Problems of Extrinsic Geometry of Foliations

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Two general points of view on variational problems.

- Given a Riemannian manifold, they look for **optimally placed objects** (submanifolds, foliations) minimizing geometric quantities.
- Many canonical Riemannian metrics are **critical points** of suitable Riemannian functionals on fixed objects.

We characterize critical points of

- **Einstein-Hilbert type actions** (for varying metric),
- **Godbillon-Vey type numbers** (for varying distributions),

in certain distinguished classes of almost-product structures.

Part 1. Einstein-Hilbert type action

The **Einstein-Hilbert action** is the most famous geometrically meaningful functional: $J : g \mapsto \int_M \{\frac{1}{2\mathfrak{a}}(\mathrm{S} - 2\Lambda) + \mathcal{L}\}\,\mathrm{d}\,\mathrm{Vol}_g$ with a cosmological constant Λ, a Lagrangian $\mathcal{L}$ describing the matter contents, and the constant $\mathfrak{a} > 0$. Critical points of J are solutions of $\mathrm{Ric} - \frac{1}{2}\,\mathrm{S}\cdot g + \Lambda\, g = \mathfrak{a}\cdot\Theta$ (the **Einstein equation**), where $\Theta_{\mu\nu} = -2\,\partial\mathcal{L}/\partial g^{\mu\nu} + g_{\mu\nu}\mathcal{L}$ is the energy-momentum tensor.

The **mixed Einstein-Hilbert action**,

$$J_{\mathcal{D}} : g \mapsto \int_M \Big\{\frac{1}{2\mathfrak{a}}\,(\mathrm{S}_{\,\mathcal{D}_1,\ldots,\mathcal{D}_k} - 2\,\Lambda) + \mathcal{L}\Big\}\,\mathrm{d}\,\mathrm{Vol}_g, \qquad (1)$$

is an analog of the Einstein-Hilbert action, where the scalar curvature on (M, g) is replaced by $\mathrm{S}_{\,\mathcal{D}_1,\ldots,\mathcal{D}_k}$ on $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$. The geometrical part of (1) is the **total mixed scalar curvature**,

$$J^{g}_{\mathcal{D}} : g \mapsto \int_M \mathrm{S}_{\,\mathcal{D}_1,\ldots,\mathcal{D}_k}\,\mathrm{d}\,\mathrm{Vol}_g\ .$$

A family of metrics $g_t \in \mathrm{Riem}(M, \mathcal{D}_1, \ldots \mathcal{D}_k)$, $|t| < \varepsilon$, is called an **adapted variation** if $\mathcal{D}_i$ and $\mathcal{D}_j$ are g_t-orthogonal for all $i \neq j$ and all time t. An adapted variation g_t is called a $\mathcal{D}_j$-**variation** if the metric g_t changes along $\mathcal{D}_j$ only.

Fig.: Salvador Dali "adapted variation".

Using **adapted** metrics preserving the volume of Ω, we obtain the **Euler-Lagrange equation** for (1) in the form similar to the Einstein equation:

$$\mathcal{R}ic_{\mathcal{D}} - (1/2)\,\mathcal{S}_{\mathcal{D}} \cdot g + \Lambda\, g = \mathfrak{a} \cdot \Theta\,, \tag{2}$$

where the Ricci-type tensor $\mathcal{R}ic_{\mathcal{D}}$ is defined by its restrictions on k subbundles $\mathcal{D}_j$ of TM, and $\mathcal{S}_{\mathcal{D}}$ is its trace.

For two complementary distributions $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ on M we have $TM \times TM = (\mathcal{D} \times \mathcal{D}) \oplus \mathrm{V} \oplus (\widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}})$, where $\mathrm{V} = (\mathcal{D} \times \widetilde{\mathcal{D}}) \oplus (\widetilde{\mathcal{D}} \times \mathcal{D})$.

$\{g_t \in \mathrm{Riem}(M) : |t| < \varepsilon\}$ are variations of $g_0 = g$ such that the induced infinitesimal variations, presented by a (0,2)-tensor $\dot{g}_t \equiv \partial_t g_t$, are supported in a relatively compact domain Ω in M. If $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ are fixed and their orthogonality is preserved by g_t, then we have **adapted variation** of metric. Adapted variations preserving metric on $\mathcal{D}$ are called $g^\top$-**variations**.

A tensor $\dot{g} = \partial_t\, g_t$ is orthogonally decomposed $\dot{g} = \dot{g}^\perp + \dot{g}^{\mathrm{V}} + \dot{g}^\top$, with (0,2)-tensors

$$\dot{g}^\top(X,Y) := \dot{g}(X^\top, Y^\top), \quad \dot{g}^\perp(X,Y) := \dot{g}(X^\perp, Y^\perp),$$
$$\dot{g}^{\mathrm{V}}(X,Y) := \tfrac{1}{2}\{\dot{g}(X^\top, Y^\perp) + \dot{g}(X^\perp, Y^\top)\}.$$

General variational formulas (for example, used in Ricci Flow):

$$(\mathrm{d}\,\mathrm{Vol}_g)^{\boldsymbol{\cdot}} = \frac{1}{2}\,(\mathrm{Tr}_g\,\dot g)\,\mathrm{d}\,\mathrm{Vol}_g\,,$$

$$2\langle(\dot\nabla_X Y), Z\rangle = (\nabla^t_X\,\dot g)(Y,Z) + (\nabla^t_Y\,\dot g)(X,Z) - (\nabla^t_Z\,\dot g)(X,Y).$$

Lemma 1 (Evolution of an adapted orthonormal frame)

Let a local $(\widetilde{\mathcal{D}}, \mathcal{D})$-adapted and orthonormal for $t=0$ frame $\{E_a(t), \mathcal{E}_i(t)\}$ be evolved by $g_t \in \mathrm{Riem}(M)$, $|t| < \varepsilon$, according to

$$\dot E_a = -(1/2)\,\dot g^\sharp_t(E_a)^\top, \quad \dot{\mathcal{E}}_i = -\frac{1}{2}\,\dot g^\sharp_t(\mathcal{E}_i)^\perp - \dot g^\sharp_t(\mathcal{E}_i)^\top. \qquad (3)$$

Then, for all t, $\{E_a(t), \mathcal{E}_i(t)\}$ is a g_t-orthonormal frame adapted to $\widetilde{\mathcal{D}}$, i.e., $\{E_a(t)\}$ belong to $\widetilde{\mathcal{D}}$.

Definition 2

Define Casorati type tensors

$$\mathcal{A}=\sum\nolimits_i \varepsilon_i\, A_i^2,\quad \mathcal{T}:=\sum\nolimits_i \varepsilon_i\,(T_i^\sharp)^2,\quad \mathcal{K}=\sum\nolimits_i \varepsilon_i\,[T_i^\sharp, A_i],$$

and a (0,2)-tensor

$$\operatorname{div}\, h=\sum\nolimits_\nu \varepsilon_\nu\, g(\nabla_\nu\, h,\ e_\nu).$$

For any $(0,2)$-tensors P, Q and S, define a tensor $\Upsilon_{P,Q}$ by

$$\langle\Upsilon_{P,Q}, S\rangle=\sum\nolimits_{\lambda,\nu}\varepsilon_\lambda\varepsilon_\nu[S(P_{e_\lambda,e_\nu}, Q_{e_\lambda,e_\nu)}+S(Q_{e_\lambda,e_\nu}, P_{e_\lambda,e_\nu})].$$

Adapted variations for $M(g,\mathcal{D},\widetilde{\mathcal{D}})$

Recall the formula $\int_M \big(\mathrm{S}_{\mathcal{D},\widetilde{\mathcal{D}}} - Q_{\mathcal{D},g}\big)\, d\,\mathrm{Vol}_g = 0$, where

$$Q_{\mathcal{D},g} = \langle\widetilde{H},\widetilde{H}\rangle + \langle H,H\rangle - \langle h,h\rangle - \langle\widetilde{h},\widetilde{h}\rangle + \langle T,T\rangle + \langle\widetilde{T},\widetilde{T}\rangle.$$

To get Euler-Lagrange equations (2) explicitly, we need adapted variations (i.e., $\mathcal{D}$- and $\widetilde{\mathcal{D}}$- variations) of 6 terms of $Q(\mathcal{D},g)$.

Proposition 1

For a $\underline{\mathcal{D}\text{-variation}}$ g_t *of a metric* $g\in\mathrm{Riem}(M,\mathcal{D},\widetilde{\mathcal{D}})$, *we obtain*

$$\partial_t\,\langle\widetilde{h},\widetilde{h}\rangle = -\langle(1/2)\Upsilon_{\widetilde{h},\widetilde{h}},\ \dot{g}\rangle,$$

$$\partial_t\,\langle h,\ h\rangle = \langle\,\mathrm{div}\,h + \mathcal{K}^\flat,\ \dot{g}\rangle - \underline{\mathrm{div}\langle h,\dot{g}\rangle},$$

$$\partial_t\,\langle\widetilde{H},\widetilde{H}\rangle = -\langle\,\widetilde{H}\otimes\widetilde{H},\ \dot{g}\rangle,$$

$$\partial_t\,\langle H,H\rangle = \langle\,(\mathrm{div}\,H)\,g,\ \dot{g}\rangle - \underline{\mathrm{div}((\mathrm{Tr}_{\mathcal{D}}\,\dot{g}^{\,\sharp})H)},$$

$$\partial_t\,\langle\widetilde{T},\widetilde{T}\rangle = \langle\,(1/2)\Upsilon_{\widetilde{T},\widetilde{T}},\ \dot{g}\rangle,$$

$$\partial_t\,\langle T,\ T\rangle = \langle 2\,\mathcal{T}^\flat,\ \dot{g}\rangle.$$

Define symmetric $(0,2)$-tensors α,θ,δ by

$$2\,\alpha(X,Y) = A_{X^\perp}(Y^\top) + A_{Y^\perp}(X^\top),$$
$$2\,\theta(X,Y) = T^\sharp_{X^\perp}(Y^\top) + T^\sharp_{Y^\perp}(X^\top),$$
$$2\,\delta_Z(X,Y) = \langle\nabla_{X^\top}Z,\ Y^\perp\rangle + \langle\nabla_{Y^\top}Z,\ X^\perp\rangle.$$

Proposition 2 (For any variation g_t:)

$$\partial_t\,\langle\tilde h,\tilde h\rangle = \langle\operatorname{div}\tilde h - 4\Upsilon_{\tilde\alpha,\theta} + \widetilde{\mathcal{K}}^\flat - (1/2)\,\Upsilon_{\tilde h,\tilde h},\ \dot g\rangle - \operatorname{div}\langle\tilde h,\dot g\rangle,$$

$$\partial_t\,g(\tilde H,\tilde H) = \langle(\operatorname{div}\tilde H)\,g^\perp + 4\langle\theta,\tilde H\rangle - \tilde H^\flat\otimes\tilde H^\flat,\dot g\rangle - \operatorname{div}((\operatorname{Tr}_{\mathcal{D}}\dot g^\sharp)\tilde H),$$

$$\partial_t\,\langle h,h\rangle = \langle\operatorname{div}h + \mathcal{K}^\flat - 2\,(\operatorname{div}\alpha)_{\,|\,\mathcal{V}} - 2\Upsilon_{\alpha,\tilde\alpha+\tilde\theta}$$
$$\qquad - (1/2)\Upsilon_{h,h},\ \dot g\rangle + 2\operatorname{div}\langle\alpha,\dot g\rangle - \operatorname{div}\langle h,\dot g\rangle,$$

$$\partial_t\,g(H,H) = \langle 2\langle\tilde\theta-\tilde\alpha,\ H\rangle + 2\,\mathrm{Sym}(H^\flat\otimes\tilde H^\flat) - 2\,\delta_H + (\operatorname{div}H)\,g^\top$$
$$\qquad - H^\flat\otimes H^\flat,\ \dot g\rangle + 2\operatorname{div}((\dot g^\sharp H)^\top) - \operatorname{div}((\operatorname{Tr}_{\widetilde{\mathcal{D}}}\dot g^\sharp)H),$$

$$\partial_t\,\langle\tilde T,\tilde T\rangle = \langle 2\,\tilde{\mathcal{T}}^\flat + 2\Upsilon_{\tilde\theta,\theta-\alpha} - 2\,(\operatorname{div}\tilde\theta)_{|\,\mathcal{V}} + (1/2)\Upsilon_{\tilde T,\tilde T},\ \dot g\rangle + 2\operatorname{div}\langle\tilde\theta,\ \dot g\rangle,$$

$$\partial_t\,\langle T,T\rangle = \langle(1/2)\Upsilon_{T,T} + 2\,\mathcal{T}^\flat,\ \dot g\rangle.$$

The **mixed Ricci curvature** of $g \in \mathrm{Riem}(M, \mathcal{D}, \widetilde{\mathcal{D}})$ is the symmetric $(0,2)$-tensor $\mathcal{R}ic_{\mathcal{D}} = \mathcal{R}ic_{\mathcal{D}}^{\perp} + \mathcal{R}ic_{\mathcal{D}}^{\mathrm{V}} + \mathcal{R}ic_{\mathcal{D}}^{\top}$ with the trace $\mathrm{S}_{\mathcal{D}} = \mathrm{Tr}_g\, \mathcal{R}ic_{\mathcal{D}}$ and the components

$$
\begin{aligned}
\mathcal{R}ic_{\mathcal{D}}^{\perp} &= \operatorname{div} \tilde{h} + \widetilde{\mathcal{K}}^{\flat} - 2\, \tilde{\mathcal{T}}^{\flat} + H^{\flat} \otimes H^{\flat} - \frac{1}{2}\Upsilon_{h,h} - \frac{1}{2}\Upsilon_{T,T} + \mu^{\perp}\, g^{\perp},\\
\frac{1}{2}\,\mathcal{R}ic_{\mathcal{D}}^{\mathrm{V}} &= \delta_H + (\operatorname{div}(\tilde{\theta} - \alpha))_{\,|\widetilde{\mathcal{D}}\times\mathcal{D}} + (\operatorname{div}(\tilde{\theta} - \alpha))_{\,|\mathcal{D}\times\widetilde{\mathcal{D}}} - 2\langle\theta, \tilde{H}\rangle\\
&\quad - \langle\tilde{\theta} - \tilde{\alpha}, H\rangle - 2\Upsilon_{\tilde{\alpha},\theta} - \Upsilon_{\alpha,\tilde{\alpha}} - \Upsilon_{\tilde{\theta},\theta} - \mathrm{Sym}(H^{\flat} \otimes \tilde{H}^{\flat}),\\
\mathcal{R}ic_{\mathcal{D}}^{\top} &= \operatorname{div} h + \mathcal{K}^{\flat} - 2\, \mathcal{T}^{\flat} + \tilde{H}^{\flat} \otimes \tilde{H}^{\flat} - \frac{1}{2}\Upsilon_{\tilde{h},\tilde{h}} - \frac{1}{2}\Upsilon_{\tilde{T},\tilde{T}} + \mu^{\top}\, g^{\top}.
\end{aligned}
$$

Here: $\mu^{\perp} = -\frac{n-1}{p+n-2}\operatorname{div}(\tilde{H} - H)$ and $\mu^{\top} = \frac{p-1}{p+n-2}\operatorname{div}(\tilde{H} - H)$.

Remark. To get $\mathcal{R}ic_{\mathcal{D}}$ of $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ with $k > 2$, we apply

$$
\int_M \big(\, 2\,\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - \sum\nolimits_i Q_{\mathcal{D}_i,g}\,\big)\,\mathrm{d}\,\mathrm{Vol}_g = 0, \quad 2\,\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} = \sum\nolimits_{i=1}^{k} \mathrm{S}_{\mathcal{D}_i,\widetilde{\mathcal{D}}_i}.
$$

(M^4, g) with a timelike vector field N is called a **space-time**.

Example 3 (A Lorentzian manifold (M^4, g))

Globally hyperbolic space-times (M^4, g) come equipped with a codimension-one foliation, since $\exists$ time function $\mathbb{T}: M \to \mathbb{R}$, $M = \mathbb{R} \times \hat{S}$, where for each t, $\{t\} \times \hat{S}$ is a Cauchy's surface.

For $\widetilde{\mathcal{D}}$ spanned by a unit vector N, set $\tau_i = \operatorname{Tr} A_N^i$. Then

$$\begin{aligned}
\mathcal{R}ic_{\mathcal{D}\,|\,\mathcal{D}\times\mathcal{D}} &= \nabla_N h_{sc} - \tau_1 h_{sc} - \varepsilon_N(2\,(T_N^\sharp)^2 + [T_N^\sharp, A_N])^\flat,\\
\mathcal{R}ic_{\mathcal{D}}(\,\cdot\,, N)_{\,|\,\mathcal{D}} &= \operatorname{div}^\perp T_N^\sharp + 2\,(T_N^\sharp H)^\flat,\\
\mathcal{R}ic_{\mathcal{D}}(N, N) &= \varepsilon_N(N(\tau_1) - \tau_2) - \|\,T\|^2,
\end{aligned}$$

which does not explicitly include the curvature tensor, with the trace $\mathrm{S}_{\mathcal{D}} = \varepsilon_N \mathcal{R}ic_{N,N} + \operatorname{div}(\varepsilon_N \tau_1 N - H)$.

Problem: Study (homogeneous) mixed E.-H. spaces: $\mathcal{R}ic_{\mathcal{D}} = \lambda\, g$.

Variable connection

The E.-L. equations for (1) with varying $K = \nabla - \nabla^g$ (which are analogous to the spin-connection equation in the Einstein-Cartan gravity) are divided into 8 sets due to decomposition $TM = \widetilde{\mathcal{D}} \oplus \mathcal{D}$.

Theorem 4

The Euler-Lagrange equation for (1) with fixed g and all variations of K is the algebraic system with **spin tensor** $s^c_{\mu\nu} = 2\,\partial\mathcal{L}/\partial K^c_{\mu\nu}$:

$$\begin{aligned}
\langle \tilde H_{K^*} - \tilde H, Z\rangle\langle X, Y\rangle + \langle \tilde H_K + \tilde H, Y\rangle\langle X, Z\rangle &= -(\mathfrak{a}/2)\,\langle s(X,Y), Z\rangle,\\
\langle H_{K^*} - H, W\rangle\langle U, V\rangle + \langle H_K + H, V\rangle\langle U, W\rangle &= (\mathfrak{a}/2)\,\langle s(U,V), W\rangle,\\
\langle \tilde H_{K^*} + H,\ U\rangle\langle X, Y\rangle - \langle (A_U - T^\sharp_U + K_U)X,\ Y\rangle &= (\mathfrak{a}/2)\,\langle s(X,Y), U\rangle,\\
\langle H_{K^*} + \tilde H, X\rangle\langle U, V\rangle - \langle (\tilde A_X - \tilde T^\sharp_X + K_X)U,\ V\rangle &= (\mathfrak{a}/2)\,\langle s(U,V), X\rangle,\\
\langle \tilde H_K - H, U\rangle\langle X, Y\rangle + \langle (A_U + T^\sharp_U - K_U)Y,\ X\rangle &= (\mathfrak{a}/2)\,\langle s(X,U), Y\rangle,\\
\langle H_K - \tilde H, X\rangle\langle U, V\rangle + \langle (\tilde A_X + \tilde T^\sharp_X - K_X)V,\ U\rangle &= (\mathfrak{a}/2)\,\langle s(U,X), V\rangle,\\
2\,\langle \tilde T^\sharp_X U,\ V\rangle + \langle K_U V + K^*_V U,\ X\rangle &= (\mathfrak{a}/2)\,\langle s(X,U), V\rangle,\\
2\,\langle T^\sharp_U X, Y\rangle + \langle K_X Y + K^*_Y X,\ U\rangle &= (\mathfrak{a}/2)\,\langle s(U,X), Y\rangle,
\end{aligned}$$

for all $X, Y, Z \in \widetilde{\mathcal{D}}$ *and* $U, V, W \in \mathcal{D}$.

Variable connection for a space-time

Remark 1

If K is critical for (1) with $\mathcal{L}=0$, then $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ are umbilical.

Corollary 5

For a space-time (M^{p+1}, g) endowed with $\widetilde{\mathcal{D}}$ spanned by a timelike unit vector field N, the system in Theorem 4 reduces to

$$\langle \tilde{H}_{K^*+K}, N\rangle = -(\mathfrak{a}/2)\,\langle s(N,N), N\rangle,$$
$$\langle H_{K^*} - H, W\rangle\langle U, V\rangle + \langle H_K + H, V\rangle\langle U, W\rangle = -(\mathfrak{a}/2)\,\langle s(U,V), W\rangle,$$
$$\langle \tilde{H}_{K^*}, U\rangle - \langle K_U\, N, N\rangle = -(\mathfrak{a}/2)\,\langle s(N,N), U\rangle,$$
$$(\langle H_{K^*}, N\rangle + \tilde{\tau}_1)\langle U, V\rangle - \langle(\tilde{A}_N - \tilde{T}^\sharp_N + K_N)U,\ V\rangle = -(\mathfrak{a}/2)\,\langle s(U,V), N\rangle,$$
$$\langle \tilde{H}_K,\ U\rangle - \langle K_U\, N, N\rangle = -(\mathfrak{a}/2)\,\langle s(N,U), N\rangle,$$
$$(\langle H_K, N\rangle - \tilde{\tau}_1)\langle U, V\rangle + \langle(\tilde{A}_N + \tilde{T}^\sharp_N - K_N)V,\ U\rangle = -(\mathfrak{a}/2)\,\langle s(U,N), V\rangle,$$
$$\langle 2\,\tilde{T}_N(U,\ V) + K_U V + K^*_V U, N\rangle = (\mathfrak{a}/2)\,\langle s(N,U), V\rangle,$$
$$\langle K_N\, N + K^*_N\, N,\ U\rangle = (\mathfrak{a}/2)\,\langle s(U,N), N\rangle \quad (\text{where } U, V, W \in \mathcal{D}).$$

$\mathcal{F}$ – transversely oriented codimension q foliation of a manifold M.
$\exists$ locally decomposable q-form ω on M: $\ker\omega$ is tangent to $\mathcal{F}$;
$\exists$ 1-form η: $\underline{d\omega = \omega\wedge\eta}$. The $(2q+1)$-form $\eta\wedge(d\eta)^q$ is closed.

The **Godbillon-Vey (GV) class** $\mathrm{gv}(\mathcal{F})$ of a foliation $\mathcal{F}$ is the de Rham cohomology class $[\eta\wedge(d\eta)^q]\in H^{2q+1}(M,\mathbb{R})$, see

★ `C. Godbillon, J. Vey, Un invariant des feuilletages de codim. 1, C. R. Acad. Sci. Paris Sér. A-B, 273 (1971), A92-A93.`

> is independent of the choice of forms ω and η.
>
> is invariant under *diffeomorphism* and foliated *concordance*.

> If $\dim M = 2q+1$, then we get a number: $\mathrm{gv}(\mathcal{F}) = \int_M \eta\wedge(d\eta)^q$.

Question: *The geometry and dynamics of a codimension q foliation $\mathcal{F}$ with* $\mathrm{gv}(\mathcal{F})\neq 0$?

$\mathrm{gv}(\mathcal{F})$ of a foliated $(2q+1)$-dimensional manifold

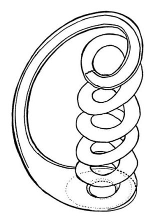
$q=1$: Codimension one foliations with $\mathrm{gv}(\mathcal{F}) \neq 0$ have nonzero **geometric entropy**.

$q=1$: $\mathrm{gv}(\mathcal{F}) \neq 0$ if $\exists$ a **resilient leaf** of exponential growth.

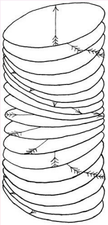
$q=1$: $\mathrm{gv}(\mathcal{F})$ measures the "**helical wobble**" of $\mathcal{F}$.

$\exists$ foliations $\mathcal{F}_t$ of $\mathbb{S}^3$ with continuously varying $\mathrm{gv}(\mathcal{F}_t) \in \mathbb{R}$.

★ `W. Thurston, Bull. AMS, 78, N 4, (1972), 511-514.`



Alejandra Ruddoff's '**Diachrony**' sculpture, 2003, Pedro de Valdivia Bridge



The disks have a constant angle of inclination to the central axis, in a direction which rotates uniformly.

The arrows represent the direction in which the leaves are most squeezed together.

Question: Which foliations are critical for $\mathrm{gv}(\mathcal{F})$ with respect to variations **among distributions**?

TM – the **tangent bundle** of M^{2q+1}

$\mathcal{D}$ – codimension q **distribution** on TM

$\mathrm{T} = \bigwedge T^i$ – q-vector field **transversal** to $\mathcal{D}$

ω – q-form such that $\mathcal{D} = \ker\omega$ and $\iota_{\mathrm{T}}\,\omega = 1$

A Riemannian **metric** g on M is **compatible** with $(\mathcal{D}, \mathrm{T})$ if $g(T^i, T^j) = \delta^{ij}$ and $T_i \perp_g \mathcal{D}$. One may take $\omega = T_1^\flat \wedge \ldots \wedge T_q^\flat$.
$\mathrm{Riem}(M, \mathcal{D}, \mathrm{T})$ – the **space of all compatible metrics**

The Lie derivative along vector fields can be generalized to a Lie derivative of differential forms along multivector fields: for a r-vector $\mathrm{X} = X_1 \wedge \ldots \wedge X_r$ on M,

$$\mathcal{L}_{\mathrm{X}} := d\,\iota_{\mathrm{X}} - (-1)^r \iota_{\mathrm{X}}\, d,$$

where $\iota_{\mathrm{X}}\,\alpha := \alpha(X_1, \ldots, X_r, \ldots)$. This yields $d\mathcal{L}_{\mathrm{X}} = (-1)^r \mathcal{L}_{\mathrm{X}}\, d$.

Lemma 6

Let T *be a* q*-vector on* M *and* α, β *differential forms. If* $\deg\alpha + \deg\beta > \dim M + q - 1$ *then*

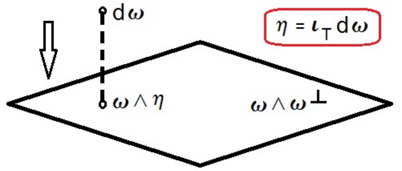
$$\iota_{\mathrm{T}}\,\alpha \wedge \beta = (-1)^{q\deg\alpha - 1} \alpha \wedge \iota_{\mathrm{T}}\,\beta.$$

Proof. By induction for q.

g – a Riemannian metric on M^{2q+1} *compatible* with $(\mathcal{D}, \mathrm{T})$

$\omega^\perp = \{\theta \in \Lambda^1(M) : \theta(T_i) = 0,\ 1 \le i \le q\}$ – subspace of 1-forms

Given ω, project $d\omega$ onto subspace $\omega \wedge \omega^\perp$ of $(q+1)$-forms. The projection $(d\omega)^\perp$ has the form $\omega \wedge \eta$, where $\eta \in \omega^\perp$ (i.e. the best approximation of $d\omega$ by the wedge-product of ω by a 1-form)



The 1-form η is unique and does not depend on g. We have

$$\eta = (-1)^{q-1}\mathcal{L}_{\mathrm{T}}\,\omega \quad (= \iota_{\mathrm{T}}\, d\omega \text{ sinse } \iota_{\mathrm{T}}\,\omega = 1).$$

The GV type invariant on M^{2q+1}

$\eta\wedge(d\eta)^q$ represents the **GV type invariant** of a pair $(\mathcal{D},\mathrm{T})$:

$$\mathrm{gv}(\mathcal{D},\mathrm{T})=\int_M \eta\wedge(d\eta)^q. \tag{4}$$

★ `V. Rovenski and P. Walczak, Variations of the Godbillon-Vey invariant for transversely parallelizable foliations, Differential Geometry & its Applications, 81 (2022) 101851`

If $\mathcal{D}$ is integrable, then $\mathrm{gv}(\mathcal{D},\mathrm{T})$ does not depend on T:

Example 7 ($q=1$: contact structure)

Let a 1-form ω be contact and T is its Reeb field, i.e. $\iota_T\, d\omega=0$. Hence $\eta=0$, and then $\mathrm{gv}(\ker\,\omega, T)=0$.

Definition 8

Let ∇ be the Levi-Civita connection of $g \in \mathrm{Riem}(M, \mathcal{D}, \mathrm{T})$.
The (non-symmetric) second fundamental form $h : \mathcal{D} \times \mathcal{D} \to \mathcal{D}^\perp$ of $\mathcal{D}$ (and similarly $h^\perp$ for the normal distribution $\mathcal{D}^\perp$) is

$$h_{X,Y} = (\nabla_X Y)^\perp \quad (X, Y \in \mathfrak{X}_\mathcal{D}).$$

A distribution $\mathcal{D}$ is called *totally geodesic* if $\mathrm{Sym}(h) = 0$, *harmonic* if $H = 0$, and *totally umbilical* if $\mathrm{Sym}(h) = g_{|\mathcal{D}} \cdot H$.

Assume that the mean curvature vector $H^\perp := \sum_i (\nabla_{T_i} T_i)^\top$ of $\mathcal{D}^\perp$ is nonzero on an open non-empty set $U \subset M$.
The **principal normal** of $\mathcal{D}^\perp$, $N = H^\perp / \|H^\perp\|$ and the **binormal distribution** $\mathcal{B} = \mathcal{D} \cap N^\perp$ are defined on U.

Proposition 3 (For a compatible metric g on M^{2q+1})

$$\eta=(-1)^q(H^\perp)^\flat\quad (\textit{for}\ q=1:\ \eta=(\nabla_T\,T)^\flat=k\,N^\flat). \tag{5}$$

Proof. Using formula for the exterior derivative of a q-form, we get for $X\in\mathcal{D}$ (for $X\in\mathcal{D}^\perp$ this is obvious) and $\mathrm{T}=T_1\wedge\ldots\wedge T_q$,

$$\begin{aligned}
d\omega(X,\mathrm{T}) &= X(\iota_{\mathrm{T}}\,\omega)+\sum\nolimits_i T_i(\omega(X,\widehat{\mathrm{T}}_i))\\
&+\sum\nolimits_{i<j}(-1)^{i+j}\omega([T_i,T_j],X,\widehat{\mathrm{T}}_{i,j})+\sum\nolimits_i(-1)^i\omega([X,T_i],\widehat{\mathrm{T}}_i)\\
&\sum\nolimits_i(-1)^i\langle[X,T_i],T_i\rangle\,\omega(T_i,\widehat{\mathrm{T}}_i)=\sum\nolimits_i(-1)^i\langle[X,T_i],T_i\rangle\,\omega(T_i,\widehat{\mathrm{T}}_i)\\
&=\sum\nolimits_i\langle\nabla_X\,T_i-\nabla_{T_i}X,T_i\rangle=-\langle X,\sum\nolimits_i\nabla_{T_i}\,T_i\rangle=-\langle X,H^\perp\rangle,
\end{aligned}$$

where the hat over a symbol denotes its omission. □

The volume form $\Omega^\pitchfork$

$\mathcal{D}^\pitchfork$ – a transverse to $\mathcal{D}$ oriented distribution on (M, g) with an orthonormal frame $\{T_i\}$ $(1 \le i \le q)$. Let $\Omega^\pitchfork$ be its **volume form**:

$$\Omega^\pitchfork(X_1, \ldots, X_q) = \det[\, g(X_i, T_j),\ i, j = 1, \ldots, q\,]$$

for all vector fields $X_1, \ldots, X_q$ on M.
Let $H^\pitchfork$ be the mean curvature vector field of $\mathcal{D}^\pitchfork$.

Proposition 4

For any vector field Z on a Riemannian manifold (M, g) one has

$$d\Omega^\pitchfork(Z, T_1, \ldots, T_q) = -g(Z, H^\pitchfork). \tag{6}$$

If $\mathcal{D}^\pitchfork$ is harmonic, i.e., $H^\pitchfork = 0$, then $\Omega^\pitchfork$ is $\mathcal{D}^\pitchfork$-*closed*, i.e., $d\Omega^\pitchfork(\,\cdot\,, X_1, \ldots, X_q) = 0$ for all $X_1, \ldots, X_q$ tangent to $\mathcal{D}^\pitchfork$.

An obstruction to tautness

Definition 9

A pair $(\mathcal{D}, \mathcal{D}^\pitchfork)$ of transversal distributions is called **geometrically taut** if there exists a Riemannian metric g on M for which $\mathcal{D}^\pitchfork$ becomes harmonic and orthogonal to $\mathcal{D}$;

Theorem 10 ($\mathrm{gv}(\omega, T)$ is an obstruction for tautness of $(\mathcal{D}, \mathcal{D}^\pitchfork)$)

Let $\mathcal{D} = \ker\omega$ be a codimension q distribution on M^{2q+1} defined by a q-form ω, and $\mathcal{D}^\pitchfork = \mathrm{span}(T_1, \ldots, T_q)$ an involutive distribution spanned by q linearly independent vector fields T_i transverse to $\mathcal{D}$. If $\mathrm{gv}(\omega, T_1 \wedge \ldots \wedge T_q) \neq 0$, then $(\mathcal{D}, \mathcal{D}^\pitchfork)$ is not geometrically taut.

Set $\mathcal{D}_t = \ker\omega_t$ and consider (ω_t, T_t) such that

$$\iota_{\mathrm{T}_t}\omega_t \equiv 1 \quad (\textit{equivalently}, \quad \omega_t(\mathrm{T}_t) \equiv 1). \tag{7}$$

Denote by $\dot{}$ the t-derivative at $t = 0$ of any quantity on M.

Lemma 11

For a variation (ω_t, T_t) obeying (7) we have

$$\dot{\mathrm{gv}} = (q+1)\int_M \dot\eta\wedge(d\eta)^q,$$

$$\ddot{\mathrm{gv}} = (q+1)\int_M \big(\ddot\eta\wedge(d\eta)^q + q\,\dot\eta\wedge d\dot\eta\wedge(d\eta)^{q-1}\big).$$

Proof. From $\eta_t = \iota_{\mathrm{T}_t} d\omega_t$ and the Taylor expansions

$$\omega_t = \omega + t\,\dot\omega + (t^2/2)\ddot\omega + O(t^3), \quad T_i(t) = T_i + t\,\dot T_i + (t^2/2)\,\ddot T_i + O(t^3),$$

we obtain

$$\begin{aligned}\eta_t\wedge(d\eta_t)^q &= \eta\wedge(d\eta)^q + \big((q+1)\dot\eta\wedge(d\eta)^q + d(\eta\wedge(d\eta)^{q-1}\wedge\dot\eta)\big)\,t\\ &+ (1/2)(q+1)\big(\ddot\eta\wedge(d\eta)^q + q\,\dot\eta\wedge d\dot\eta\wedge(d\eta)^{q-1}\\ &+ d((q/2)\ddot\eta\wedge\eta\wedge(d\eta)^{q-1} + q(q-1)\,\dot\eta\wedge d\dot\eta\wedge(d\eta)^{q-2})\big)\,t^2 + O(t^3).\end{aligned}$$

First examples of critical distributions

- If T spans a **harmonic distribution** ($H^\perp = 0$) on (M^{2q+1}, g), then $\eta = 0$, and $(\mathcal{D}, \mathrm{T})$ is **critical** for (4) and (ω_t, T_t) obeying (7).

- $q = 1$: If ω is a **contact 1-form** and T is its Reeb field, then $(\ker\omega, T)$ is **critical** for (4) for (ω_t, T_t) obeying (7).

Indeed, $\eta = 0 \;\Rightarrow\; d\eta = 0$. Hence, $\dot{\mathrm{gv}} = 0$.

Theorem 12

Let $\mathcal{D}$ be integrable *and $g \in \mathrm{Riem}(M, \mathcal{D}, \mathrm{T})$. Then $(\mathcal{D}, \mathrm{T})$ is critical for the functional* (4) *for variations (ω_t, T_t) obeying* (7) *iff*

$$\iota_{\mathrm{T}}\mathcal{L}_{\mathrm{T}}(d\eta)^q = 0 \quad (\textit{for } q = 1: \ (\mathcal{L}_T)^3\,\omega = 0).$$

Proof. $q = 1$: Using $d\omega = \omega \wedge \eta$ (integrability of $\mathcal{D}$) we obtain

$$\tfrac{1}{2}(\eta \wedge d\eta)'_{|t=0} = \dot\omega \wedge \big(\eta \wedge \iota_T d\,\eta - d\,\iota_T\,d\eta\big) + d\,\big(\tfrac{1}{2}\,\dot\eta \wedge \eta - \dot\omega \wedge \iota_T\,d\eta\big).$$

For critical $(\mathcal{D}, T)$, by this and definition of η, the 2-form vanishes:

$$\Omega := \mathcal{L}_T\,\omega \wedge (\mathcal{L}_T)^2\omega - d\,(\mathcal{L}_T)^2\omega = 0.$$

Thus $\iota_T\,\Omega = (\mathcal{L}_T)^3\,\omega = 0.$ □

One can restrict the space of distributions under consideration to any reasonable subspace, e.g. to "integrable in average" ones.

Let $\mathcal{D} = \ker \omega$ be framed, and $(\omega_1, \dots, \omega_q)$ – the frame.
Set $\omega_0 = \bigwedge_{i=1}^q d\omega_i$ and define functionals

$$J_i(\omega) := \int_M \omega_i \wedge \omega_0, \quad i = 1, \dots, q,$$

on the space $(\Lambda^1(M^{2q+1})^q$ of q-tuples of 1-forms.

Definition 13

The space $\Lambda^q_{\rm av}(M^{2q+1})$ of **q-forms integrable in average** is defined as the following extension of the space of decomposable q-forms with integrable $\ker \omega_i$: $\omega \in \Lambda^q_{\rm av}(M)$ if and only if

$$J_i(\omega) = 0, \quad i \in \{1, \dots q\}. \tag{8}$$

For $q = 1$, (8) reads as one equality $\int_M \omega \wedge d\omega = 0$.

q-forms "integrable in average"

Set $\widehat{\omega}_i = \bigwedge_{j\neq i}\omega_j$ and $\widehat{\omega}_{0i} = \bigwedge_{j\neq i} d\omega_j$ for $i = 1,\ldots,q$,
so that $\omega_0 = \widehat{\omega}_{0i}\wedge d\omega_i$ for any i.

Theorem 14

Let $\mathcal{D} = \ker\omega$ be an integrable framed distribution on M^{2q+1}. Then (ω, T) is critical for (4) for variations obeying (7) and (8) iff

$$\widehat{\omega}_{\,i}\wedge\Omega = \sum\nolimits_j \lambda_j\,(\delta_{ij}\,\omega_0 - d\omega_j\wedge\widehat{\omega}_{\,0i}),\quad i\in\{1,\ldots,q\}. \tag{9}$$

for some constants $\lambda_j\in\mathbb{R}$ and $j = 1,\ldots,q$.

Corollary 15 (Sufficient condition for infinitesimal rigidity of $\mathcal{F}$)

Let $\mathcal{F}$ be a q-dimensional foliation of M^{2q+1}. If (9) holds for any (ω,T) such that $T\mathcal{F} = \ker\omega$ and $\omega(\mathrm{T}) = 1$, then $\mathrm{gv}(\mathcal{F})$ is infinitesimally rigid, i.e., $\dot{\mathrm{gv}}(\mathcal{F}) = 0$ for any infinitesimal deformation of $\mathcal{F}$.

Corollary 16 ($q = 1$)

Let $\mathcal{D} = \ker\omega$ on M^3 be integrable and $g \in \mathrm{Riem}(M, \mathcal{D}, T)$. Then (ω, T) is critical for the functional (4) with respect to all variations obeying (7) and (8) if and only if the following holds for some $\lambda \in \mathbb{R}$:

$$(\mathcal{L}_T)^3\,\omega = \lambda\,\mathcal{L}_T\,\omega. \tag{10}$$

Moreover, (10) yields (but is not equivalent to) the equality

$$\mathcal{L}_T(\eta \wedge d\eta) = 0,$$

telling that the three-form representing $\mathrm{gv}(\omega, T)$ is invariant under the flow of T.

Example: critical twisted products

Corollary 17

Let (M^{2q+1}, g) be a twisted product $B^{q+1} \times_\phi F^q$.
(i) If $\phi = \phi_1\phi_2$ with $\phi_1 \in C^\infty(B)$ and $\phi_2 \in C^\infty(F)$ then (ω, T) is critical for (4).

(ii) For a warped product $B^{q+1} \times_\phi F^q$, (ω, T) is critical for (4).

Proof. Use the fact (by B.Y. Chen): $b \times F^q$ have parallel mean curvature iff $\phi = \phi_1\phi_2$ with $\phi_1 \in C^\infty(B)$ and $\phi_2 \in C^\infty(F)$.

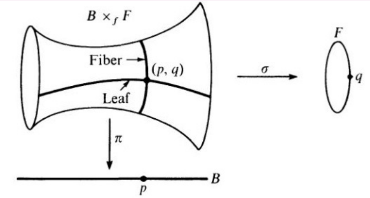


Figure: Twisted product.

Critical (among foliations) Reeb foliations are singular

(r,θ) – polar coordinates in disc $D^2: 0\le r\le r_0$, parameter t on S^1, convex $f(r)$ $(f(0)=0)$ has vertical asymptote at $r=r_0$. The foliation (the Reeb component) in $D^2\times S^1$ is given by $\omega=0$,

$$\omega(r,t)=\cos\mu(r)\,dt-\sin\mu(r)\,dr,\quad \mu=\arctan f',\ \mu'=\frac{f''}{1+(f')^2}$$

Since $\mu(0)=0$ and $\mu(r_0)=\pi/2$, then $\omega(0,t)=dt$, $\mathcal{D}(0,t)=\{\partial_r,\partial_\theta\}$ and $\omega(r_0,t)=-dr$, $\mathcal{D}(r_0,t)=\{\partial_t,\partial_\theta\}$.

Proposition 5

The Reeb foliation of a sphere S^3 produced by a function $f=f(r)$ is critical for (4) and all variations obeying (7), (8), if and only if $((\mu'\sin^2\mu)'\sin\mu)'=\lambda\,\mu'\sin\mu$, i.e., f solves the Cauchy's problem:

$$f^{(4)}=\frac{(6(f')^2-7)}{f'(1+(f')^2)}f'''f''-\frac{2(3(f')^4-9(f')^2+2)}{(1+(f')^2)^2(f')^2}(f'')^3+\lambda\frac{(1+(f')^2)}{(f')^2}f'',$$
$$f(0)=0,\quad f'(0)\ne 0,\quad f''(0)\ge 0,\quad f'''(0)\in\mathbb{R},$$

with real parameter λ.

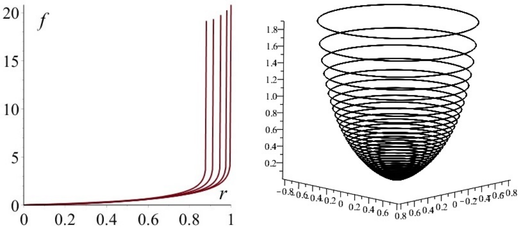
More critical Reeb foliations

Proposition 6

The Reeb foliation of a sphere S^3 produced by function $f = f(r)$ is critical for (4) iff $(\mu' \sin^2 \mu)' \sin \mu \equiv A_0 \in \mathbb{R}$, i.e., f solves ODE

$$f''' = \frac{2((f')^2 - 1)}{(1 + (f')^2)f'}\,(f'')^2 + \frac{A_0(1 + (f')^2)^{5/2}}{(f')^3},$$
$$f(0) = 0, \quad f'(0) \neq 0, \quad f''(0) \geq 0. \qquad (11)$$

Solutions $f(r)$ to (11) with $A_0 = 1$, $f''(0) = 0$, $f'(0) = i/8$ $(i = 1 \dots 5)$, producing singular Reeb foliations by rotation about f-axis.



The following symmetric bilinear form (depending on T):

$$I_{\mathrm{T}}(\alpha,\beta) = \int_M (\mathcal{L}_{\mathrm{T}}(\mathcal{L}_{\mathrm{T}} d\,\alpha \wedge (d\eta)^{q-1}) \wedge \beta$$

is the **index form** of our variational problem with integrable $\mathcal{D}$, on the space of 1-forms α, β on M.

For $q = 1$: $I_T(\alpha,\beta) = \int_M (\mathcal{L}_T)^2 d\,\alpha \wedge \beta$.

Proposition 7

Let $g \in \mathrm{Riem}(M, \mathcal{D}, \mathrm{T})$ and $\mathcal{D} = \ker\omega$ be integrable on M. For a critical pair (ω, T) for the functional (4) and all variations obeying (7), we have $\ddot{\mathrm{gv}} = I_{\mathrm{T}}(\dot\omega, \dot\omega)$.

Proof. $q = 1$: When $\mathcal{D}$ is integrable, only variations (ω_t, T) are essential. We get

$$(\eta \wedge d\eta)'' = ((d\iota_T)^2 d\,\dot\omega) \wedge \dot\omega + d(\dot\omega \wedge (\iota_T\, d)^2\dot\omega + \ddot\omega \wedge \iota_T\, d\eta).$$

From the above and Lemma 1 the claim follows. $\square$

Case 2: GV type action for a.c.m. manifolds

D. Chinea-C. Gonzalez (1990) got 12 classes of **almost contact metric (a.c.m.)** manifolds, which in dim=3 reduce to 5 classes:

C_5 of β-Kenmotsu manifolds,
C_6 of α-Sasakian manifolds,
C_9-manifolds,
C_{12}-manifolds (called generalized Sasakian space forms),
$|C| = C_5 \cap C_{12}$ of cosymplectic manifolds.

Many works on a.c.m. manifolds are devoted to normal structures, that is to the first three Chinea-Gonzalez classes.

Class $C_5 \oplus C_{12}$ consists of integrable non-normal manifolds (locally **double-twisted products** $B^2 \times_{(u,v)} I$, where B is a Kähler manifold).

Let $g \in \mathrm{Riem}(M^3, \mathcal{D}, T)$, and ∇ Levi-Civita connection of g.

$h_{X,Y} = g(\nabla_X Y,\ T)$ – **scalar 2nd fundamental form** of $\mathcal{D}$.

k – **curvature** and τ – **torsion** of T-curves.

$\{T,\ N\ (\mathrm{normal}),\ B\ (\mathrm{binormal})\}$ – **Frenet frame**.

Frenet formulas on a domain $U = \{k \neq 0\}$:

$$\nabla_T T = k\,N, \quad \nabla_T N = -kT + \tau B, \quad \nabla_T B = -\tau N.$$

$\mathcal{T}_{X,Y} = \frac{1}{2}\,g([X, Y], T)$ – the **integrability tensor** of $\mathcal{D}$.

Many a.c.m. manifolds are critical points of curvature functionals restricted to associated metrics.

A **question** arises: *can one use a GV-type functional to find optimal 3-dimensional a.c.m. manifolds, e.g., in the class* $C_5 \oplus C_{12}$?

To answer the question, we define a GV type functional for a.c.m. manifolds, presented it in Reinhart-Wood form and find its Euler-Lagrange equations for variations of metric preserving T.

★ Rovenski, V. *Godbillon-Vey type functional for almost contact manifolds*, J. of Geometry, 2024, 115:32.

An almost contact structure (φ, ω, T) on a smooth manifold M^{2n+1} consists of an endomorphism φ of TM, a 1-form $\omega \in \Lambda^1(M)$ and a vector field T satisfying

$$\varphi^2(X) = -X + \omega(X)\, T \quad (X \in TM), \quad \omega(T) = 1. \tag{12}$$

The plane field $\mathcal{D} = \ker \omega$ is called the contact distribution. By (12), we get $\omega \circ \varphi = 0$ and $\varphi\, T = 0$.

Consider a 1-form $\eta^* = \eta \circ \varphi$, i.e.,

$$\eta^*(X) := d\omega(T, \varphi X) \quad (X \in TM).$$

Define (similarly to gv) the following functional:

$$gv^*(\mathcal{D}, T) = \int_M \eta^* \wedge d\eta^*. \tag{13}$$

The 1-form η^* for a.c.m. manifolds

If $M^3(\varphi, \omega, T)$ admits a Riemannian metric $g = \langle\cdot,\cdot\rangle$ such that

$$\langle\varphi X, \varphi Y\rangle = \langle X, Y\rangle - \omega(X)\,\omega(Y), \quad \eta(X) = g(T, X),$$

then g is a compatible metric, and we get an a.c.m. manifold $M^3(\varphi, \omega, T, g)$, e.g., hypersurfaces of Hermitian manifolds.

We get

$$\eta^* = -k(\varphi N)^\flat.$$

If T is a geodesic field (i.e., $k = 0$) then $\eta^* = 0$ and $gv^* = 0$.

Let an orientable M^3 be equipped with a vector field $T \neq 0$.
For any Riemannian metric g on M such that $g(T,T)=1$
there is a unique $\omega \in \Lambda^1(M)$ (given by $\omega = g(T,\cdot)$)
and a unique $\varphi \in End(TM)$ such that $\varphi|_{\ker\omega}$ gives a right rotation
and $M^3(\varphi,\omega,T,g)$ is an a.c.m. manifold.

$\mathcal{C}(M,T)$ – all such a.c.m. structures (φ,T,ω,g) on (M,T).

From (13), we obtain

$$gv^*_T = -\int_M k^2(\tau + h_{N,B})\,\mathrm{d}V_g.$$

Let $(\varphi_t,\omega_t,T,g_t) \in \mathcal{C}(M,T)$.

Since $\dot{T}=0$ and $\dot{g}_{T,T}=0$, the symmetric $(0,2)$-tensor
$\dot{g} = (dg_t/dt)|_{t=0}$ has 5 independent components on $U=\{k\neq 0\}$:

$$\dot{g}_{T,N},\quad \dot{g}_{T,B},\quad \dot{g}_{N,N},\quad \dot{g}_{N,B},\quad \dot{g}_{B,B}.$$

General variational formulas (for example, used in Ricci Flow):

$$(\mathrm{d}\,\mathrm{Vol}_g)^{\cdot} = \tfrac{1}{2}\,(\mathrm{Tr}_g\,\dot g)\,\mathrm{d}\,\mathrm{Vol}_g\,,$$

$$2\langle(\dot\nabla_X\,Y),Z\rangle = (\nabla^t_X\,\dot g)(Y,Z) + (\nabla^t_Y\,\dot g)(X,Z) - (\nabla^t_Z\,\dot g)(X,Y).$$

Theorem 18

The Euler-Lagrange equations on U of the action gv^*_T with respect to variations of metric in $\mathcal{C}(M,T)$ are the following:

$$\begin{aligned}
&\tau + h_{N,B} = 0,\\
&T(k) - k\,h_{B,B} = 0,\\
&(h_{N,B} + h_{B,N})\,(h_{N,N} + h_{B,B}) = 0,\\
&h^2_{N,N} - h^2_{B,B} - h_{N,N}h_{B,B} - T(h_{N,N}) - \frac{1}{4}\,k^2 = 0. \qquad (14)
\end{aligned}$$

What are non-trivial ($k \neq 0$) solutions to (14)?

Proposition 8

Let an a.c.m. manifold $M^3(\varphi, \omega, T, g)$ be critical for the action gv_T^* with respect to all variations in $\mathcal{C}(M, T)$. If $\ker\omega$ *is* <u>*involutive*</u> *and* <u>*totally umbilical*</u> *with the mean curvature $H = \frac{1}{2}\operatorname{Tr}_g h \not\equiv 0$, then the distributions Span(T, N) and Span(T, B) on U are also integrable and* (14) *reduce to $\tau = 0$ and the following ODE:*

$$T(k) = k\,H, \quad T(H) = -H^2 - (1/4)\,k^2. \tag{15}$$

General solution

The E.-L. equations (15) can be seen on T-curves (parameterized by t) as a dynamical system for functions $k(t)$ and $H(t)$:

$$(d/dt)k = k\,H, \quad (d/dt)H = -H^2 - (1/4)\,k^2. \tag{16}$$

Proposition 9

The general solution of the system (16) *has the following form:*

$$t = \int (H^4 + C_1)^{-1/2}\,dH + C_2, \quad k(t) = \pm 2\sqrt{-\frac{\mathrm{d}}{\mathrm{d}t}H(t) - H^2(t)}.$$

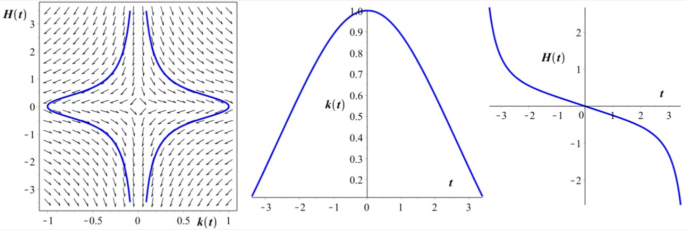


Figure: Solution $k(t), H(t)$ of (16) with $k_0 = 1$ and $H_0 = 0$ for $|t| < 3.4$

Theorem 19

There exist 3-dimensional a.c.m. manifolds of a class $C_5 \oplus C_{12}$ critical for the action gv_T^* with respect to variations in $\mathcal{C}(M, T)$. These manifolds have integrable distributions $Span(T, N)$ and $Span(T, B)$ on U, and are presented locally as double twisted products $B \times_{(u,v)} I$, where the following functions satisfy (15):

$$H = -2\,\langle\nabla(\log u), T\rangle, \quad k = -\langle\nabla(\log v), N\rangle \neq 0.$$

References: Variarions of the GV class

[1] V. Rovenski and P. Walczak, Variations of the Godbillon-Vey invariant of foliated 3-manifolds, Complex Analysis and Operator Theory, 13(6), (2019), 2917–2937.

[2] V. Rovenski and P. Walczak, A Godbillon-Vey type invariant for a 3-dimensional manifold with a plane field, Differential Geometry and its Applications, 66, (2019), 212–230.

[3] V. Rovenski and P. Walczak, Variations of the Godbillon-Vey invariant for transversely parallelizable foliations. Differential Geometry and its Applications, Vol. 81 (2022), 101851.

[4] V. Rovenski, Godbillon-Vey type functional for almost contact manifolds, Journal of Geometry, 2024, 115 : 32.