Symplectic Lie Algebras, Lie–Hamilton Systems and Geometric Structures

Rutwig Campoamor-Stursberg

Instituto de Matemática Interdisciplinar
Univ. Complutense, E-28040 Madrid, Spain

Lie–Hamilton systems constitute a relevant and distinguished class of ODE and PDE systems admitting (nonlinear) superposition rules [1, 2, 3], where their compatibility with a symplectic structure allow the use of external techniques, such as mixed superposition rules or the so-called coalgebra formalism developed in the context of superintegrability [4, 5]. This method allows to find systematically and explicitly a superposition rule, based on families of constants of the motion that arise naturally from the formalism. LH systems have been classified and their explicit integrability studied in the real plane, with higher-dimensional cases not yet fully characterized. In recent years, the LH formalism has been combined with representation theory of Lie algebras and their realization by vector fields [6], leading to new hierarchies of LH systems on Riemannian and pseudo-Riemannian manifolds, where the real symplectic group has been shown to play a fundamental role [7, 8]. This generalized approach also allows combination with other related geometric structures, such as nonlinear LH systems and contact structures [9, 10], from which new applications can be derived.

The objective of the lectures is to review the current status of Lie and Lie–Hamilton systems, as well as to discuss recent extrapolations and generalizations to higher dimensions and their quantum counterparts, with several applications ins classical and quantum mechanics, superintegrable systems, as well as thermodynamics.

Lecture 1. Classical Lie systems,

- Superposition rules for ODEs.
- Vessiot–Guldberg algebras. Lie–Scheffers theorem.
- Lie algebras of vector fields.

Lecture 2. Lie–Hamilton systems.

- Time-dependent Hamiltonians. Symplectic compatibility.
- Geometric theory of LH systems.
- Classification in low dimensions.
- LH hierarchies based on $\mathfrak{sp}(2n, \mathbb{R})$.

Lecture 3. Deformation of Lie–Hamilton systems.

- The extended coalgebra formalism.
- Quantum deformations of LH systems.
- Perturbations of LH systems and constants of the motion.

Lecture 4. LH compatible structures.

- Contact Lie systems. Reductions.
- Dirac–Lie systems.
- Jacobi–Lie systems.
- Lie systems and k-symplectic structures.

Lecture 5. Current research problems

- Generalized LH systems on Lorentzian manifolds.
- Nonlinear Lie systems.
- Mixed superposition principles.

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