Problems of Extrinsic Geometry of Foliations

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XXVI-th International Conference on Geometry, Integrability and Quantization

June 4–11, 2026 Varna, Bulgaria

Metric-affine geometry generalizes Riemannian geometry: it uses a linear connection ∇ with torsion $T(X,Y){=}\nabla_X Y{-}\nabla_Y X{-}[X,Y]$,

$$\nabla_{fX_1+X_2} Y{=}f\nabla_{X_1} Y{+}\nabla_{X_2} Y,\ \ \nabla_X(fY_1{+}Y_2){=}f\nabla_X Y_1{+}X(f)Y_1{+}\nabla_X Y_2$$

instead of the Levi-Civita connection ∇^g of g, and has applications in the theory of relativity, computer science, etc.

The important cases of (M, g, ∇) are:

• **Riemann-Cartan manifolds** with metric connections ($\nabla g = 0$), relate torsion to spin of matter,

• **statistical manifolds**, with torsionless ∇ and symmetric ∇g.

• **totally skew-symmetric torsion** $T(X,Y,Z) := g(T(X,Y),Z)$.

• **non-symmetric Riemannian geometry** $G = g + F$ with skew-symmetric F, and ∇ satisfying the Einstein's metricity condition: $(\nabla_X G)(Y,Z) = G(T(Y,X),Z)$.

Weak metric structures are suitable for studying $(M, G = g{+}F)$ with an arbitrary skew-symmetric tensor $F(X,Y) = g(X, fY)$.

Given complementary orthogonal distributions $\mathcal{D}, \mathcal{D}^\perp \subset TM$.

$$\overline{\mathrm{S}}_{\rm mix} := \frac{1}{2}\sum\nolimits_{a,i} \varepsilon_a\,\varepsilon_i \left\{\langle\overline{R}(E_a,\mathcal{E}_i)E_a,\,\mathcal{E}_i\rangle + \langle\overline{R}(\mathcal{E}_i,E_a)\mathcal{E}_i,\,E_a\rangle\right\},$$

$$\mathrm{S}_{\,\rm ex} := g(H,H) - g(h,h).$$

Then

$$\overline{S}_{\rm mix} = S_{\rm mix} + Q, \tag{1}$$

$$\mathrm{S}_{\rm mix} = \mathrm{S}_{\,\rm ex} + \mathrm{S}^\perp_{\rm ex} + \langle T, T\rangle + \langle T^\perp, T^\perp\rangle \underline{+\,\mathrm{div}(H+\tilde{H})}, \tag{2}$$

where, using the **difference (contorsion) tensor** $K = \nabla - \nabla^g$,

$$Q = \frac{1}{2}\sum\nolimits_{i,a} \varepsilon_i\varepsilon_a\big\{\langle\big((\nabla_i K)_a - (\nabla_a K)_i + [K_i, K_a]\big)E_a, \mathcal{E}_i\rangle$$
$$+\langle\big((\nabla_a K)_i - (\nabla_i K)_a + [K_a, K_i]\big)\mathcal{E}_i, E_a\rangle\big\}.$$

Lemma 1 (Non-essential singularities)

Let X be a smooth vector field on $M \setminus \Sigma$, where Σ be a closed Riemannian submanifold of (M, g) of codimension $k \geq 2$, and $(k-1)(q-1) \geq 1$ and $\int_{M\setminus\Sigma} \|X\|^q \,\mathrm{d}\,\mathrm{vol}_g < \infty$. Then

$$\int_{M\setminus\Sigma} (\operatorname{div} X) \,\mathrm{d}\,\mathrm{vol}_g = 0.$$

Let $\langle B, C\rangle_{|V}$ be the inner product of tensors along the subbundle $V = (\mathcal{D} \times \mathcal{D}^\perp) \oplus (\mathcal{D}^\perp \times \mathcal{D})$. Define (1,2)-tensors K^* and $\widehat{K}$,

$$\langle K^*_X Y, Z\rangle = \langle K_X Z, Y\rangle, \quad \widehat{K}_X Y = K_Y X, \quad X, Y, Z \in \mathfrak{X}_M.$$

Define the "mean curvature type" vector fields related to K by

$$H_K := \sum\nolimits_a \varepsilon_a K_a E_a, \quad \tilde{H}_K := \sum\nolimits_i \varepsilon_i K_i \mathcal{E}_i,$$

where $\{E_a, \mathcal{E}_i\}$ is an adapted orthonormal frame. For projections of these vectors we use notations $H^\perp_K = (H_K)^\perp$, etc.

$$\underline{\operatorname{div}\big((H^\top_K - H^\top_{K^*})^\perp + (H^\perp_K - H^\perp_{K^*})^\top\big)} = 2\,(\overline{\rm S}_{\,\rm mix} - {\rm S}_{\,\rm mix})$$
$$-\ \langle H^\top_K,\, H^\perp_{K^*}\rangle - \langle H^\perp_K,\, H^\top_{K^*}\rangle - \langle H^\top_K - H^\perp_K + H^\perp_{K^*} - H^\top_{K^*},\, H^\perp - H\rangle$$
$$-\ \langle K - K^* + \widehat{K} - \widehat{K^*},\, A^\perp - \tilde{T}^\sharp + A - T^\sharp\rangle + \langle K^*,\, \widehat{K}\rangle_{\,|\,V}.$$

Here, $g(A_\xi X, Y) = g(h(X,Y),\xi)$, $g(T^\sharp_\xi X, Y) = g(T(X,Y),\xi)$.

Theorem 2

Let $(M, g, \overline{\nabla})$ be a closed metric-affine space, $\mathcal{D}$ a distribution defined on the complement to the "set of singularities" Σ of codimension k, and $(k-1)(q-1)\ge 1$ and $\|\xi\|_g \in L^q(M,g)$, where $\xi = H + H^\perp + \frac12(H^\top_K - H^\top_{K^*})^\perp + \frac12(H^\perp_K - H^\perp_{K^*})^\top$. Then

$$\int_M \{\overline{\rm S}_{\rm mix} - \|T\|^2 - \|T^\perp\|^2 + \|h\|^2 + \|h^\perp\|^2 - |H^\perp|^2 - |H|^2$$
$$-\tfrac12\,\big[\,\langle H^\top_K, H^\perp_{K^*}\rangle + \langle H^\perp_K, H^\top_{K^*}\rangle + \langle H^\top_K - H^\perp_K + H^\perp_{K^*} - H^\top_{K^*}, H^\perp - H\rangle\,\big]$$
$$-\tfrac12\langle K - K^* + \widehat{K} - \widehat{K^*},\, A^\perp - T^{\perp\sharp} + A - T^\sharp\rangle + \tfrac12\langle K^*,\, \widehat{K}\rangle_{|\,V}\}\,{\rm d}\,{\rm vol}_g = 0.$$

Notice that the condition

$$2H^{\perp} = (H^{\top}_{K^*} - H^{\top}_{K})^{\perp} \qquad (3)$$

for $K = 0$ reduces to $H^{\perp} = 0$.

Theorem 3 (IF on a compact leaf)

Let a distribution $\mathcal{D}$ on a metric-affine space $(M, g, \overline{\nabla})$ has a compact leaf (M', g') with condition (3) on its neighborhood. Then the following integral formula along the leaf holds:

$$\int_{M'} \big\{\overline{\mathrm{S}}_{\,\mathrm{mix}} - \|T^{\perp}\|^2 + \|h^{\perp}\|^2 + \|h\|^2 + \tfrac{1}{2}\,\big[\langle H^{\top}_{K} - H^{\top}_{K^*}, H\rangle + \langle H^{\perp}_{K} - H^{\perp}_{K^*}, H^{\perp}\rangle - \langle H^{\top}_{K}, H^{\perp}_{K^*}\rangle - \langle H^{\perp}_{K}, H^{\top}_{K^*}\rangle + \langle K^*, \widehat{K}\rangle_{\,|\,V} - \langle K - K^* + \widehat{K} - \widehat{K^*}, A^{\perp} - T^{\perp\sharp} + A\rangle\big]\big\}\,\mathrm{d}\,\mathrm{vol}_{g'} = 0.$$

For statistical manifolds we have

$$\overline{\mathrm{S}}_{\,\mathrm{mix}} - \mathrm{S}_{\,\mathrm{mix}} - \langle H_K,\, H_K^\perp\rangle + \tfrac{1}{2}\,\langle K,\, K\rangle_{\,|\,V} = 0.$$

Corollary 4

Let $(M, g, \overline{\nabla})$ be a closed **statistical manifold** *and $\mathcal{D}^\perp$ a distribution on the complement to a "singularity set" Σ. If $\|H^\perp + H\|_g \in \mathrm{L}^2(M, g)$, then the following* **IF** *is true:*

$$\int_M \big\{\,\overline{\mathrm{S}}_{\,\mathrm{mix}} - \langle T, T\rangle - \langle T^\perp, T^\perp\rangle + \langle h, h\rangle + \langle h^\perp, h^\perp\rangle - \langle H, H\rangle - \langle H^\perp, H^\perp\rangle - \langle H_K,\, H_K^\perp\rangle + \tfrac{1}{2}\,\langle K,\, K\rangle_{\,|\,V}\big\}\,\mathrm{d}\,\mathrm{vol}_g = 0.$$

Applying S.T.Yau version of Stokes' theorem on a complete open pseudo-Riemannian manifold (M, g) yields the following.

Lemma 5

Let (M, g) *be a* **complete open pseudo-Riemannian manifold** *with a vector field* ξ *such that* $\operatorname{div}\xi \geq 0$. *If* $\|\xi\|_g \in L^1(M, g)$ *then*

$$\operatorname{div}\xi \equiv 0\,.$$

The following conditions simplify the presentation of results:

$$K_X Y = 0 = K_Y X, \quad K^*_X Y = 0 = K^*_Y X \quad (X \in \mathcal{D},\ Y \in \mathcal{D}^\perp), \tag{4}$$
$$g(H^\top_K - H^\top_{K^*},\ H^\perp) = 0. \tag{5}$$

Theorem 6

Let $\mathcal{D}$ and $\mathcal{D}^\perp$ be complementary orthogonal integrable distributions with (3), (4) and (5) on a **complete open metric-affine manifold** (M, g, ∇).
Suppose that the leaves (M', g') of $\mathcal{D}$ obey condition

$$\|\xi_{\,|M'}\|_{g'} \in L^1(M', g'), \quad \textit{where} \ \ \xi = H^\perp + \frac{1}{2}\,(H^\perp_K - H^\perp_{K^*})^\top.$$

If $\bar{\mathrm{S}}_{\,\rm mix} \ge 0$ then M splits.

Theorem 7

Let (M, g, ∇) *be a* **complete open (or closed) metric-affine manifold** *endowed with complementary orthogonal umbilical distributions* $\mathcal{D}$ *and* $\mathcal{D}^\perp$ *defined on the complement to the "set of singularities"* Σ. *If conditions* (4),

$$H^\top_K = 0 = H^\perp_K, \quad H^\top_{K^*} = 0 = H^\perp_{K^*} \tag{6}$$

and

$$\|\xi\|_g \in L^1(M, g), \quad \textit{where} \;\; \xi = H^\perp + H^\top,$$

are satisfies and $\bar{\mathrm{S}}_{\,\rm mix} \le 0$ *then* M *splits.*

Definition 8

A **doubly-twisted product** $B \times_{(v,u)} F$ of metric-affine spaces (B, g_B, K_B) and (F, g_F, K_F) is a manifold $M = B \times F$ with metric $g = g^\top + g^\perp$ and contorsion tensor $K = K^\top + K^\perp$, where

$$\begin{aligned} g^\top(X,Y) &= v^2 g_B(X^\top, Y^\top), \quad g^\perp(X,Y) = u^2 g_F(X^\perp, Y^\perp), \\ (K^\top)_X Y &= u^2 (K_B)_{X^\top} Y^\top, \quad (K^\perp)_X Y = v^2 (K_F)_{X^\perp} Y^\perp, \end{aligned}$$

and the warping functions $u, v \in C^\infty(M)$ are positive.

Let $\mathcal{D}$ be tangent to *fibers* $\{x\} \times F$ and $\mathcal{D}^\perp$ tangent to *leaves* $B \times \{y\}$. The leaves and the fibers are umbilical w.r.t. ∇ and ∇^g. Conditions (4) are obviously satisfied for $B \times_{(v,u)} F$.

Umbilical distributions appear on double-twisted products.

Corollary 9

Let $B \times_{(v,u)} F$ be a double-twisted product, and (3), (5) *and*

$$\|\xi_{\,|M'}\|_{g'} \in L^1(M', g'), \quad \textit{where} \quad \xi = H^\perp + \frac{1}{2}\,(H^\perp_K - H^\perp_{K^*})^\top,$$

are satisfied on its fibres, where (F, g_F) is a **complete open (or closed) leaf**. *If $\overline{\mathrm{S}}_{\,\rm mix} \le 0$ then M is the product.*

Corollary 10

Let $M = B \times_{(v,u)} F$ be a **complete open (or closed)** *double-twisted product, with conditions* (5), (6) *and*

$$\|\xi\|_g \in L^1(M, g), \quad \textit{where} \quad \xi = H^\perp + H^\top.$$

If $\overline{\mathrm{S}}_{\,\rm mix} \le 0$ then M is the direct product.

In his attempt to construct a Unified Field Theory, (Nonsymmetric Gravitational Theory – **NGT**), A. Einstein considered a differentiable manifold $(M, G = g + F)$ equipped with a linear connection ∇ with torsion satisfying the **metricity condition**

$$(\nabla_X G)(Y, Z) = G(T(Y, X), Z) \qquad (X, Y, Z \in \mathfrak{X}_M). \tag{7}$$

He associated with gravity the symmetric part g of (0,2)-tensor G, and with electromagnetism the skew-symmetric part F.

Example. The Einstein connection with totally skew-symmetric torsion on a manifold $(M, G = g + F)$ is unique, its torsion (0,3)-tensor $T(X, Y, Z) := g(T(X, Y), Z)$ is given by $T = -\frac{1}{3}dF$, and the difference tensor $K := \nabla - \nabla^g$ is given by

$$K_X Y = \tfrac{1}{2}\{T(f X, Y) - T(X, f Y) + T(X, Y)\},$$

where $F(X, Y) = g(X, fY)$. We cannot directly apply statistical structures to NGT: if the cubic form $K(X, Y, Z) := g(K_X Y, Z)$ is symmetric, then $T = K = 0$ is true; hence, $\nabla = \nabla^g$.

The f^2-T-condition

We explicitly present the Einstein connection of an NGT space with a non-degenerate F (i.e., a weak almost Hermitian manifold) satisfying a natural and powerful structural assumption,

$$T(f^2X, Y) = T(X, f^2Y) = f^2\,T(X, Y), \qquad (8)$$

called the f^2-T-**condition**, or, equivalently,

$$T(f^2X, Y, Z) = T(X, f^2Y, Z) = T(X, Y, f^2Z).$$

This assumption disappears for the classical case, $f^2 = -I$.

The $(\lambda_1,\ldots,\lambda_k)$-weighed products of nearly Kähler manifolds serve as new models for NGT.

Theorem 11

Let (M,f,Q,g) be a weak almost Hermitian manifold with a fundamental 2-form F, considered as a non-symmetric Riemannian manifold $(M, G=g+F)$. Suppose that an Einstein's connection ∇ on M with totally skew-symmetric torsion satisfies the f^2-T-condition (8). Then the following properties are true.
(i) If $Q=\lambda\, I$ with $\lambda\in C^\infty(M)$, then $\lambda=const>0$ and $(\lambda^{-1/2}f,g)$ is a nearly Kähler structure.
(ii) If $Q\ne\lambda\, I$ for $\lambda\in C^\infty(M)$, then there exist $k>1$ mutually orthogonal even-dimensional distributions $\mathcal{D}_i\subset TM$ $(1\le i\le k)$ such that $\bigoplus_i\mathcal{D}_i=TM$ and $\mathcal{D}_i$ are the eigen-distributions of Q with constant eigenvalues $\lambda_i: 0<\lambda_1<\ldots<\lambda_k$; moreover, each $\mathcal{D}_i$ defines a ∇^g-totally geodesic foliation and (M,f,Q,g) is locally the $(\lambda_1,\ldots,\lambda_k)$-weighed product of nearly Kähler manifolds.

The following theorem is similar to Theorem 11.

Theorem 12

Let (M, f, Q, ξ, η, g) be a weak a.c.m. manifold, considered as a non-symmetric Riemannian manifold $(M, G = g + F)$, and ∇ be an Einstein's connection with totally skew-symmetric torsion satisfying the f^2-T-condition (8). Then the following properties are true.
(i) If $Q|_{\mathcal{D}} = \lambda\, I_{\mathcal{D}}$, then $\lambda = const > 0$ and $(M, \lambda^{-1/2} f, \xi, \eta, g)$ splits as the product of $\mathbb{R}$ and a nearly Kähler manifold.
(ii) If $Q|_{\mathcal{D}} \neq \lambda\, I_{\mathcal{D}}$ where $\lambda \in C^\infty(M)$, then there exist $k > 1$ mutually orthogonal even-dimensional distributions $\mathcal{D}_i \subset \mathcal{D}$ such that $\bigoplus_{i=1}^k \mathcal{D}_i = \mathcal{D}$ and $\mathcal{D}_i$ are eigen-distributions of Q with constant eigenvalues $0 < \lambda_1 < \ldots < \lambda_k$; moreover, the distributions $\mathcal{D}_i$ are involutive and define ∇^g-totally geodesic foliations and (M, f, Q, ξ, η, g) is locally a $(1, \lambda_1, \ldots, \lambda_k)$-weighed product of $\mathbb{R}$ and k nearly Kähler manifolds.

Separating symmetric and skew-symmetric parts of (7), we express the covariant derivatives ∇g and ∇F in terms of the torsion:

$$2\,(\nabla_X\, g)(Y,Z) = T(Z,X,\ Y+fY) - T(X,Y,\ Z+fZ),$$
$$2\,(\nabla_Z\, F)(X,Y) = -T(Z,X,\ Y+fY) - T(Y,Z,\ X+fX).$$

We also have two fundamental relations:

$$\begin{aligned} dF(X,Y,Z) &= -\,T(X,Y,Z) - T(Y,Z,X) - T(Z,X,Y),\\ 2(\nabla^g_X F)(Y,Z) &= -\,T(Z,X,Y) - T(X,Y,Z) - T(fZ,X,fY)\\ &\quad -\,T(X,fY,fZ) + T(Y,fZ,fX) + T(fY,Z,fX). \end{aligned}$$

Lemma 13

For an Einstein connection ∇, these conditions are equivalent:
(i) $(\nabla_X F)(Y,Z) = -(\nabla_Y F)(X,Z)$ *or* $g((\nabla_X f)Z,Y) = -g((\nabla_Y f)Z,X)$,
(ii) $K(X,Y,Z) = -K(X,Z,Y)$,
(iii) $\nabla g = 0$.

The property $K_X Y = -K_Y X$ of the difference tensor reduces to

$$T(Y,Z,fX) + T(X,Z,fY) = 0, \tag{9}$$

and characterizes **special Einstein connections**, i.e., the symmetric part of an Einstein connection ∇ coincides with ∇^g.

Theorem 14 (see Prvanovic (1995))

For an Einstein connection of almost Hermitian manifold (M^{2n}, J, g), we have

$$\begin{aligned} 2\,T(Y,Z,X) = 2\,(\nabla^g_{JX}F)(JY,Z) - (\nabla^g_{JY}F)(JZ,X) - (\nabla^g_{JZ}F)(X,JY) \\ - (\nabla^g_{Y}F)(Z,X) - (\nabla^g_{Z}F)(X,Y); \end{aligned} \tag{10}$$

in particular, the torsion of a special Einstein connection ∇ is

$$2\,T(X,Y,Z) = (\nabla^g_{X}F)(Y,Z) - (\nabla^g_{JZ}F)(JX,Y) - (\nabla^g_{JY}F)(JX,Z).$$

The operator $P = I - f^2$ is fundamental in a pseudo-Riemannian weak almost Hermitian manifold.

Example. For a nonsymmetric pseudo-Riemannian space $(M, G = g + F)$, the tensor $P = I - f^2$ can be degenerate. For example, consider the Lorentzian metric $g = (-, +, \ldots, +)$ on $\mathbb{R}^{2n}$ with an orthonormal basis $(e_1, \ldots, e_{2n})$, that is, $g(e_i, e_j) = \epsilon_i\,\delta_{ij}$ and $\epsilon_1 = -1,\ \epsilon_2 = \ldots = \epsilon_{2n} = 1$. Let a skew-symmetric (1,1)-tensor f have a block form with two parts: $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and J_{2n-2}. Then f^2 has block form with two parts: $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $-I_{2n-2}$. Therefore, $\mathrm{rank}(P) = 2n - 2$, and $Pe_1 = Pe_2 = 0$.

In an almost Hermitian manifold with $g > 0$ one has $f^2 = -I$, hence $P = 2I$ is non-degenerate.

Define a "small" tensor: $\widetilde{Q} := -f^2 - I$.

Theorem 15

Let (M^{2n}, f, g) be a weak almost Hermitian manifold, considered as a nonsymmetric pseudo-Riemannian manifold $(M, G = g + F)$, where $F(X,Y) = g(X, fY)$. Suppose that an Einstein connection ∇ on M satisfies the f^2-T-condition (8), and the (1,1)-tensor $P := I - f^2$ is non-degenerate. Then the torsion T of ∇ is given by

$$
\begin{aligned}
&2\,T(Y,Z,(I+\tfrac{1}{2}\,\widetilde{Q})^2 f^4 X) = (\nabla^g_X F)((f+f^3)Y, fZ) + (\nabla^g_Y F)(f^2 Z, X)\\
&+(\nabla^g_Z F)(f^2 X, Y) - (\nabla^g_{fX} F)(f^3 Y, Z) - (\nabla^g_{fX} F)(f^2 Y, fZ)\\
&+(\nabla^g_{fY} F)(f^3 Z, X) + (\nabla^g_{fZ} F)(f^2 X, fY) + (\nabla^g F)(P^{-1}(2\,\widetilde{Q}+\widetilde{Q}^2) f^2 X, fY, Z)\\
&-(\nabla^g F)(P^{-1}(2\,\widetilde{Q}+\widetilde{Q}^2) f^2 X, Y, fZ) - (\nabla^g F)(\widetilde{Q} f^2 X, Y, Z)\\
&-\tfrac{1}{2} dF((3\,\widetilde{Q}+2\,\widetilde{Q}^2)X, fY, fZ) + \tfrac{1}{2} dF((3\,\widetilde{Q}+\widetilde{Q}^2) f^2 X, Y, Z). \qquad (11)
\end{aligned}
$$

Example

If an Einstein connection ∇ satisfies the f^2-T-condition (8), then $T(X,Y)=0$ for $X\in TM_i$, $Y\in TM_j$ $(i\neq j)$. Hence, the torsion tensor splits into the direct sum $T=T|_{TM_1}\oplus\ldots\oplus T|_{TM_m}$. Substituting $\widetilde{Q}|_{TM_j}=(\lambda_j-1)\,I_j$ into (11), we uniquely determine the torsion component $T\,|_{TM_j}$ by

$$\tfrac12(\lambda_j+1)^2\lambda_j\,T\,|_{TM_j}(Y,Z,X)=-(\nabla^{g_j}_Y F_j)(Z,X)-(\nabla^{g_j}_Z F_j)(X,Y)$$
$$+\,\lambda_j\big\{2\,(\nabla^{g_j}_{J_jX}F_j)(Y,J_jZ)-(\nabla^{g_j}_{J_jY}F_j)(J_jZ,X)-(\nabla^{g_j}_{J_jZ}F_j)(X,J_jY)\big\}$$
$$+(\lambda_j-1)\big\{2(\nabla^{g_j}_X F_j)(Y,Z)-(\lambda_j+\tfrac12)dF_j(X,J_jY,J_jZ)-\tfrac12(\lambda_j+2)dF_j(X,Y,Z)\big\}.$$

The weighted product of almost Hermitian manifolds produces a family of Einstein connections parametrized by the constants λ_j, measuring the deviation from the almost Hermitian manifolds $(\lambda_j=1)$. If each (M_j,J_j,g_j) is a Kähler manifold, then $\nabla^{g_j}F_j=0$ and $dF_j=0$. In this case, all torsion components vanish, and the Einstein connection reduces to the Levi-Civita connection.
The formula above for $\lambda_j=1$ gives the solution (10).

Definition 16

Let ∇ be a linear connection with torsion T on a weak a.c.m. manifold $(M^{2n+1}, f, Q, \xi, \eta, g)$, hence $Q = -f^2 - \eta \otimes \xi$.
We introduce a natural and powerful structural assumption:

$$T(QX, Y, Z) = T(X, QY, Z) = T(X, Y, QZ), \tag{12}$$

called the **Q-T-condition**, that disappears when $Q = I$.

If the Q-T-condition is true, then

$$dF(QX, Y, Z) = dF(X, QY, Z) = dF(X, Y, QZ),$$
$$(\nabla^g_{QX} F)(Y, Z) = (\nabla^g_X F)(QY, Z) = (\nabla^g_X F)(Y, QZ).$$

Theorem 17

Let (f, Q, ξ, η, g) *be a weak a.c.m. structure on an NGT space* $(M^{2n+1}, G = g + F, \nabla)$ *with* $F(X, Y) = g(X, fY)$*. If the* Q-T*-condition* (12) *holds, then* $\nabla^g_\xi\,\xi = 0$*, and we have*

$$T(\xi, Y, \xi) = T(Y, \xi, \xi) = 0, \tag{13}$$

$$T(\xi, Y, X) = dF((I + f)^{-1}Y, \xi, X) - 2\,(\nabla^g_X F)((I + f)^{-1}Y, \xi), \tag{14}$$

$$\begin{aligned} T(X, Y, \xi) = &-dF(X, Y, \xi) + dF((I+f)^{-1}Y, \xi, X) - dF((I+f)^{-1}X, \xi, Y) \\ &-2(\nabla^g_X F)((I + f)^{-1}Y, \xi) + 2(\nabla^g_Y F)((I + f)^{-1}X, \xi). \end{aligned} \tag{15}$$

If $P := I - f^2$ *has rank* $2n + 1$*, then for* $X, Y, Z \in \ker\eta$ *we have*

$$\begin{aligned} 2\,T(Y, Z, (I + \tfrac{1}{2}\,\widetilde{Q})^2 f^4 X) &= (\nabla^g_X F)((f + f^3)Y, fZ) + (\nabla^g_Y F)(f^2 Z, X) \\ &+ (\nabla^g_Z F)(f^2 X, Y) - (\nabla^g_{fX} F)(f^3 Y, Z) - (\nabla^g_{fX} F)(f^2 Y, fZ) \\ &+ (\nabla^g_{fY} F)(f^3 Z, X) + (\nabla^g_{fZ} F)(f^2 X, fY) \\ &+ (\nabla^g F)(P^{-1}(2\widetilde{Q} + \widetilde{Q}^2) f^2 X, fY, Z) - (\nabla^g F)(P^{-1}(2\widetilde{Q} + \widetilde{Q}^2) f^2 X, Y, fZ) \\ &-(\nabla^g F)(\widetilde{Q} f^2 X, Y, Z) - \tfrac{1}{2} dF((3\widetilde{Q} + 2\widetilde{Q}^2)X, fY, fZ) + \tfrac{1}{2} dF((3\widetilde{Q} + \widetilde{Q}^2), Y, Z). \end{aligned}$$

Example

Any weak contact metric manifold (for example, weak K-contact and Sasakian manifolds) satisfies $\nabla^g_\xi\,\xi=0$, and by Theorem 17, admits an Einstein connection. Another important weak metric structure (f,Q,ξ,η,g) admitting Einstein connection appears on weak KM, given by $(\nabla^g_X\,f)Y=g(fX,Y)\,\xi-\eta(Y)\,fX$. These manifolds are locally warped products of weak Kähler manifolds and a real line (in the ξ-direction), and $\nabla^g_X\,\xi=X-\eta(X)\,\xi$ holds, hence $\nabla^g_\xi\,\xi=0$.

Theorem 18

Let ∇ be an <u>Einstein connection</u> of an a.c.m. manifold $(M^{2n+1}, f, \xi, \eta, g)$ considered as a nonsymmetric manifold $(M, G = g + F)$, where $F(X, Y) = g(X, fY)$. Then $\nabla^g_\xi\, \xi = 0$ and (13)–(15) *are true, and for $X, Y, Z \in \ker\eta$ we get*

$$2\,T(Y, Z, X) = 2(\nabla^g_{fX} F)(fY, Z) - (\nabla^g_{fY} F)(fZ, X) - (\nabla^g_{fZ} F)(X, fY) \\ - (\nabla^g_{Y} F)(Z, X) - (\nabla^g_{Z} F)(X, Y).$$

An Einstein connection on a.c.m. manifold is <u>special</u>, see (9)*, iff*

$$(\nabla^g_{X} F)(Y, Z) + (\nabla^g_{Y} F)(X, Z) = (\nabla^g_{fX} F)(fY, Z) + (\nabla^g_{fY} F)(fX, Z)$$

holds for all $X, Y, Z \in \ker\eta$; in this case, we have

$$2\,T(X, Y, Z) = (\nabla^g_{fZ} F)(fX, Y) + (\nabla^g_{fY} F)(fX, Z) - (\nabla^g_{X} F)(Y, Z).$$

A **singular distribution** $\mathcal{D}$ on a manifold M assigns to each point $x \in M$ a linear subspace $\mathcal{D}_x$ of the tangent space T_xM in such a way that for any $v \in \mathcal{D}_x$ there exists a smooth vector field V defined in a neighborhood U of x and such that $V(x) = v$ and $V(y) \in \mathcal{D}_y$ for all y of U. A priori, the dimension $\dim \mathcal{D}_x$ depends on $x \in M$. If $\dim \mathcal{D}_x = \mathrm{const}$ then we obtain a regular distribution.

Singular foliations are defined as families of maximal integral submanifolds (leaves) of integrable singular distributions.
Singular foliations appear in the theory of Riemannian foliations.

The tools of Geometric Analysis (the Divergence Theorem, Bochner method) can be extended for singular distributions.

Frobenius Theorem still holds if one removes the constant-rank assumption, but the involutivity assumption does not suffice.
A sufficient condition, in the case when E is spanned by a family $\{Y_1, \ldots, Y_p\}$ of vector fields, is the **finite-type condition**

$$[Y_i, Y_j] = \sum\nolimits_k c_{ij}^k Y_k,$$

where the coefficients $\{c_{ij}^k\}$ are **locally bounded**.

Example 19

The above condition is not a property of the distribution, but it concerns the spanning family. Let $x = (x_1, x_2) \in \mathbb{R}^2$ and $E = \mathrm{span}(Y_1, Y_2)$, where $Y_1 = e^{-1/|x|^2}\partial_1$ and $Y_2 = |x|^2\partial_2$. Then

$$[Y_1, Y_2] = 2x_2/|x|^2\, Y_1 + 2x_1/|x|^2 e^{-1/|x|^2}\, Y_2$$

with unbounded (near the origin) function $x \mapsto 2x_2/|x|^2$.

$\mathcal{D} = f(TM)$, where $f \in \mathrm{End}(TM)$, is a **singular distribution**.

Example 20

(i) Let $f \in \mathrm{End}(TM)$ on (M, g) be of **constant rank** and

$$f^2 = f, \quad f^* = f \quad \text{(adjoint endomorphism to } f\text{)},$$

and we have an **almost-product structure** on (M, g).
$f(TM)$ and $(I - f)(TM)$ are complementary $\perp$ distributions.
If $\lambda \in C^\infty(M)$ with $\lambda^{-1}(0) \neq \emptyset$, then $(\lambda f)(TM)$ is singular.

(ii) Let $(M, G = g + F)$ be a nonsymmetric Riemannian manifold with $F(X, Y) = g(X, fY)$, Then $\mathcal{D} = f(TM)$ is a singular distribution.

f-connection for tensors

Given $f \in \mathrm{End}(TM)$, define the map

$$\nabla^f_X\, Y := \nabla_{fX}\, Y,$$

which is **not a linear connection** on TM. Set $\nabla^f_X\, f = (fX)f$ for $f \in C^\infty(M)$. For an (s,k)-tensor S with $s = 0,1$, put

$$(\nabla^f S)(Y, X_1, \ldots, X_k) = \nabla^f_Y(S(X_1, \ldots, X_k)) \\ -\sum\nolimits_{i=1}^{k} S(X_1, \ldots, \nabla^f_Y X_i, \ldots, X_k).$$

This makes sense for all $s, k \geq 0$, using the product rule with respect to tensors. If $\nabla^f S = 0$ then S is called f-**parallel**.

The modified divergence

Definition 21

The f-**divergence** of a $(1, k)$-tensor S is a $(0, k)$-tensor on M,

$$\operatorname{div}_f S = \mathrm{trace}(Y \to \nabla_{fY} S),$$

where $f \in \mathrm{End}(TM)$. For a vector field X we get a function

$$\operatorname{div}_f X = \mathrm{trace}(Y \to \nabla_{fY} X).$$

The generalized Divergence Theorem

Proposition 1

Given $f \in \mathrm{End}(TM)$, *the condition*

$$\mathrm{div}\, f = 0 \tag{16}$$

is equivalent to the following:

$$\mathrm{div}_f\, X = \mathrm{div}(f\, X), \quad X \in \mathcal{X}_M.$$

Theorem 22 (Generalized Divergence Theorem)

If (16) *holds on a compact* (M, g), *then for any* $X \in \mathcal{X}_M$,

$$\int_M (\mathrm{div}_f\, X)\, d\,\mathrm{vol} = \int_{\partial M} \langle X,\, f\,\nu\rangle\, d\omega.$$

Define the f-**connection Laplacian** on tensors by $\nabla^{*f}\nabla^{f}$, where

$$(\nabla^{*f} S)(X_2,\ldots,X_k) = -\sum\nolimits_i (\nabla^f_{e_i} S)(e_i, X_2,\ldots,X_k).$$

Theorem 23

Let the condition (16) *be true:* $\operatorname{div} f = 0$. *Then*

- ∇^{*f} *is L^2-adjoint to* ∇^f *on compactly supported tensors.*
- *for a closed* (M,g), *if* $\langle\nabla^{*f}\nabla^f S,\ S\rangle_{L^2} \le 0$, *then S is f-parallel.*

Proof. For any compactly supported (s,t)-tensor S_1 and $(s,t+1)$-tensor S_2, define a 1-form ω by $\omega(Y) = \langle\iota_Y S_2,\ S_1\rangle_{L^2}$. To simplify calculations assume that $s=t=1$, then at $x\in M$:

$$\operatorname{div}_f \omega^\sharp = -\nabla^{*f}\omega = \langle S_2, \nabla^f S_1\rangle_{L^2} - \langle\nabla^{*f} S_2, S_1\rangle_{L^2}.$$

Put $S_2{=}\nabla^f S_1$ and use the Generalized Divergence Theorem 22. $\square$

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