $M \setminus \Sigma$, M a closed oriented Riemannian manifold and Σ – a closed submanifold of M.
Denote by $\Sigma_r = \partial \mathcal{N}_r$ the tubular surface at distance r from Σ.

Lemma 8

If $\operatorname{codim} \Sigma \geq 2$ *and* $\liminf\limits_{r \to 0^+} \int_{\Sigma_r} f \, \mathrm{d}\mu > 0$, *then* $\int_M f^2 \, \mathrm{d}\,\mathrm{vol}_g = \infty$.

Technical lemma

Also, let X be a vector field defined on $M \setminus \Sigma$ and $f = \|X\|$.

Lemma 9

If $\operatorname{codim}\Sigma \geq 2$ *and* $\int_M \|X\|^2 \,\mathrm{d}\,\mathrm{vol}_g < \infty$, *then*

$$\int_M (\operatorname{div} X)\,\mathrm{d}\ \mathrm{vol}_g = 0.$$

Proof.

Let ν_r be the suitably oriented unit vector field orthogonal to the tubular surface Σ_r. By Stokes's theorem and Lemma 8 applied to $f = \|X\|$, we obtain for $r \to 0$,

$$\Big|\int_{M\setminus\mathcal{N}_r} (\operatorname{div} X)\mathrm{d}\,\mathrm{vol}_g\Big| = \Big|\int_{\Sigma_r} \langle X, \nu_r\rangle\,\mathrm{d}\mu\Big| \leq \int_{\Sigma_r} \|X\|\,\mathrm{d}\mu \longrightarrow 0.$$

$\square$

Application: the Reeb formula

We have

$$\operatorname{div} N = -H,$$

and $H = \sigma_1(A)$ is the mean curvature of the leaves.
Since $\|N\| = 1$, it is integrable over M when M is compact, Lemma 9 implies directly the following.

Theorem 10

If $\mathcal{F}$ is a singular codimension-one foliation on a closed (M, g) and the singular locus Σ of $\mathcal{F}$ is of codimension ≥ 2, then **IF** *holds:*

$$\int_M H \, \mathrm{d}\,\mathrm{vol}_g = 0;$$

thus, either $H \equiv 0$ or $H(x)H(x') < 0$ for some points $x \neq x'$.

If M is closed, then the equality holds:

$$\operatorname{div}(\sigma_1(A)\,N + \nabla_N\,N) = \operatorname{Ric}_{N,N} - 2\,\sigma_2(A). \tag{11}$$

Since $\|\sigma_1(A)\,N + \nabla_N\,N\|^2 = \sigma_1^2(A) + \kappa^2$, where $\kappa = \|\nabla_N N\|$ is the curvature of the curves orthogonal to the leaves of $\mathcal{F}$, Lemma 9 and (11) yield the following.

Theorem 11

If $\mathcal{F}$ is a singular codimension-one foliation on a closed Riemannian manifold (M, g) and the singular locus Σ of $\mathcal{F}$ has codimension ≥ 2 and $\int_M(\sigma_1^2(A) + \kappa^2)\,\mathrm{d}\,\mathrm{vol}_g < \infty$, then the **IF** *holds:*

$$\int_M (2\,\sigma_2(A) - \operatorname{Ric}_{N,N})\,\mathrm{d}\,\mathrm{vol}_g = 0.$$

.

Minkowski norm

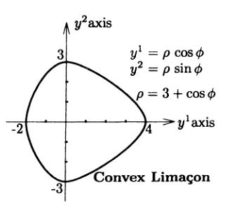
A **Minkowski norm** $\Phi : \mathbb{R}^{m+1} \to [0, \infty)$:

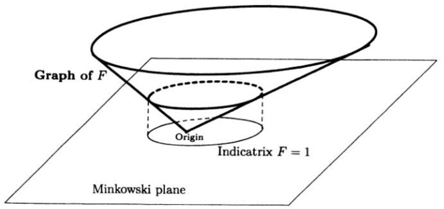
– **Regularity**: $\Phi \in C^\infty(\mathbb{R}^{m+1} \setminus \{0\})$,

– **1-homogeneity**: $\Phi(\lambda\, y) = \lambda \Phi(y)$ for $\lambda > 0$ and $y \in \mathbb{R}^{m+1}$,

– **Strong convexity**: For any $y \in \mathbb{R}^{m+1} \setminus \{0\}$, the following symmetric bilinear form is positive definite on $\mathbb{R}^{m+1}$:

$$g_y(u, v) = \frac{1}{2}\,\frac{\partial^2}{\partial s \partial t}\left[\Phi^2(y + su + tv)\right]_{|\, s=t=0}\,.$$





Metric defined by a vector field

Definition 12

Given a unit vector field n on a Finsler space (M^{m+1}, Φ), define a **Riemannian metric** by

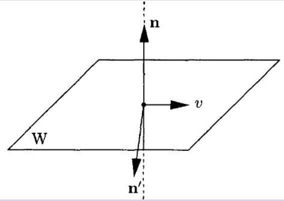
$$g := g_n.$$

∇ – the Levi-Civita connection of g. Set $\nu = n/\Phi(n)$.

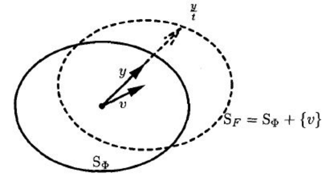
$n \in \mathbb{R}^{m+1}$ is a **normal** to a hyperplane $W^m \subset \mathbb{R}^{m+1}$ if

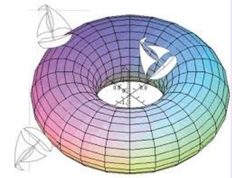
$$g_n(n, w) = 0, \quad w \in W.$$

There are exactly **two** normal directions to W (which are opposite when Φ is **reversible**: $\Phi(-y) = \Phi(y) \quad \forall\, y \in \mathbb{R}^{m+1}$).



One may construct Minkowski norms by shifting a Minkowski norm.





E. Zermelo's navigation problem (1931).

Let $a(\cdot\,,\cdot) = \langle\cdot\,,\cdot\rangle$ be a scalar product and $\alpha(y) = \|y\|_\alpha = \sqrt{\langle y, y\rangle}$ for $y \in \mathbb{R}^{m+1}$. If β is a linear form on $\mathbb{R}^{m+1}$ with $\|\beta\|_\alpha < 1$ then

$$\Phi(y) = \alpha(y) + \beta(y)$$

is called the **Randers norm** (i.e. shifted Euclidean norm), first introduced by physicist G. Randers in 1941.
$\bar\nabla\beta = 0$ – of Berwald type (investigated by L. Berwald).

IF for Randers metric $\Phi = \alpha + \beta$

Given a unit vector field n on $(M, \alpha+\beta)$, for $g = g_n$ we get
$n = c_1 N - \beta^\sharp,\ s = \beta(n) = c\,c_1 - 1,\ \phi(s) = c\,c_1$,
where $c_1 = c + \beta(N)$ and $c = \sqrt{1 - b^2 + \beta(N)^2} \in (0, 1]$.
Set $\bar Z = \bar\nabla_N N$ and $Z = \nabla_\nu \nu$, where $\nu = n/|n|_g$, $\overline{A}X = -\overline{\nabla}_X N$.
Next, $\mu_g(n) = (c\,c_1)^{m+2}$ and

$$g(u,v) = (1+s)\langle u,v\rangle - s\langle n,u\rangle\langle n,v\rangle + \beta(u)\langle n,v\rangle + \beta(v)\langle n,u\rangle + \beta(u)\beta(v).$$

Theorem 13

For Randers metric on M and a unit vector field n, we get the **IF***:*

$$\int_M (c\,c_1)^{m+1} c^{-1}\Big((c\,c_1)\ \mathrm{Tr}(\overline{A}) - \frac{m+2}{2}\,(N + c_1^{-1}\beta^\sharp)(c\,c_1) + c\,N(c) - (c_1 - c)\langle c^{-1}\overline{A}(\beta^{\sharp\top}) + \overline{Z},\ \beta^\sharp\rangle\Big)\,\mathrm{d}\,\mathrm{vol}_a = 0,$$

which reduces to the **IF** (6) *(by Reeb) when* $\beta = 0$.