Problems of Extrinsic Geometry of Foliations

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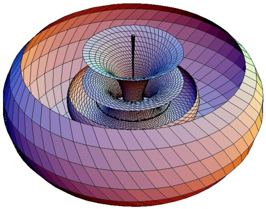
Almost Hermitian Structure.

A Riemannian manifold (M, g) with an **almost complex structure** J (a skew-symmetric $(1{,}1)$-tensor such that $J^2 = -I$) is called an **almost Hermitian manifold**; and a **Kähler manifold** if $\nabla J = 0$.

For an oriented (M^{2n}, g) its **twistor space** $J(M)$ (R. Penrose and M. MacCallum in 1960s; E. Calabi in 1967) can be understood as the **space of all almost complex structures** on TM.

The twistor space is useful in constructing harmonic maps.

A smooth mapping $\phi : (M, g) \to (M', g')$ is called **harmonic** if it yields extrema for the energy $E(\phi) = \frac{1}{2}\int_M \|d\phi\|^2\, d\,\mathrm{vol}_g$.



An f-**structure** (K.Yano, 1963) on a smooth manifold M^{2n+s} generalizes almost complex structure ($s=0$) and almost contact structure ($s=1$). It is defined by a (1,1)-tensor f of rank $2n$ such that $f^3+f=0$. The tangent bundle splits: $TM = f(TM)\oplus\ker f$. The restriction of f to $f(TM)$ is an **almost complex structure**.

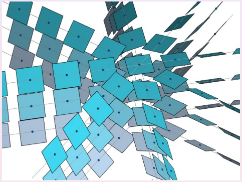
A **framed** f-**structure** occurs if $\ker f$ is **parallelizable**: there exist vector fields $\xi_1,\ldots\xi_s$ that span $\ker f$, and dual 1-forms η^i such that

$$f^2=-I+\sum\nolimits_{i=1}^{s}\eta^i\otimes\xi_i,\quad \eta^i(\xi_j)=\delta^i_j.$$

If there is a Riemannian metric g such that

$$g(fX,fY)=g(X,Y)-\sum\nolimits_{i=1}^{s}\eta^i(X)\,\eta^i(Y),$$

then we get a **metric** f-**structure**.



Example. Orientable hypersurfaces of almost Hermitian manifolds have an **almost contact metric (a.c.m.) structure** ($\xi=J(N)$):

$$g(fX,\ fY)=g(X,Y)-\eta(X)\,\eta(Y). \qquad (1)$$

Contact metric structure: $d\eta(X,Y)=g(X,fY)$, $\eta\wedge(d\eta)^n\neq 0$.

Interest in metric f-structures is due to their applications in CR-integrability, symplectization, topology and dynamics of contact foliations as well as in mathematical physics.

An **almost $\mathcal{S}$-structure** on M^{2n+s} is a metric f-structure (f, ξ_i, η^i, g) satisfying $g(X, fY) = d\eta^1(X, Y) = \ldots = d\eta^s(X, Y)$.

An involutive distribution is **regular** if every point of the manifold has a neighborhood such that any integral submanifold passing through the neighborhood passes through only once.

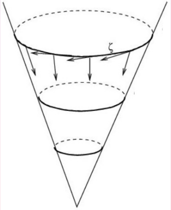
The "generalized Boothby-Wang theorem" shows a similarity between almost $\mathcal{S}$- and contact manifolds: "*A compact manifold M^{2n+s} with a regular almost $\mathcal{S}$-structure is a principal torus bundle over an almost Hermitian manifold of dimension $2n$*".

A contact metric structure is called a K-**contact structure**, if ξ is a Killing vector field, i.e., ξ generates infinitesimal isometries.
Two subclasses:
Sasakian and **cosymplectic** manifolds are defined by (1) and

$$(\nabla_X \varphi)Y = \begin{cases} \langle X, Y\rangle\,\xi - \eta(Y)X\,, & \text{Sasakian},\\ 0\,, & \text{cosymplectic}.\end{cases}$$

• Every **cosymplectic** manifold is locally the metric product of a Kähler manifold and a real line $\mathbb{R}$.

• A Riemannian manifold (M^{2n+1}, g) with a contact 1-form η (i.e., $\eta\wedge(d\eta)^n \ne 0$) is **Sasakian**, if its Riemannian cone $\widetilde{M} = M\times\mathbb{R}^{>0}$ with the cone metric $\tilde g = t^2 g + dt^2$ is a Kähler manifold, see Fig.:



Example:

Hopf fibration $S^{2n+1} \overset{S^1}{\mapsto} P\mathbb{C}^n$ gives a Sasakian structure on S^{2n+1}.

Weak almost Hermitian manifolds

Definition 1

A Riemannian manifold (M^{2n}, g) equipped with a skew-symmetric (1,1)-tensor J of rank $2n$ (other than the complex structure) is called a **weak almost Hermitian manifold**. This manifold is called **weak Kähler** if $\nabla^g J = 0$, where ∇^g is the Levi-Civita connection of g.

Example. Take two or more almost Hermitian (or, Kähler) manifolds (M_j, J_j, g_j), where $J_j^2 = -I_j$. The weighted product

$$\prod\nolimits_j (M_j, \sqrt{\lambda_j}\, J_j, g_j),$$

where $\lambda_j \neq 1$ are different positive constants, is a **weak almost Hermitian** (or **weak Kähler**) **manifold** with $J^2 = -\bigoplus_j \lambda_j I_j$.

Weak framed/metric f-manifolds

Definition 2

A **weak framed f-structure** on a differentiable manifold M^{2n+s} is a set (f, Q, ξ_i, η^i), where f is a (1,1)-tensor of rank $2\,n$, $\xi_1, \ldots, \xi_s$ are vector fields such that $\ker f = \mathrm{Span}(\xi_1, \ldots, \xi_s)$, $\eta^1, \ldots, \eta^s$ are dual 1-forms such that $\bigcap_i \ker \eta^i$ is f-invariant, and

$$Q = -f^2 + \sum\nolimits_{i=1}^{s} \eta^i \otimes \xi_i, \quad \eta^i(\xi_j) = \delta^i_j.$$

We get a **weak metric f-structure**, if there exists a (pseudo-)Riemannian metric g satisfying

$$g(fX, fY) = g(X, Q\,Y) - \sum\nolimits_{i=1}^{s} \eta^i(X)\,\eta^i(Y) \quad (X, Y \in \mathfrak{X}_M).$$

$f^3 + fQ = 0$, $[Q,\, f] = 0$, $Q\,\xi_i = \xi_i$, $f\,\xi_i = 0$, $\eta^i \circ f = 0$ hold.
A weak metric f-structure is a special case of an almost product structure, defined by two complementary orthogonal distributions of a Riemannian manifold, with Naveira's 36 distinguished classes.

How rich are weak structures compared to the classical ones?

The **Nijenhuis torsion** of f and derivatives $d\eta^i$ and dF are

$$[f,f](X,Y) = f^2[X,Y] + [fX,fY] - f[fX,Y] - f[X,fY],$$

$$d\eta^i(X,Y) = \frac{1}{2}\,\{X(\eta^i(Y)) - Y(\eta^i(X)) - \eta^i([X,Y])\},$$

$$\begin{aligned} dF(X,Y,Z) = \tfrac{1}{3}\big\{&X\,F(Y,Z) + Y\,F(Z,X) + Z\,F(X,Y) \\ &- F([X,Y],Z) - F([Z,X],Y) - F([Y,Z],X)\big\}. \end{aligned}$$

Since $dF(X,Y,Z) = -F([X,Y],Z)$ $(X,Y \in \ker f)$, for a weak framed f-structure, $\ker f$ **is involutive if and only if** $dF = 0$.

Example. Given a weak framed f-manifold $M^{2n+s}(f, Q, \xi_i, \eta^i)$, consider the **product** $\bar M = M^{2n+s} \times \mathbb{R}^s$. Define the tensor J on $\bar M$:

$$J(X, \sum_i a^i \partial_i) = (fX - \sum_i a^i \xi_i, \sum_j \eta^j(X)\partial_j),$$

where $\partial_1, \ldots, \partial_s$ is an orthonormal basis of $\mathbb{R}^s$.

The **integrability condition** $[J, J] = 0$ is used to express the **normality condition** $N^{(1)} = 0$, where $N^{(1)} = [f, f] + 2\sum_i d\eta^i \otimes \xi_i$.

Proposition 1

The normality condition for a weak metric f-structure implies

$$\mathcal{L}_{\xi_i} f = 0, \quad d\eta^j(\xi_i, \cdot) = 0 \quad (1 \le i, j \le s),$$

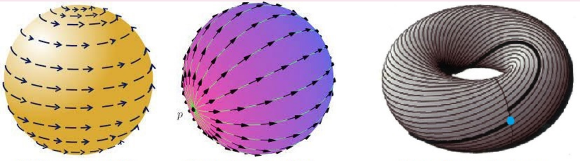
Moreover, $\ker f$ is a totally geodesic distribution.

A weak metric f-manifold (f, Q, ξ_i, η^i, g) is called
(i) a **weak almost $\mathcal{K}$-structure** if $dF = 0$. Two subclasses:
(ii) a **weak almost $\mathcal{C}$-structure** if $dF = 0$ and $d\eta^i = 0$ for any i;
(iii) a **weak almost $\mathcal{S}$-structure** if $d\eta^i = F$ for any i.

If in addition, $N^{(1)} = 0$, then we have, respectively,
(i) **weak $\mathcal{K}$-structure**; (ii) **weak $\mathcal{C}$-structure**; (iii) **(weak) $\mathcal{S}$-structure**.

Theorem 3

On a weak $\mathcal{K}$-manifold, $\xi_1, \ldots, \xi_s$ are Killing vector fields and $\ker f$ is tangent to a foliation with flat totally geodesic leaves.



Killing vector field Geodesic vector field Linear flow on the torus

Calculation of ∇f and the tensor $N^{(5)}$

Proposition 2

For a weak almost $\mathcal{S}$-structure, we get

$$2\,g((\nabla_X f)Y,Z) = g(N^{(1)}(Y,Z), fX) + 2\,g(fX,fY)\,\bar\eta(Z) - 2\,g(fX,fZ)\,\bar\eta(Y) + N^{(5)}(X,Y,Z),$$

where $\bar\eta=\sum_i \eta^i$ and the **new tensor** $N^{(5)}(X,Y,Z)$ *is defined by*

$$\begin{aligned}N^{(5)}(X,Y,Z) &= fZ\,(g(X,\tilde QY)) - fY\,(g(X,\tilde QZ)) \\ &\quad + g([X,fZ],\tilde QY) - g([X,fY],\tilde QZ) - X(g(\widetilde QY,fZ)) \\ &\quad + g\big([Y,fZ]-[Z,fY]-f[Y,Z],\ \tilde QX\big).\end{aligned}$$

In particular, $2\,g((\nabla_{\xi_i} f)Y,Z) = N^{(5)}(\xi_i,Y,Z)$ $(1\le i\le s)$.
Thus, $\nabla_{\xi_i}\xi_j=0$ $(1\le i,j\le s)$, and $\ker f$ is tangent to a **totally geodesic foliation** *with zero sectional curvature: $K(\xi_i,\xi_j)=0$.*

The particular values: $N^{(5)}(\xi_i,\xi_j,Z)=N^{(5)}(\xi_i,Y,\xi_j)=0$ and

$$N^{(5)}(X,\xi_i,Z) = -N^{(5)}(X,Z,\xi_i) = g((\mathcal{L}_{\xi_i} f)(Z),\ \tilde QX),$$

$$N^{(5)}(\xi_i,Y,Z) = g([\xi_i,fZ],\tilde QY) - g([\xi_i,fY],\tilde QZ) - \xi_i(g(\widetilde QY,fZ)).$$

Theorem 4 (Rigidity of an $\mathcal{S}$-structure)

A weak metric f-structure is a weak $\mathcal{S}$-structure if and only if it is an $\mathcal{S}$-structure (i.e., $Q = I$).

Recall that "*a $\mathcal{K}$-structure is a $\mathcal{C}$-structure $\Leftrightarrow$ $\nabla f = 0$*".

Theorem 5 (Characterization of a weak $\mathcal{C}$-structure)

A weak metric f-structure with conditions $\nabla f = 0$ and $[\xi_i, \xi_j]^\perp = 0$ for $1 \le i, j, k \le s$, is a weak $\mathcal{C}$-structure with $N^{(5)} = 0$.

Example. Consider (M^{2n}, g) and a skew-symmetric endomorphism $J : TM \to TM$ of rank $2n$ with $\nabla J = 0$. To construct a weak $\mathcal{C}$-structure on the product $M \times \mathbb{R}^s$, take any point $(x, t_1, \ldots, t_s)$ of the space and set $\xi_i = (0, d/dt_i)$, $\eta^i = (0, dt_i)$ and

$$f(X, Y) = (JX, 0), \;\; Q(X, Y) = (-J^2 X, \; Y) \;\; (X \in T_x M, \; Y \in \mathbb{R}^s_t).$$

Then $\nabla f = 0$ holds and Theorem 5 can be applied.

A weak f-K-contact structure

Definition 6 (Generalization of a weak $\mathcal{K}$-structure)

A **weak f-K-contact structure** is a weak almost $\mathcal{S}$-structure ($F{=}d\eta^1{=}\ldots{=}d\eta^s$) such that all vector fields ξ_i are Killing.

Theorem 7

A weak almost $\mathcal{S}$-manifold is weak f-K-contact if and only if

$$\nabla\,\xi_i = -f \qquad (1 \le i \le s).$$

The mapping $R_{\xi_i}: X \mapsto R_{X,\xi_i}\,\xi_i$ $(X \in \mathcal{D})$ is called **Jacobi operator** in the ξ_i-direction. For a $\mathcal{K}$-manifold: $R_{\xi_i} = I$.

Theorem 8

A Riemannian manifold (M^{2n+s}, g) with orthonormal Killing vector fields ξ_i $(1 \le i \le s)$ such that $d\,\eta^1 = \ldots = d\,\eta^s$ (where η^i is the 1-form dual to ξ_i) and positive definite Jacobi operators, $R_{\xi_i} > 0$ along $\mathcal{D} = \bigcap_i \ker \eta^i$, is a **weak f-K-contact manifold** *with*

$$f := -\nabla\,\xi_i, \quad QX := R_{\xi_i}(X) \quad (X \in \mathcal{D}, \quad i = 1, \ldots, s).$$

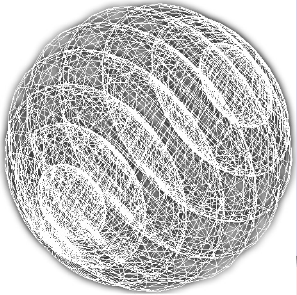
Example of a weak f-K-contact manifold ($s = 1$)

For $0 < a = const \neq 1$, consider an ellipsoid of $\mathbb{R}^{2n+2}$:

$$M^{2n+1} = \Big\{(u_1, \ldots, u_{2n+2}) : \sum_{i=1}^{n+1} u_i^2 + a \sum_{i=n+2}^{2n+1} u_i^2 = 1\Big\}.$$

$\xi = (-u_2, u_1, \ldots, -u_{n+1}, u_n, -\sqrt{a}\, u_{n+3}, \sqrt{a}\, u_{n+2}, \ldots,$
$-\sqrt{a}\, u_{2\,n+2}, \sqrt{a}\, u_{2\,n+1})$

is a Killing vector field on $\mathbb{R}^{2n+2}$, its restriction to M^{2n+1} has unit length. Sectional curvature of M^{2n+1} is > 0. Since M^{2n+1} is ξ-invariant, ξ is a unit Killing vector field on (M, g).



Projection of 3-ellipsoid into 3D space and covered with surface grid

Weak nearly $\mathcal{C}$-manifolds are defined by condition that the symmetric part of ∇f vanishes:

$$(\nabla_X f)Y + (\nabla_Y f)X = 0.$$

The product of a weak nearly Kähler manifold and $\mathbb{R}^s$ is a weak nearly $\mathcal{C}$-manifold. The following condition is trivial when $Q = \mathrm{Id}$:

$$((\nabla_X Q)Y)^\top = 0, \quad (\nabla_Y Q)Y = 0 \quad (X \in TM,\ Y \in \mathcal{D}). \tag{2}$$

Weak almost $\mathcal{C}$-manifolds satisfy the following conditions:

$$[\xi_i, \xi_j] = 0 \quad (1 \le i, j \le s), \tag{3}$$

$$g(\nabla_X \xi_i,\ \xi_j) = 0 \quad (X \in \mathfrak{X}_M,\ 1 \le i, j \le s). \tag{4}$$

For a weak nearly $\mathcal{C}$-manifold:

- (3) and (4) $\Rightarrow$ $\ker f$ defines a flat totally geodesic foliation;
- (2) and (4) $\Rightarrow$ ξ_i are Killing vector fields.

For a nearly cosymplectic manifold, $g(R_{\xi,Z}\, fX, fY) = 0$, thus $\ker\eta$ is a **curvature invariant** distribution. Assume a weaker condition

$$R_{\widetilde{Q}X,Y}Z \in \mathcal{D} \quad (X,Y,Z\in\mathcal{D}). \tag{5}$$

Define the tensor fields $h_1,\ldots,h_s$,

$$h_i = \nabla\xi_i \quad (1\le i\le s).$$

So, $\eta^j\circ h_i = 0$, $h_i\xi_j = 0$, h_i skew-symmetric (totally geodesic $\mathcal{D}$):

$$g(h_iX,\,X) = g(\nabla_X\,\xi_i, X) = \frac12\,(\mathcal{L}_{\xi_i}\,g)(X,X) = 0 \quad (1\le i\le s).$$

Proposition 3

For a weak nearly $\mathcal{C}$-manifold with conditions (2), (3), (4) and (5), the operators h_i and h_j commute and each h_ih_j is self-adjoint. Moreover, the eigenvalues and their multiplicities of h_i^2 are constant, and h_i^2 share the same eigen-frame. The spectrum of the self-adjoint operator h_i^2 has the form

$$Spec\; h_i^2 = \{0,-\lambda_{i,1}^2,\ldots-\lambda_{i,r_i}^2\} \qquad (1\le i\le s).$$

Two theorems on weak nearly $\mathcal{C}$-manifolds

Theorem 9 (Splitting when $n>2$)

Let $(M^{2n+s}, f, Q, \vec\xi, \vec\eta, g)$, $\underline{n>2}$, be a weak nearly $\mathcal{C}$-manifold with conditions (2), (3), (4) and (5). If the coframe $\vec\eta$ has a common exterior derivative, $d\eta^i = d\eta^j$ $(1\le i,j\le s)$, then M is locally isometric to one of the Riemannian products

$$(i)\ \mathbb{R}^s\times\bar M^{2n},\quad (ii)\ B^{4+s}\times\bar M^{2n-4},$$

where the induced structure $(\bar f,\bar g)$ on $\bar M$ is weak nearly Kähler satisfying $\bar\nabla(\bar f^{\,2})=0$, and the induced structure on B is a weak nearly $\mathcal{C}$-structure satisfying (2) and (5).

Theorem 10 (Characterization of $n=2$)

Let a weak nearly $\mathcal{C}$-manifold $(M^{2n+s}, f, Q, \vec\xi, \vec\eta, g)$ satisfy (2)–(5), $d\eta^i = d\eta^j$ $(i\ne j)$, and rank $h_i = 2n$ $(1\le i\le s)$. Then $\mathbf{n=2}$ and M admits an f-K-contact structure $(\widehat f, \vec\xi, \vec\eta, g)$ with $\widehat f = -\nabla\xi_i$; that is, ξ_i are Killing vector fields and $g(X,\widehat f Y) = d\eta^i(X,Y)$.

Remark. When $s=1$, (3) and (4) are satisfied automatically.

Proposition 4

For a weak f-K-contact manifold, the following equalities hold:

$$R_{\xi_i, X} = \nabla_X f,$$
$$\mathrm{Ric}(\xi_i, \xi_j) = \operatorname{Tr} Q > 0.$$

The next theorem generalizes a well-known result (with $Q = I$).

Theorem 11

A weak f-K-contact manifold $M^{2n+s}(f, Q, \xi_i, \eta^i, g)$ satisfying

$$(\nabla \,\mathrm{Ric})(\xi_i, \cdot) = 0 \ \ (1 \le i \le s), \quad \operatorname{Tr} Q = const,$$

is an Einstein manifold and $s = 1$.

The self-adjoint **partial Ricci curvature tensor** is given by

$$\mathrm{Ric}^\top(X) = \sum\nolimits_{i=1}^{s} (R_{X^\top, \xi_i}\, \xi_i)^\top \quad (\text{hence} \ \ \mathrm{S}_{\,\rm mix} = \mathrm{Tr}\,\mathrm{Ric}^\top).$$

For weak f-K-contact manifolds, $\mathrm{Ric}^\top > 0$ holds (see Theorem 8).

The **partial Ricci flow** of metrics on $(M, g, \mathcal{D}, \mathcal{D}^\perp)$ is defined as

$$\partial_t\, g_t = -2\,\mathrm{Ric}^\top(g_t) + 2\,s\,g_t^\top, \tag{6}$$

where $g^\top(X,Y) := g(X^\top, Y^\top)$. When the flow (6) is applied to weak f-K-contact manifolds, (6) reduces to ODEs, and we get

$$\partial_t\,\mathrm{Ric}^\top = 4\,\mathrm{Ric}^\top(\mathrm{Ric}^\top - s\,I^\top). \tag{7}$$

The f-K-contact structure is a fixed point of (7): $\mathrm{Ric}^\top = s\,I^\top$.

We also have along $\mathcal{D}$:

$$\partial_t T^\sharp_{\xi_i} = 2(\mathrm{Ric}^\top - s\, I^\top)\, T^\sharp_{\xi_i},$$

where (1,1)-tensors $T^\sharp_{\xi_i}$ are given by $g(T^\sharp_{\xi_i} X, Y) = \frac{1}{2}\, g([X,Y], \xi_i)$.

Theorem 12

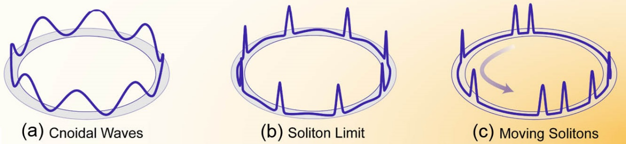
Let $M^{2n+s}(f_0, Q_0, \xi_i, \eta^i, g_0)$ be a weak f-K-contact manifold. Then there exists a flow of metrics g_t $(t\in\mathbb{R})$ such that each $(f_t, Q_t, \xi_i, \eta^i, g_t)$ is a weak f-K-contact structure on M with

$$Q_t\,|_{\mathcal{D}} = (1/s)\,\mathrm{Ric}^\top_t, \quad f_t\,|_{\mathcal{D}} = T^\sharp_{\xi_i}(t).$$

Moreover, g_t converges exponentially fast, as $t\to-\infty$, to a limit metric $\hat g$ that provides an f-K-contact structure on M.

Generalized Ricci Solitons

Solutions of some non-linear PDEs decompose at large time into solitary waves (called **solitons**) that retain their shape and speed.



Ricci solitons (**RS**) $\frac{1}{2}\mathcal{L}_V\, g + \mathrm{Ric} = \lambda\, g$ (where V is a vector field, $\mathcal{L}$ is the Lie derivative and $\lambda \in \mathbb{R}$), are self-similar solutions $g_t = \sigma(t)\psi_t^*(g_0)$ of the Ricci flow $\partial g/\partial t = -2\,\mathrm{Ric}_g$. When V is a Killing field ($\mathcal{L}_V = 0$), **RS** are Einstein metrics $\mathrm{Ric} = \lambda\, g$ (e.g., of constant curvature).

Since some compact manifolds don't admit Einstein metrics, **Einstein-type metrics** and **generalized RS** are studied.

For example:

η-**RS**: $\frac{1}{2}\mathcal{L}_V\, g + \mathrm{Ric} = \lambda\, g + \mu\, \eta \otimes \eta$, $\quad \eta$ is a 1-form, $\mu \in C^\infty(M)$,

$*$-**RS**: $\frac{1}{2}\mathcal{L}_V\, g + \mathrm{Ric}^* {=} \lambda g$, $\ \mathrm{Ric}^*(X, Y) = \frac{1}{2}\mathrm{Tr}\{Z \to R_{X,JY} JZ\}$.

How generalized RS interact with weak metric structures?

Recall the following property of f-K-contact manifolds:

$$\mathrm{Ric}(Y,\xi)=0 \quad (Y\in\mathcal{D},\ \xi\in\mathcal{D}^\perp). \tag{8}$$

Theorem 13

Let a weak f-K-contact manifold with the properties $\operatorname{Tr} Q = const$ and (8) satisfy the following **generalized gradient RS equation** *with $c_1(\lambda + c_2 \operatorname{Tr} Q) \neq -1$:*

$$\mathrm{Hess}_\sigma - c_2\,\mathrm{Ric} = \lambda\, g - c_1 d\sigma\otimes d\sigma$$

for some $\sigma\in C^\infty(M)$ and $c_1, c_2, \lambda\in\mathbb{R}$. Then $\sigma = const$, and if $c_2\neq 0$, then our manifold is an **Einstein manifold** *and $s=1$.*

How η-**RS** and η-Einstein metrics interact?

Some compact manifolds have no Einstein metrics, $\mathrm{Ric} = \lambda g$.

Definition 14

A weak metric f-manifold $M^{2n+s}(f, Q, \xi_i, \eta^i, g)$ is said to be **η-Einstein**, if for some $a, b \in C^\infty(M)$,

$$\mathrm{Ric} = a\, g + b \sum\nolimits_i \eta^i \otimes \eta^i + (a+b) \sum\nolimits_{i\neq j} \eta^i \otimes \eta^j. \tag{9}$$

An **η-RS** is a weak metric f-manifold satisfying

$$\frac{1}{2}\mathcal{L}_V\, g + \mathrm{Ric} = \lambda\, g + \mu \sum\nolimits_i \eta^i \otimes \eta^i + (\lambda+\mu) \sum\nolimits_{i\neq j} \eta^i \otimes \eta^j \tag{10}$$

for some smooth vector field V and functions λ, μ on M.

For a Killing vector field V, i.e., $\mathcal{L}_V\, g = 0$, (10) reduces to (9).
For $s = 1$:
(9) gives $\mathrm{Ric} = a\, g + b\eta \otimes \eta$; (10) gives $\frac{1}{2}\mathcal{L}_V\, g + \mathrm{Ric} = \lambda\, g + \mu\eta \otimes \eta$.

When a metric f-manifold equipped with a Ricci-type soliton, carries an Einstein-type metric?

Integral formula

Lemma 15 (IF by Pohozaev-Schoen)

Let (M^n, g) be a compact Riemannian manifold without boundary, E a divergence free symmetric $(0,2)$-tensor on M. Then

$$\int_M V(\mathrm{Tr}_g\, E)\,\mathrm{d}\,\mathrm{vol} = \frac{n}{2}\int_M g(E^0, \mathcal{L}_V\, g)\,\mathrm{d}\,\mathrm{vol}, \tag{11}$$

where $E^0 = E - \frac{1}{n}(\mathrm{Tr}_g\, E)g$, and V a vector field on M.

Theorem 16

*Let a compact weak f-K-contact manifold $M^{2n+s}(f, \xi_i, \eta^i, Q, g)$ satisfy $r = const$ and $\mathrm{Tr}\, Q = const$. If (g, V, λ, ν) represents an η-**RS**, then M is an η-**Einstein manifold**.*

Proof. We take the trace of (9): $r=(2n+s)\,\alpha+s\,\beta$. Using $\mathrm{Ric}(\xi_i,\xi_j)=\operatorname{Tr}Q$ in (9) yields $\alpha+\beta=\operatorname{Tr}Q$. By the above: $\alpha=\frac{r-s\operatorname{Tr}Q}{2\,n}$, $\beta=\frac{(2\,n+s)\operatorname{Tr}Q-r}{2\,n}$. Define a traceless tensor

$$E=\mathrm{Ric}-\frac{r-s\operatorname{Tr}Q}{2\,n}g+\frac{r-(2\,n+s)\operatorname{Tr}Q}{2\,n}\sum_{i}\eta^i\otimes\eta^i-\operatorname{Tr}Q\sum_{i\neq j}\eta^i\otimes\eta^j.$$

We get the equality $\operatorname{div}_g E=0$.
Since E is traceless, we get $E^0=E$ and $V(\operatorname{Tr}_g E)=0$.
Then, using **integral formula** (11), we find $\int_M g(E,E)\,\mathrm{d}\,\mathrm{vol}=0$.
From this we find $E=0$ – our manifold is η-**Einstein**. $\square$

Weak β-f-Kenmotsu-manifolds (β-f-KM)

Definition 17

A normal (i.e., $N^{(1)} = 0$) weak metric f-manifold M is called a **weak β-f-Kenmotsu manifold (KM)** with a smooth function $\beta \neq 0$ on M (a **weak f-KM** when $\beta \equiv 1$) if

$$(\nabla_X f)Y = \beta\{g(fX, Y)\,\bar\xi - \bar\eta(Y)fX\} \quad (X, Y \in \mathfrak{X}_M), \tag{12}$$

where $\bar\xi = \sum_i \xi_i$ and $\bar\eta = \sum_i \eta^i$.

Proposition 5

A weak metric f-manifold with condition (12) *is a weak β-f-KM if and only if the following formula holds:*

$$\nabla_X\,\xi_i = \beta\{X - \sum\nolimits_j \eta^j(X)\,\xi_j\} \quad (1 \le i \le s, \quad X \in \mathfrak{X}_M). \tag{13}$$

By (13), $\nabla_{\xi_i}\xi_j = 0$, thus, $\ker f$ defines a totally geodesic $\mathfrak{g}$-foliation with an abelian Lie algebra.
By (13), every vector field ξ_i is not a Killing vector field.

Geometry of weak β-f-KM

Theorem 18

A normal weak metric f-manifold is a weak β-f-KM if and only if $d\eta^i = 0$ $(1 \le i \le s)$ and

$$dF = 2\,\beta\,\bar\eta \wedge F, \quad N^{(5)}(X,Y,Z) = 2\,\beta\,\bar\eta(X) g(fY, \widetilde{Q}Z).$$

Theorem 19

A weak β-f-KM is locally a twisted product $\mathbb{R}^s \times_\sigma \bar M$ (a warped product when $X(\beta) = 0$ for $X \in \mathcal{D}$), where $\bar M(\bar g, J)$ is a weak Kähler manifold.

A weak Hermitian manifold is called **weak Kähler** if $\tilde\nabla J = 0$, where $\tilde\nabla$ is the Levi-Civita connection of $\tilde g$.

Theorem 20

Let $M^{2n+s}(f, Q, \xi_i, \eta^i, g)$ be a weak β-f-KM satisfying $\beta = const$. If $\nabla_{\xi_i} \mathrm{Ric}^\sharp = 0$, then (M, g) is an η-Einstein manifold (9) of scalar curvature $r = -2\,s\,n(2\,n+1)\,\beta^2$.

A vector field V on $M^{2n+s}(f,Q,\xi_i,\eta^i,g)$ is called a **contact vector field**, if there exists $\rho\in C^\infty(M)$ such that $\mathcal{L}_X\,\eta^i=\rho\,\eta^i$; and if $\rho=0$, then V is called a *strict contact vector field*.

Do weak β-f-KM admit η-RS?

Theorem 21

Let $M^{2n+s}(f,Q,\xi_i,\eta^i,g)$ be a weak β-f-KM with $\beta=const$ and $\dim M>3$. *If g represents an η-RS* (10) *with any of conditions:*

(i) with a contact potential vector field V,

(ii) with $V=\delta\,\bar\xi$ for a smooth function $\delta\neq 0$ on M,

then M is η-Einstein (9) *with $a=-2\,s\,n\beta^2$, $b=2(s-1)n\beta^2$ of constant scalar curvature $r=-2\,s\,n(2\,n+1)\beta^2$.*

S. Tashibana (1959) defined the $*$-Ricci tensor on almost Hermitian manifolds. T. Hamada (2002) applied this to real hypersurfaces in non-flat complex space forms. G. Kaimakamis and K. Panagiotidou (2014) introduced $*$-**RS** by $(1/2)\,\mathcal{L}_V\, g + \mathrm{Ric}^* = \lambda\, g$.

We extend the notion of $*$-**Ricci tensor** to weak almost Hermitian and weak metric f-manifolds:

$$\mathrm{Ric}^*(X,Y) = (1/2)\ \mathrm{Tr}\{Z \to f\, R_{X,fY} Z\}\quad (X,Y\in\mathfrak{X}_M).$$

Note that Ric^* is not symmetric.

Proposition 6

For a weak β-KM, Ric^* *and* $\mathrm{Scal}^* = \mathrm{Tr}_g\,\mathrm{Ric}^*$ *are given by*

$$\begin{aligned}\mathrm{Ric}^*(X,Y) &= \mathrm{Ric}(X,QY) + \beta^2\big\{s(2\,n-1)\,g(QX,Y)\\ &\quad + 2\,n\,\bar\eta(X)\bar\eta(Y) - s(2\,n-1)\sum\nolimits_j \eta^j(X)\eta^j(Y)\big\},\\ \mathrm{Scal}^* &= \mathrm{Tr}(Q\,\mathrm{Ric}^\sharp) + 4\,n^2 + (2n-1)\,\mathrm{Tr}\,\widetilde{Q}.\end{aligned}$$

$*$-η-**RS** and Weak β-KM

A weak metric f-manifold $M(f, Q, \xi_i, \eta^i, g)$ represents an $*$-η-**RS** if

$$(1/2)\mathcal{L}_V\, g + \mathrm{Ric}^* = \lambda\,\{g - \textstyle\sum_i \eta^i \otimes \eta^i\} + (\lambda+\mu)\,\bar\eta \otimes \bar\eta. \quad (14)$$

where V is a smooth vector field and λ, μ are real constants. In this case, Ric^* is symmetric and $[Q, \mathrm{Ric}^\sharp] = 0$, where $\mathrm{Ric}^\sharp$ is a (1,1)-tensor adjoint to Ric. For $\mu = 0$, (14) defines a $*$-**RS**.

Proposition 7

*Let a weak β-KM with $\beta = const$ be a $*$-η-**RS**, then $\lambda + \mu = 0$.*

Theorem 22

*Let a weak β-KM ($\beta = const$) be a $*$-η-**RS** with any of conditions:*

(i) the potential vector field V is contact,

(ii) $V = \sum_i \delta^i \xi_i \ne 0$ for smooth functions δ^i on M.

*Then M is an η-**Einstein manifold** of $\mathrm{Scal} = -2s\,n(2n+1)\beta^2$.*

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