Problems of Extrinsic Geometry of Foliations

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XXVI-th International Conference on Geometry, Integrability and Quantization

June 4–11, 2026 Varna, Bulgaria

- Willmore-type functional
- Mutual curvature invariants
- Geometric inequalities for submanifolds
- Optimal immersions of almost product manifolds
- Geometric inequalities for CR-submanifolds

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1. Variational problem for submanifolds

Many authors were looking for an immersion of a differentiable manifold M^n ($n \geq 2$) into a Riemannian manifold $(\bar{M}, \bar{g})$, which is a critical point of the following functional:

$$W_{n,p} = \int_M H^p \,\mathrm{d}V, \quad p > 0, \tag{1}$$

Here, $\mathrm{d}V$ is the volume form of induced metric $g = \langle \cdot\,, \cdot \rangle$ on M, h the second fundamental form, $H = \frac{1}{n}\,\mathrm{Tr}_g\, h$ the mean curvature. Functional (1) measures how much an immersed manifold differs from a minimal ($H = 0$) submanifold.

Variational problems for (1) were first posed by T. Willmore (1965) for the action $W_{2,2}$, which belongs to conformal geometry.
The E.-L. equation for $W_{2,2}$ (Willmore energy) is an elliptic PDE

$$\Delta H + 2\,H(H^2 - K) = 0,$$

where Δ is the Laplacian, K – the gaussian curvature of $M^2 \subset \mathbb{R}^3$.
Willmore energy is used in such areas as biophysics, elastic theory, liquid crystallography, image processing and general relativity.

Willmore-type functionals for foliated submanifolds

An interesting task is the generalization of $W_{n,p}$ to manifolds with additional structures such as **distributions** and **foliations**.
The study may be used for the design of layered materials.

Let M^n equipped with an s-dimensional foliation $\mathcal{F}$ be immersed into $(\bar{M}, \bar{g})$, and $H_{\mathcal{F}} = \frac{1}{s}\operatorname{Tr}_g h|_{T\mathcal{F}}$ the mean curvature along $\mathcal{F}$.
Consider a **Willmore-type functional**:

$$W_{n,p,s} = \int_M \|H_{\mathcal{F}}\|^p \,\mathrm{d}V, \qquad (2)$$

which reduces to $W_{n,p}$ of (1) when $s = n$.

Willmore-type functionals for foliated hypersurfaces

A foliated hypersurface in $\mathbb{R}^{n+1}$, whose leaves $\{L\}$ are minimal submanifolds in $\mathbb{R}^{n+1}$ is a <u>minimizer</u> for $W_{n,p,s}$ with even p.
It is interesting to find critical hypersurfaces for the action (2) with $H_{\mathcal{F}} \neq 0$ on an open dense set U of M.

"Willmore Surface" sculpture at Durham University in memory of Thomas Willmore

Denote $\Delta_{\mathcal{F}} = \mathrm{Tr}_g\,\mathrm{Hess}^{\mathcal{F}}$ the **leafwise Laplacian** on functions, where $\mathrm{Hess}^{\mathcal{F}}$ is the **Hessian on the leaves** of $\mathcal{F}$.

Consider a one-parameter family of hypersurfaces in $\mathbb{R}^{n+1}$ $\mathrm{r}_t = \mathrm{r} + t\,u\,\mathrm{N}$ ($|t| < 1$), and $u : U \to \mathbb{R}$ is a smooth function supported on a relatively compact neighborhood U. We get a t-variation $\delta\,\mathrm{r} = u\,\mathrm{N}$. Derivatives $(\delta\,\mathrm{r})_i = u_i\,\mathrm{N} + u\,\mathrm{N}_i$. Hessian of u is $(\mathrm{Hess}\,_u)(X, Y) = X(Y(u)) - (\nabla_X\,Y)u = (u_{ij} - \Gamma^k_{ij}u_k)dx^i dx^j$. Laplacian is $\Delta u = \mathrm{Tr}_g\ \mathrm{Hess}\,_u = g^{ij}u_{ij}$, hence $\langle g, \mathrm{Hess}\,_u\rangle = \Delta\,u$. The divergence of a vector field $X = X^i\partial_i$ on M is $\mathrm{div}\,X = \nabla_i X^i$.

Lemma 1 (The following evolution equations are true:)

$$\delta\,g_{ij} = -2\,u\,h_{ij}, \tag{3}$$

$$\delta\,g^{ij} = 2\,u\,h^{ij}, \tag{4}$$

$$\delta\,h_{ij} = u_{ij} - u h^l_i\,h_{jl} \ \Leftrightarrow\ \delta\,h = \mathrm{Hess}\,_u - u\,h^2, \tag{5}$$

$$\delta\,\|h\|^2 = 2\,\langle h,\ u h^2 + \mathrm{Hess}\,_u\rangle, \tag{6}$$

$$\delta(nH) = \Delta\,u + u\,\|h\|^2, \tag{7}$$

$$\delta\,\mathrm{d}V = -n\,uH\,\mathrm{d}V. \tag{8}$$

The first variation

We calculate for a foliated hypersurface $(M,\mathcal{F})$ and a foliated neighborhood $U\subset M$ with normal coordinates $(x^1,\ldots,x^n)$ adapted to $\mathcal{F}$, i.e., $(x^1,\ldots,x^s)$ are variables along the leaves.

Lemma 2

The following evolution equations are true:

$$\delta\,(s\,H_{\mathcal F}) = \Delta_{\,\mathcal F}\,u + u\,(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2), \tag{9}$$

$$\delta\,\|h_{\mathcal F}\|^2 = 2\,\langle h_{\mathcal F},\ u\,(h^2_{\mathcal F} + h^2_{\rm mix}) + {\rm Hess}^{\,\mathcal F}_{\,u}\rangle, \tag{10}$$

$$\delta\,\|h_{\rm mix}\|^2 = u\,\langle h_{\mathcal F} + h_{\mathcal F^\perp},\, h^2_{\rm mix}\rangle + 2\,\langle h_{\rm mix}, {\rm Hess}^{\,\rm mix}_{\,u}\rangle. \tag{11}$$

For any smooth function $f: M\times\mathbb{R}\to\mathbb{R}$, we obtain

$$\begin{aligned}\delta\,(\Delta_{\,\mathcal F} f) = {}&\Delta_{\,\mathcal F}\,\delta f + 2\,u\langle h_{\mathcal F}, {\rm Hess}^{\mathcal F}_f\rangle + s\,u\,\langle\nabla^{\mathcal F} H_{\mathcal F}, \nabla^{\mathcal F} f\rangle\\ &+ 2\,h(\nabla^{\,\mathcal F} u, \nabla^{\mathcal F} f) - sH_{\mathcal F}\langle\nabla^{\mathcal F} u, \nabla^{\mathcal F} f\rangle.\end{aligned} \tag{12}$$

Let $P: TM \to T\mathcal{F}$ be the orthoprojector.

Lemma 3 (helps to find the Euler–Lagrange equations)

A foliated Riemannian manifold $(M, g, \mathcal{F})$ satisfies condition

$$(\operatorname{div} P) \circ P = 0 \tag{13}$$

iff $\mathcal{F}$ is transversally harmonic (e.g., a Riemannian foliation).

Proof. Using a local orthonormal frame on M such that $e_i \in T\mathcal{F}$ $(1 \le i \le s)$, we calculate:

$$\begin{aligned}
&(\operatorname{div} P)(PX) = \sum\nolimits_{i=1}^{n} \langle (\nabla_{e_i} P)(PX), e_i \rangle \\
&= \sum\nolimits_{i=1}^{n} \big\{ \langle \nabla_{e_i} (P^2 X), e_i \rangle - \langle P \nabla_{e_i} (PX), e_i \rangle \big\} \\
&= \sum\nolimits_{i=1}^{n} \big\{ \langle \nabla_{e_i} (PX), e_i \rangle - \langle \nabla_{e_i} (PX), P e_i \rangle \big\} = \sum\nolimits_{i>s} \langle \nabla_{e_i} (PX), e_i \rangle \\
&= -\langle X, (n-s) H^{\perp} \rangle,
\end{aligned}$$

where $(n-s)H^{\perp} = P \sum_{i>s} \nabla_{e_i} e_i$ is the mean curvature vector of $(T\mathcal{F})^{\perp}$ and $X \in \mathfrak{X}_M$. $\square$

The adjoint (of $\nabla^{\mathcal F}$) operator

The adjoint of ∇ operator: $\nabla^* B = -\sum_i (\nabla_i B)(e_i, \cdot)$ for any 2-tensor B. Then $\int_M \langle B, \nabla C\rangle \,\mathrm{d}V = \int_M \langle \nabla^* B, C\rangle \,\mathrm{d}V$.

Lemma 4

If $(M, g, \mathcal F)$ satisfies (13), then the following formulas are valid for any compactly supported functions u, f and a 2-tensor B:

$$\int_M f(\Delta_{\mathcal F}\, u)\mathrm{d}V = \int_M u(\Delta_{\mathcal F}\, f)\mathrm{d}V, \quad \int_M \langle B, \mathrm{Hess}^{\mathcal F}_u\rangle \mathrm{d}V = \int_M u(\nabla^{\mathcal F *})^2(B)\mathrm{d}V.$$

Proof. We have $\mathrm{div}_{\mathcal F}(PX) = \mathrm{div}\,(PX) - (\mathrm{div}\,P)(PX)$ and $\Delta_{\mathcal F}\, f_2 = \mathrm{div}_{\mathcal F}(\nabla^{\mathcal F} f_2)$. Using $\nabla^{\mathcal F} f = P\nabla f$ and (13), we get

$$f_1 \Delta_{\mathcal F}\, f_2 = f_1\, \mathrm{div}_{\mathcal F}(\nabla^{\mathcal F} f_2) = f_1\big\{\mathrm{div}(P\nabla f_2) - (\mathrm{div}\,P)(P\nabla f_2)\big\}$$
$$= \mathrm{div}\,(f_1\, P\nabla f_2) - \langle P\nabla f_1, P\nabla f_2\rangle = \mathrm{div}\,(f_1\, P\nabla f_2) - \langle \nabla^{\mathcal F} f_1, \nabla^{\mathcal F} f_2\rangle.$$

Thus $\int_M f_1(\Delta_{\mathcal F}\, f_2)\mathrm{d}V = -\int_M \langle \nabla^{\mathcal F} f_1, \nabla^{\mathcal F} f_2\rangle \mathrm{d}V$ by the divergence theorem. By this and $f_1 \leftrightarrows f_2$, the 1st formula is true. $\square$

Theorem 5

If (13) is true, then the Euler-Lagrange equations for $W_{n,p,s}$ are:

$$\Delta_{\mathcal{F}}(H_{\mathcal{F}}^{p-1}) + H_{\mathcal{F}}^{p-1}\big(\|h_{\mathcal{F}}\|^2 - \|h_{\rm mix}\|^2 - \frac{n\,s}{p}\,H\,H_{\mathcal{F}}\big) = 0.$$

- For $W_{n,2,1}$ ($s=1,\ p=2$) we get the leaf-wise elliptic PDE:

$$\Delta_{\mathcal{F}}\,\kappa + (\kappa^2 - \|h_{\rm mix}\|^2 - \frac{n}{2}\,H\kappa)\,\kappa = 0.$$

- For $W_{2,2,0}$ ($s=0,\ p=n=2$) we get $\Delta H + 2H(H^2-K) = 0$.

Corollary 6

At a critical hypersurface, satisfying (13), *the 2nd variation of the functional* $W_{n,p,s}=\int_M H_{\mathcal F}^p\,\mathrm{d}V$ *is*

$$
\begin{aligned}
&\delta^2 W_{n,p,s} = -\int_M \frac{np}{s}\Big\{H_{\mathcal F}^{p-1}\Delta_{\mathcal F}\,u - u\,\Delta_{\mathcal F}\,(H_{\mathcal F}^{p-1})\Big\}uH\,\mathrm{d}V\\
&+\int_M \frac{p}{s}H_{\mathcal F}^{p-2}\Big\{H_{\mathcal F}\big(2\,u\langle h_{\mathcal F},\mathrm{Hess}_u^{\mathcal F}\rangle + s\,u\langle\nabla^{\mathcal F}H_{\mathcal F},\nabla^{\mathcal F}u\rangle + 2h(\nabla^{\mathcal F}u,\nabla^{\mathcal F}u)\\
&-s\,H_{\mathcal F}\|\nabla^{\mathcal F}u\|^2\big)+\frac{p-1}{s}\Delta_{\mathcal F}u\big(\Delta_{\mathcal F}u+u\,(\|h_{\mathcal F}\|^2-\|h_{\rm mix}\|^2)\big)\Big\}\mathrm{d}V\\
&+\int_M H_{\mathcal F}^{p-2}u\Big\{\frac{p}{s}\Big(\frac{p-1}{s}(\|h_{\mathcal F}\|^2-\|h_{\rm mix}\|^2)-n\,HH_{\mathcal F}\Big)\big(\Delta_{\mathcal F}\,u+u\,(\|h_{\mathcal F}\|^2\\
&-\|h_{\rm mix}\|^2)\big)-H_{\mathcal F}^2(\Delta\,u+u\|h\|^2)+\frac{p}{s}\,H_{\mathcal F}\big(2\,\langle h_{\mathcal F},u\,(h_{\mathcal F}^2+h_{\rm mix}^2)+\mathrm{Hess}\,_u^{\mathcal F}\rangle\\
&\quad -u\,\langle h_{\mathcal F}+h_{\mathcal F^\perp},h_{\rm mix}^2\rangle-2\,\langle h_{\rm mix},\mathrm{Hess}\,_u^{\rm mix}\rangle\big)\Big\}\mathrm{d}V.
\end{aligned}
$$

Reilly-type functionals for foliated hypersurfaces

R.C. Reilly (1973) and some mathematicians studied variations of

$$WF_n = \int_M F(\sigma_1,\ldots,\sigma_n)\,\mathrm{d}V,\quad JF_n = \int_M F(\tau_1,\ldots,\tau_n)\,\mathrm{d}V, \quad (14)$$

more general functionals than (1), where $F \in C^3(\mathbb{R}^n)$. Here $\sigma_r = \sum_{i_1<\ldots<i_r} k_{i_1}\ldots k_{i_r}$ $(0 \le r \le n)$ are the elementary symmetric functions of the principal curvatures k_i; $\tau_i = k_1^i + \ldots + k_n^i = \operatorname{Tr} A^i$.

In [3], we define functionals for compactly supported variations of $(M^n, \mathcal{F})$ immersed in $\mathbb{R}^{n+1}$ (which, for $s = n$, reduces to (14)):

$$WF_{n,s} = \int_M F(\sigma_1^{\mathcal{F}},\ldots,\sigma_s^{\mathcal{F}})\,\mathrm{d}V,\ JF_{n,s} = \int_M F(\tau_1^{\mathcal{F}},\ldots,\tau_s^{\mathcal{F}})\,\mathrm{d}V. \quad (15)$$

Theorem 7

If (13) *is valid, then Euler-Lagrange equations for* (15) *are*

$$\sum\nolimits_{r=1}^{s}\big\{(\nabla^{\mathcal{F}*})^2(F'_r\cdot T_{r-1}(A_{\mathcal{F}})) + F'_r\big(\sigma_1^{\mathcal{F}}\sigma_{r-1}^{\mathcal{F}} - (r+1)\sigma_{r+1}^{\mathcal{F}}$$
$$+\,\langle T_{r-1}(A_{\mathcal{F}}), A^2_{\rm mix}\rangle\big)\big\} - n\,F\cdot H = 0,$$
$$\sum\nolimits_{i=1}^{s}\tfrac{1}{i}\big\{(\nabla^{\mathcal{F}*})^2\big(F'_i\cdot A_{\mathcal{F}}^{i-1}\big) + F'_i\big(\tau_{i+1}^{\mathcal{F}} + \langle h_{\mathcal{F}}^{i-1}, h^2_{\rm mix}\rangle\big)\big\} - n\,F\cdot H = 0.$$

A rotational hypersurface $M^n \subset \mathbb{R}^{n+1}$ is foliated by $(n-1)$-spheres (parallels) and has a warped product metric $g = d\rho^2 + \rho^2 ds^2_{n-1}$.
This is locally a graph $x_{n+1} = F(\rho)$, where $F \in C^2$ is monotonic, $\rho = \sqrt{\sum_i x_i^2}$, and (x_i) are coordinates in $\mathbb{R}^{n+1}$.
For spherical coordinates $(\phi_1, \ldots, \phi_{n-1}; \rho)$ in $\mathbb{R}^n$, the parametrization of M^n is $\mathrm{r} = \mathrm{r}(\phi_1, \ldots, \phi_{n-1}; F(\rho))$.
The principal curvatures are $k_1 = \ldots = k_{n-1} = \frac{F'}{\rho(1+(F')^2)^{1/2}}$ for parallels and $k_n = \frac{F''}{(1+(F')^2)^{3/2}}$ for profile curves.

We get $n\,H = (n-1)\,k_1 + k_n$, $\quad \|h_{\mathcal{F}}\|^2 = (n-1)k_1^2$, $\quad H_{\mathcal{F}} = k_1$,
$\|h\|^2 = (n-1)k_1^2 + k_n^2$, $\quad \langle h, h^2\rangle = (n-1)\,k_1^3 + k_n^3$,
$\langle h_{\mathcal{F}}, h^2_{\mathcal{F}}\rangle = (n-1)k_1^3$, $\quad h_{\rm mix} = h^2_{\rm mix} = 0$.

Theorem 8

A hypersurface of revolution $M : x_{n+1} = F(\rho)$ $(F'' \neq 0)$ *in* $\mathbb{R}^{n+1}$ *foliated by* $(n-1)$*-spheres-parallels* $\{L_\rho\}$ *is a critical point of the action* $W_{n,p,n-1}$ *if and only if* F *satisfies the following ODE:*

$$\rho F'' = (p-n+1)\, F'(1+(F')^2). \tag{16}$$

The solution of (16) *is given by*

$$F(\rho) = \int \frac{\sqrt{C_1 \rho^{2p-2n+2} - \rho^{4p-4n+4}}}{C_1 - \rho^{2p-2n+2}}\,\mathrm{d}\rho + C_2, \quad C_1, C_2 \in \mathbb{R}.$$

• *A critical hypersurface is* <u>*not a local minimum*</u> *of* $W_{n,p,n-1}$ *for* $p > n \geq 2$ *with respect to general variations,*

• *but it is* <u>*a local minimum*</u> *of* $W_{n,p,n-1}$ *for* $p > n \geq 2$ *with respect to variations* $u = u(\phi_1, \ldots, \phi_{n-1})$ *satisfying* $u|_{L_\rho} \perp_{L^2} \ker \Delta_{\mathcal{F}}$.

Example

Let $M^2 \subset \mathbb{R}^3$ be critical for the functional $W_{2,p,1}$ with $p \geq 2$.
Then $k_2 = (p-1)k_1 \neq 0$; hence, $H^2/K = \frac{p^2}{4\,(p-1)} = const.$
The solution of (16), $F(\rho) = \int \frac{\sqrt{C_1 \rho^{2p+2} - \rho^{4p}}}{\rho^{2p} - C_1\,\rho^2}\,\mathrm{d}\rho + C_2,\ \ C_1, C_2 \in \mathbb{R}.$

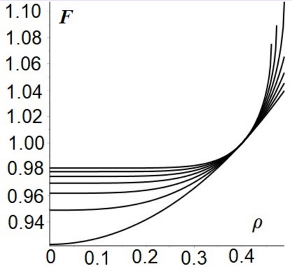

Figure: Graphs of $F(\rho)$ for $F(\frac{2}{5}) = 1$, $F'(\frac{2}{5}) = \frac{2}{5}$, $n = 2$, $p = 2, 3, \ldots, 8$.

2. Geometric inequalities for submanifolds

The embedding theorem of J.F. Nash (1954) gave hope for the use of **Extrinsic Geometry** when (M^n, g) is smoothly isometrically embedded in $\mathbb{R}^m$, but it does not indicate the relationship between m and n in light of **curvature invariants**.

The fundamental problem: **Find optimal connections between intrinsic and extrinsic invariants of a submanifold in a Riemannian manifold; in particular, in a real space form**. This led to a question on obstacles for a Riemannian manifold to admit an isometric minimal immersion in a Euclidean space.

B.-Y. Chen proved geometrical inequalities for submanifolds in terms of δ-invariants of curvature and studied "ideal immersions".

We defined and studied new curvature invariants of $(M, g, \mathcal{D})$:

- δ_m-invariants (mutual curvature) alternative to Chen's invariants.
- $\delta_{\mathcal{D}}$-invariants adapted to $\mathcal{D}$ (for $\mathcal{D} = TM$ reduce to δ-invariants).

Mutual curvature of $k \ge 2$ orthogonal subspaces

Tracing the **Gauss equation** for a submanifold (M, g) in $(\bar M, \bar g)$:

$$\bar g(\bar R_{Y,Z}U, V) - g(R_{Y,Z}U, V) = \bar g(h(Y,U), h(Z,V)) - \bar g(h(Z,U), h(Y,V)),$$

and using $h^2 \ge \frac{1}{n} H^2$ with $H = \operatorname{Tr} h$, we obtain the inequality

$$2\,S - 2\,\bar S|_M = H^2 - h^2 \le \tfrac{n-1}{n}\, H^2.$$

The case of equality – **ideal immersions**. Hence, (M, g) with $S > 0$ does't admit an isometric minimal immersion in any $\mathbb{R}^m$.

Definition 9 (see Lecture 1)

The **mutual curvature** $\mathrm{S}_{V,\tilde V}$ of two orthogonal subspaces $V, \tilde V \subset T_xM$ of a Riemannian manifold (M, g) is defined (see Lecture 1) as an averaged mixed sectional curvature. Let $V_1, \ldots, V_k$ $(k \ge 2)$ be mutually orthogonal subspaces of T_xM at a point $x \in M^n$ with $\dim V_i = n_i \ge 1$ and $\sum_i n_i \le n$. The **mutual curvature** of $k \ge 2$ orthogonal subspaces $\{V_1, \ldots, V_k\}$ is defined by $\mathrm{S}_{V_1,\ldots,V_k} = \sum_{i<j} \mathrm{S}_{V_i,V_j}$.

For integers $n,k\geq 2$, denote by $S(n,k)$ the set of unordered k-tuples $(n_1,\ldots,n_k)$ of natural numbers satisfying $\sum_i n_i\leq n$.

Definition 10

For $(n_1,\ldots,n_k)\in S(n,k)$, define the functions on M by

$$\begin{aligned}\delta_{\rm m}(n_1,\ldots,n_k)(x) &= \max \mathrm{S}_{V_1,\ldots,V_k},\\ \hat\delta_{\rm m}(n_1,\ldots,n_k)(x) &= \min \mathrm{S}_{V_1,\ldots,V_k},\end{aligned} \tag{17}$$

where $V_1,\ldots,V_k$ run over all k mutually orthogonal subspaces of T_xM with $\dim V_i=n_i$ $(i=1,\ldots,k)$.

If $c\leq K\leq C$ (the sectional curvature K) and $\sum_{i=1}^k n_i=s\leq n$, then

$$\frac{c}{2}\big(s^2-\sum\nolimits_i n_i^2\big)\leq\hat\delta_{\rm m}(n_1,\ldots n_k)\leq\delta_{\rm m}(n_1,\ldots n_k)\leq\frac{C}{2}\big(s^2-\sum\nolimits_i n_i^2\big)\,.$$

For a k-tuple ($k \ge 0$) and $x \in M$, Chen's invariants are

$$
\begin{aligned}
2\,\delta(n_1,\ldots,n_k)(x) &= S(x) - \min\{S(V_1)+\ldots+S(V_k)\},\\
2\,\hat\delta(n_1,\ldots,n_k)(x) &= S(x) - \max\{S(V_1)+\ldots+S(V_k)\}, \qquad (18)
\end{aligned}
$$

where $V_1,\ldots,V_k$ run over all k mutually orthogonal subspaces of T_xM with $\dim V_i = n_i$ ($i=1,\ldots,k$).

Proposition 1 (The following inequalities are valid:)

$\delta_{\mathrm{m}}(n_1,\ldots,n_k) \ge \delta(n_1,\ldots,n_k) - \delta(s)$ *for* $\sum_i n_i = s < n$,

$\delta_{\mathrm{m}}(n_1,\ldots,n_k) = \delta(n_1,\ldots,n_k)$ *for* $s = n$,

$\delta_{\mathrm{m}}(n_1,\ldots,n_k) \ge \delta(n_1,\ldots,n_k) - \max \mathrm{Ric}$ *for* $s = n-1$.

Similarly for $\hat\delta_{\mathrm{m}}$ and $\hat\delta$.

We define the curvature invariants δ_D and $\delta_{\mathrm{m},D}$ of $(M^n, g, \mathcal{D})$, $\dim \mathcal{D} = d < n$ similarly as (20) and (18) using subspaces $V_i \subset \mathcal{D}$.

$$
\begin{aligned}
2\,\delta_D(n_1,\ldots,n_k)(x) &= S_D(x) - \min\{S(V_1)+\ldots+S(V_k)\},\\
2\,\hat\delta_D(n_1,\ldots,n_k)(x) &= S_D(x) - \max\{S(V_1)+\ldots+S(V_k)\}, \qquad (19)\\
\delta_{\mathrm{m},D}(n_1,\ldots,n_k)(x) &= \max \mathrm{S_m}(V_1,\ldots,V_k),\\
\hat\delta_{\mathrm{m},D}(n_1,\ldots,n_k)(x) &= \min \mathrm{S_m}(V_1,\ldots,V_k), \qquad (20)
\end{aligned}
$$

Proposition 2

For $k \ge 2$ and $n_1+\ldots+n_k < d$, the following inequalities hold:

$$
\begin{aligned}
\delta_{\mathrm{m},D}(n_1,\ldots,n_k) &\ge \delta_D(n_1,\ldots,n_k) - \delta_D(n_1+\ldots+n_k)\,,\\
\hat\delta_{\mathrm{m},D}(n_1,\ldots,n_k) &\le \hat\delta_D(n_1,\ldots,n_k) - \hat\delta_D(n_1+\ldots+n_k)\,,
\end{aligned}
$$

and if $n_1+\ldots+n_k = d$, then

$$
\hat\delta_D(n_1,\ldots,n_k) = \hat\delta_{\mathrm{m},D}(n_1,\ldots,n_k) \le \delta_{\mathrm{m},D}(n_1,\ldots,n_k) = \delta_D(n_1,\ldots,n_k).
$$

Let $F:(M^n,g)\to(\bar M^m,\bar g)$ be an isometric immersion, and $h: TM\times TM\to TM^\perp$ the **second fundamental form** of F, where $TM^\perp$ is the normal bundle of the submanifold $F(M)$.

The **mean curvature vector of a subspace** $V\subset T_xM$ is given by $H_V=\sum_i h(e_i,e_i)$, where (e_i) is an orthonormal basis of V. Set

$$\mathcal{H}_x(s)=\max\{\,\|H_V\|:\ V\subset T_xM,\ \dim V=s>0\}. \tag{21}$$

Theorem 11

For $(M,g) \overset{F}{\to} (\bar M,\bar g)$ and any $n_i\in\mathbb{N}$ with $\sum_i n_i = s\le n$ we get

$$\delta_{\rm m}(n_1,\ldots,n_k) \le \bar\delta_{\rm m}(n_1,\ldots,n_k) + \frac{k-1}{2\,k}\begin{cases} \mathcal{H}(s)^2, & s<n,\\ \|H\|^2, & s=n,\end{cases} \qquad (22)$$

where $\bar\delta_{\rm m}(n_1,\ldots,n_k)$ are defined similarly for $(\bar M,\bar g)$. The equality in (22) holds at a point $x\in M$ if and only if there exist mutually orthogonal subspaces $V_1,\ldots,V_k$ of T_xM with $\sum_i n_i = s\le n$ such that F is mixed totally geodesic on $V=\bigoplus_{i=1}^k V_i$, $H_1=\ldots=H_k$, $\|H_V\| = \mathcal{H}_x(s)$ and $\bar{\rm S}_{V_1,\ldots,V_k} = \bar\delta_{\rm m}(n_1,\ldots,n_k)(x)$.

Corollary 12

A Riemannian manifold (M^n,g) with $\delta_{\rm m}(n_1,\ldots,n_k)>0$ for some $(n_1,\ldots,n_k)$ and $\sum_i n_i = n$ does not admit minimal isometric immersions into Euclidean space $\mathbb{R}^m$.

Function $\mathcal{H}_{\mathcal{D}}(s)$ is defined for subspaces of $\mathcal{D}$ similarly to $\mathcal{H}(s)$.

Theorem 13

Let $F:(M,g;\mathcal{D})\to(\bar M,\bar g)$ be an isometric immersion, and $\sum_i n_i = s\le d$, where $d=\dim\mathcal{D}$. Then,

$$\delta_{\mathrm{m},\mathcal{D}}(n_1,\ldots,n_k)\le\bar\delta_{\mathrm{m}}(n_1,\ldots,n_k)+\frac{k-1}{2\,k}\begin{cases}\mathcal{H}_{\mathcal{D}}(s)^2, & \text{if } s<d,\\ \|\,H_{\mathcal{D}}\,\|^2, & \text{if } s=d.\end{cases}$$

The equality holds at $x\in M$ if and only if there exist mutually orthogonal subspaces $V_1,\ldots,V_k$ of $\mathcal{D}_x$ with $\sum_i n_i = s$ such that the immersion is mixed totally geodesic on $V=\bigoplus_{i=1}^k V_i$, $H_1=\ldots=H_k$, $\|H_V\|=\mathcal{H}_{\mathcal{D}}(s)$ and $\bar{\mathrm{S}}_{\,V_1,\ldots,V_k}=\bar\delta_{\mathrm{m}}(n_1,\ldots,n_k)$.

Corollary 14

A Riemannian manifold $(M,g;\mathcal{D})$ with $\delta_{\mathrm{m},\mathcal{D}}(n_1,\ldots,n_k)>0$ for some $(n_1,\ldots,n_k)$ and $\sum_i n_i = d$ does not admit minimal isometric immersions into Euclidean space $\mathbb{R}^m$.

Define the $\delta_{\mathcal{D}}$-**invariants** of $(M^n, g, \mathcal{D})$ similarly to (18) as:

$$2\,\delta_{\mathcal{D}}(n_1,\ldots,n_k)(x) = S_{\,|\,\mathcal{D}}(x) - \min\{S(V_1)+\ldots+S(V_k)\}\,,$$
$$2\,\hat{\delta}_{\mathcal{D}}(n_1,\ldots,n_k)(x) = S_{\,|\,\mathcal{D}}(x) - \max\{S(V_1)+\ldots+S(V_k)\}\,,$$

where $V_1,\ldots,V_k$ run over all k mutually orthogonal subspaces of $\mathcal{D}_x$ with $\dim V_i = n_i$ $(i=1,\ldots,k)$.

Certain **extremal immersions** in $\mathbb{R}^m$ in terms of $\delta_{\mathcal{D}}$-invariants are an analogue of Chen's "**ideal immersions**".

Proposition 3

For each k-tuple $(n_1,\ldots,n_k)\in S(d,k)$, we obtain

$$\delta_{\mathcal{D}}(n_1,\ldots,n_k) \le \frac{d^2(d+k-1-\sum_i n_i)}{2(d+k-\sum_i n_i)}\|H_{\mathcal{D}}\|^2 + \frac12\big[d(d-1)-\sum_i n_i(n_i-1)\big]\max\bar K.$$

Consider the case when $\mathcal{D}$ is the sum $\mathcal{D} = \mathcal{D}_1 \oplus \ldots \oplus \mathcal{D}_k$ of $k \ge 2$ mutually orthogonal distributions of ranks $n_i > 0$.
The **mutual curvature of** $\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} = \sum_{i<j} \mathrm{S}_{\mathcal{D}_i,\mathcal{D}_j}$ is a function on M (the **mixed scalar curvature** when $\mathcal{D} = TM$).

Theorem 15

Let $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ be isometrically immersed in $(\bar M, \bar g)$, and $\mathcal{D} = \bigoplus_{i=1}^{k} \mathcal{D}_i$. Then the poinwise inequality is true:

$$\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} \le \frac{k-1}{2\,k}\,\|H_{\mathcal{D}}\|^2 + \bar\delta_{\mathrm{m}}(n_1, \ldots, n_k)\,. \tag{23}$$

The equality in (23) holds at a point $x \in M$ if and only if the immersion is mixed totally geodesic on $\mathcal{D}_x$, $H_1(x) = \ldots = H_k(x)$ and $\bar{\mathrm{S}}_{\mathcal{D}_1(x),\ldots,\mathcal{D}_k(x)} = \bar\delta_{\mathrm{m}}(n_1, \ldots, n_k)(x)$.

Corollary 16

$(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ with $\mathrm{S}_{\mathrm{m}}(\mathcal{D}_1, \ldots, \mathcal{D}_k) > 0$ and $\mathcal{D} = \bigoplus_{i=1}^{k} \mathcal{D}_i$ does not admit $\mathcal{D}$-minimal isometric immersions into $\mathbb{R}^m$.

Corollary 17

Let $\psi:(M,g;\xi_1,\ldots\xi_k)\to(\bar M,\bar g)$ *be an isometric immersion,* $\mathcal{D}=\mathrm{Span}(\xi_1,\ldots\xi_k)$. *Then we obtain the pointwise inequality*

$$S(\mathcal{D})\le\frac{k-1}{2\,k}\,\|H_{\mathcal{D}}\|^2+\max\bar S_k\,. \tag{24}$$

The equality in (24) *holds at a point* $x\in M$ *iff* ψ *is totally umbilical on* $\mathcal{D}_x$, *i.e.,* $h_{\,|\,\mathcal{D}_x}=H_{\mathcal{D}_x}\cdot g_{\,|\,\mathcal{D}_x}$, *and* $S(\mathcal{D}_x)=\max\bar S_k(x)$.

Holomorphic mutual curvature invariants

Given two J-invariant planes σ and σ' in T_xM of an almost Hermitian manifold (M,J,g), and unit vectors $X\in\sigma$ and $Y\in\sigma'$, the **holomorphic bisectional curvature** $K_{\rm h}(\sigma,\sigma')$ is defined by

$$K_{\rm h}(\sigma,\sigma') = R(X,JX,Y,JY), \tag{25}$$

and depends on σ and σ' only. For $\sigma=\sigma'$ gives the **holomorphic sectional curvature**. Set $\mathrm{S}_{\rm h}(\sigma_1,\ldots,\sigma_k)=\sum_{i<j}K_{\rm h}(\sigma_i,\sigma_j)$.

Definition 18

Let $\mathcal{D}$ be a d-dimensional complex distribution of a Riemannian manifold (M,g), i.e., there is a skew-symmetric (1,1)-tensor $J:\mathcal{D}\to\mathcal{D}$ such that $J^2X=-X$ and $g(JX,JY)=g(X,Y)$ for $X,Y\in\mathcal{D}$. **Holomorphic mutual curvature invariants** are

$$\delta^{\,k}_{\,{\rm h},\mathcal{D}}(x)=\max\mathrm{S}_{\rm h}(\sigma_1,\ldots,\sigma_k),\quad \hat\delta^{\,k}_{\,{\rm h},\mathcal{D}}(x)=\min\mathrm{S}_{\rm h}(\sigma_1,\ldots,\sigma_k)$$

where $\sigma_1,\ldots,\sigma_k$ run over all k mutually orthogonal J-invariant planes of $\mathcal{D}_x$ at a point $x\in M$.

A submanifold M^{d+l} of an almost Hermitian manifold $(\bar M, \bar J, \bar g)$ is called a **CR-submanifold** if $\mathcal{D} = \bar J(TM) \cap TM$ is a complex distribution of constant real dimension d, see [4].

Theorem 19

Let (M^{d+l}, g) be a CR-submanifold of an almost Hermitian manifold $(\bar M, \bar J, \bar g)$, and $\mathcal{D} = \bar J(TM) \cap TM$. For any natural number $k \in [2, d/2]$, we obtain

$$\delta^k_{\mathrm{h},\mathcal{D}} \le \bar\delta^{\,k}_{\mathrm{h}} + \frac{k-1}{4\,k}\left\{\begin{array}{ll} \mathcal{H}_{\mathcal{D}}(2k)^2, & \text{if } 2k < d,\\ \|\, H_{\mathcal{D}}\,\|^2, & \text{if } 2k = d,\end{array}\right. \tag{26}$$

where $\bar\delta^{\,k}_{\mathrm{h}}$ are defined for $(\bar M, \bar g)$ similarly to $\delta^k_{\mathrm{h},\mathcal{D}}$.
The equality in (26) holds at $x \in M^{d+l}$ if and only if there exist mutually orthogonal J-invariant planes $\{\sigma_1, \ldots, \sigma_k\}$ of $\mathcal{D}_x$ such that M^{d+l} is mixed totally geodesic on $V = \bigoplus_{i=1}^k \sigma_i$, $H_1 = \ldots = H_k$, $\|H_V\| = \mathcal{H}_{\mathcal{D}_x}(2k)$ and $\bar{\mathrm{S}}_{\mathrm{h}}(\sigma_1, \ldots, \sigma_k) = \bar\delta^{\,k}_{\mathrm{h}}(x)$.