The Fuzzy S^7 and Beyond

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arXiv : 2205.09019 (Nucl. Phys.)

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1. Intro, Motivation

1-0. Back ground

1-1 Semisimple Lie alg is solution of IKKT.

1-2 Fuzzy S^2

2. Matrix regularization (MR) for $\mathbb{C}[x]$

3. MR for $\mathbb{C}[x]/I(C)$

$$\mathbb{C}[x]/I(C) \xrightarrow{\xi_{U}/I} U_{\mathfrak{g}}[\hbar]/I(C) \xrightarrow{\rho} \mathfrak{gl}(V^{n}) \xrightarrow{R} \mathfrak{gl}(V^{n})$$

4. Reducible representation

5. Fuzzy S^7

6. Summary

1-0. Back ground

String theory : candidate of Quantum gravity

↓ non-perturbative

- Matrix model. (IKKT matrix model)
  [Ishibashi-Kawai-Kitazawa-Tsuchiya '97)

Solution of E.O.M. is given by "Matrix".

⟶ N.C. geometry with some matrix rep.

We have to find some connection
between "Matrix geometry" & commutative geo.
〃 Poisson alg.

A method is "Matrix regularization" (Hoppe, Madore)

1-1 Semisimple Lie alg is solution of Mass-def. IKKT

▷ Matrix model (M.M.) of $\mathfrak{g}$-solution

$$\boxed{S = -\frac{1}{4} g^{ij} g^{kl} \operatorname{tr} [X_i, X_k][X_j, X_l] + \frac{\hbar^2}{2} g^{ij} \operatorname{tr} X_i X_j}$$

$\rightsquigarrow$ E.O.M. $\qquad$ X_i: Hermitian Mat.

$$\underline{[X^i, [X_i, X_j]] = -\hbar^2 X_j}$$

Solution

$\mathfrak{g}$: semi simple Lie alg

basis :

$$[X_i, X_j] = \hbar f_{ij}^{\ \ k} X_k \quad , \quad \operatorname{tr} X^i X_j = c\, \delta^i_{\ j}$$

$$g_{ij} := - f_{il}^{\ \ m} f_{jm}^{\ \ \ l} \quad : \text{Killing metric.}$$

(Non-degenerate for semisimple)

$$\boxed{S = -\frac{1}{4} g^{ij} g^{kl} \operatorname{tr}[X_i, X_k][X_j, X_l] + \frac{\hbar^2}{2} g^{ij} \operatorname{tr} X_i X_j}$$

$\cup$ a special case $\left(g \sim \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix} \right)$

IKKT matrix model

mass-deformed

$$S = S_{IKKT} = \operatorname{tr}\Big(-\frac{1}{4} [X_\mu, X_\nu]^2 + \underbrace{\hbar^2 \frac{1}{2} X^\mu X_\mu}_{\text{mass-deformed}} \Big)$$

"Mat rep of Lie alg is a solution of IKKT!"

$\downarrow$

Q. How can we find $\underbrace{\text{classical physics on smooth mfd}}^{\text{poisson geometry}}$ from the Lie alg?

$\rightsquigarrow$ Matrix regularization.

1-3

▷ Matrix regularization (Hoppe 89, Madore 92)

$N_1, N_2, N_3, \cdots$: increasing seq of positive integers

$\hbar(N)$: $|\lim_{N\to\infty} N\hbar(N)| < \infty$ converge.

(M, ω): Symplectic mfd.

$\mathcal{T}_k : C^\infty(M) \longrightarrow \{N_k \times N_k \text{ Hermitian}\} \subset Mat_{N_k}(\mathbb{C})$

s.t.
- $\lim_{k\to\infty} \|\mathcal{T}_k(f)\| < \infty$ ✓
- $\lim_{k\to\infty} \|\mathcal{T}_k(fg) - \mathcal{T}_k(f)\mathcal{T}_k(g)\| = 0$ ✓
- $\lim_{k\to\infty} \left\| \frac{1}{i\hbar(N)} [\mathcal{T}_k(f), \mathcal{T}_k(g)] - \mathcal{T}_k(\{f,g\}) \right\| = 0$ ✓

- $\lim_{k\to\infty} 2\pi\hbar(N_k) \operatorname{Tr}(\mathcal{T}_k(f)) = \int_M f\omega$

Except for B-T quantization, only a few examples are known.

Def) Quantization

A : Poisson alg. $(A, \cdot, \{,\})$

$T \subset M$: R-module in R-alg M (R is $\mathbb{C}$ or $\mathbb{C}[\hbar]$)

$q : A \to T$ linear, with $\hbar(q) \in \mathbb{C}$

for $\forall f, g \in A$,

$$[q(f), q(g)] = i\hbar\, q(\{f, g\}) + O(\hbar^{1+\varepsilon})$$

Def Weak Matrix regularization

$\{q_k : A \to \mathrm{Mat}_{N_k}(\mathbb{C})\}$ is a series of quantizations

with $\hbar \to 0$ as $k \to \infty$ ($N_k \to \infty$)

1-2. Fuzzy S^2, $\mathbb{R}^3$ (Madre, Hoppe)

~ As a Hint of such classical physics ~

Consider $su(2)$ case

$(x_1, x_2, x_3) \in \mathbb{R}^3$ with $su(2)$ Poisson str.

$$\{x_a, x_b\} = i\epsilon^{abc} x_c$$ ← Kirillov-Kostant $\mathfrak{g}(\mathfrak{g}^*) \to$ Poisson alg.

$A_{su(2)} = (\mathbb{C}[x], \cdot, \{,\})$: Poisson-Lie alg.

$f \in A_{su(2)}$: Polynomial on $\mathbb{R}^3$

$$f = f_0 + f_a x_a + \frac{1}{2} f_{ab} x_a x_b + \cdots$$

$f_{a_1 \cdots a_n}$: completely sym.

V^k : k-dim Vect. sp.

$\rho_k : A_{su(2)} \longrightarrow gl(V^k)$: linear

V^k : k-dim Vect. sp.

$\mathcal{S}_k : A_{su(2)} \longrightarrow gl(V^k)$: linear

$$x^A = x_{a_1} \cdots x_{a_m} \mapsto \mathcal{S}_k(x^A) = \begin{cases} \dfrac{\hbar^m}{m!} \displaystyle\sum_{\sigma \in Sym(m)} J_{a_{\sigma(1)}} \cdots J_{a_{\sigma(m)}} & (m < k) \\ 0 & (m \geq k) \end{cases}$$

where J_i $(i = 1 \sim 3)$ is $su(2)$ basis

$$[J_i, J_j] = i \epsilon^{ijk} J_k \quad (\text{Spin } s, \ \dim = 2s+1 = k \text{ rep.})$$

$$\boxed{\mathcal{S}_k(x_i) = \hbar J_i}$$

Not S^2 yet (Quantization of $\mathbb{R}^3$)

◎ $\mathfrak{su}(2)$ has a quadratic Casimir

Def). Casimir Polynomial

$\mathfrak{g}$: d-dim Lie alg.

$A_{\mathfrak{g}} := (\mathbb{C}[x], \cdot, \{\ ,\ \})$

$x = (x_1, \cdots, x_d)$

Kirillov-Kostant

$[X_i, X_j] = f_{ij}^{k} X_k$

$\{x_i, x_j\} = f_{ij}^{k} x_k$

Lie-Poisson alg.

$f(x)$ is a Casimir poly. $\overset{\text{def}}{\iff}$ $\{x_i, f(x)\} = 0$

$i = 1, 2, \cdots, d$

1-7.

For $A_{su(2)}$ $f^c = \delta^{ab} x_a x_b$ is the Casimir

$\mathcal{Q}_k(f^c(x)) = \hbar^2 \delta^{ab} J_a J_b$: Casimir op.

$$= \hbar^2 \frac{1}{4}(k^2 - 1) \mathbb{1}_k$$

$\rightsquigarrow$ If we fix $\hbar(\mathcal{Q}_k) = \sqrt{\dfrac{4\lambda^2}{k^2-1}}$

then $\lim_{k\to\infty} \delta^{ab} \mathcal{Q}_k(x_a) \mathcal{Q}_k(x_b) = \underline{\lambda^2}$

fixed

$[\mathcal{Q}_k(x_a), \mathcal{Q}_k(x_b)] = \hbar\, \mathcal{Q}_k(\{x_a, x_b\}) \longrightarrow 0$ commutative limit

$(k \to \infty)\ \hbar \to 0$

$\delta^{ab} x_a x_b = \lambda^2$: S^2

$su(2)$ rep is quantization of S^2

◎ Polynomial on S^2

i) a formulation : spherical harmonics

$\uparrow$

I don't know how to generalize this.

ii) function on S^2

$\mathbb{C}[x]$: functions on $\mathbb{R}^3$

$$I(C) := \{ g(x)(\underbrace{\delta^{ab}x_a x_b - \lambda^2}_{\text{Eq of } S^2}) \mid g(x) \in \mathbb{C}[x] \}$$

I(C) is an ideal.

i.e.) $I(C) \subset \mathbb{C}[x]$, module

$\forall f \in I(C), \forall g(x) \in \mathbb{C}[x] \Rightarrow f(x) g(x) \in I(C)$

1-9

$$I(C) := \{\, g(x)(\delta^{ab} x_a x_b - \lambda^2) \mid g(x) \in \mathbb{C}[x] \,\}$$

$$\Downarrow$$

$\mathbb{C}[x]/I(C)$: functions on S^2

$\Downarrow$

$$[f] = \{\, f + h \mid f \in \mathbb{C}[x],\ h \in I(C) \,\}$$

$$\left.\begin{aligned} [f] + [g] &:= [f+g] \\ [f]\cdot[g] &:= [f \cdot g] \\ \{[f],[g]\} &:= [\{f,g\}] \end{aligned}\right\} \text{well-defined}$$

Variety S^2 is given by $\mathbb{C}[x]/I(C)$

$\rightarrow$ generalize this picture.

2 M.R. for $\mathbb{C}[x]$ with $\mathfrak{g}$

$\{e_1, \cdots, e_d \mid \text{basis of } \mathfrak{g} \quad [e_i, e_j] = f_{ij}^k e_k\}$

$\rho^\mu : \mathfrak{g} \to \mathfrak{gl}(V^\mu)$: irreducible rep.

$e_i^{(\mu)} := \hbar \rho^\mu(e_i)$, $[e_i^{(\mu)}, e_j^{(\mu)}] = \hbar f_{ij}^k e_k^{(\mu)}$

$T^\mu =$ alg generated by $\{e_1^{(\mu)}, \cdots, e_d^{(\mu)}\}$.

<u>Notation</u>

$$e^{(\mu)}_{I_i^m} := e^{(\mu)}_{(i_1, \cdots, i_m)} = \frac{1}{m!} \sum_{\sigma \in Sym(m)} e^{(\mu)}_{i_{\sigma(1)}} \cdots e^{(\mu)}_{i_{\sigma(m)}}$$

$I_j^k = \{\{j_1, \cdots, j_k\}\}$, $I_i^k \sqcup I_j^l = \{\{i_1, \cdots, i_k, j_1, \cdots, j_l\}\}$

We can make a basis of T_μ by $E_{I_i^m} := e^{(\mu)}_{I_i^m}$.

D) "Weak" M.R.

Def). $\mathcal{g}_\mu : A_g \longrightarrow T_\mu$ linear s.t. $\quad := (\mathbb{C}[x], \{,\})$

for $x^I = x_{i_1} \cdots x_{i_m}$

$$\mathcal{g}_\mu(x^I) = \begin{cases} e^{(\mu)}_{(i_1 \cdots i_m)} = \frac{1}{m!} \sum_{\sigma \in S_m} e^{(\mu)}_{i_{\sigma(1)}} \cdots e^{(\mu)}_{i_{\sigma(m)}} & (m \leq n_\mu) \\ 0 & (m > n_\mu) \end{cases}$$

$$\boxed{\begin{array}{l} \text{Def. } \mathcal{g}_k : \text{Poisson alg} \longrightarrow gl(V_k) \text{ is Quantization} \\ [\mathcal{g}_k(f), \mathcal{g}_k(g)] = \hbar\, \mathcal{g}_k(\{f,g\}) + \hbar^2 (\Sigma a_I E_I), \\ \text{with } \hbar \to 0 \text{ when } \dim V_k \to \infty. \text{ We say } \mathcal{g}_k \text{ is} \\ \text{"Weak M.R."} \end{array}}$$

We can prove the above $\mathcal{g}_\mu$ is "Weak M.R."

3. M.R. for $A_g / I(C)$

3-0. Preparation

▷ Casimir (k-th degree) C_k

For Semisimple Lie alg

There exists $\{V^\mu : \text{Ir. rep. sp. of } \mathfrak{g}\}$ s.t.

$$\dim V^\mu \xrightarrow[\mu\to\infty]{} \infty \quad \cap \quad |\underset{\substack{\| \\ \Lambda^k \, Id(V^\mu)}}{C_k(V^\mu)}| \xrightarrow[\mu\to\infty]{} \infty$$

Assume

$\mathfrak{g}$ has series of Ir. rep. $\{V^\mu\}$ s.t.

k-th degree Casimir

$$C_{ki} = \Lambda^k_i(V^\mu)\, Id_\mu, \quad \lim_{\dim V^\mu\to\infty} |\Lambda^k_i(V^\mu)| = \infty$$

<u>Prop.</u> $C_i^k(x) \in A_{\mathfrak{g}}$ is Casimir poly of deg k

i.e. $\{x_i, C_i^k(x)\} = 0 \quad (i = 1 \sim d)$

$\Rightarrow \quad \delta_\mu(C_i^k(x))$ is Casimir op.

We denote $\quad C_i^k(e^\mu) := \delta_\mu(C_i^k(x)) \sim \hbar^k$

We fix $\hbar(\mu)$ by $|C_i^k(e^\mu)| = |\hbar^k \Lambda_i^k| = \text{const.}$

$$\boxed{C_i^k(e^\mu) = \lambda_i^k}$$

▷ Gröbner basis

3-3

Fact. Fix a mono.-order on $\mathbb{C}[x_1, \cdots, x_d]$.

1. Every ideal has a unique reduced Gröbner basis $\rightarrow$ I's element is expanded by this basis

   $G = \{g_1, \cdots, g_m\}$

2. $\forall f \in \mathbb{C}[x_1, \cdots, x_d]$

   $$\left.\begin{aligned} f &= h + r \\ h &\in I, \ r \notin I \end{aligned}\right\} \text{ is uniquely determined}$$

s.t. no monomial in r is divisible by g_i.

$$
\begin{tikzcd}
\mathbb{C}[x] \arrow[r, "g_\mu"] & gl(V^\mu) & & \\
\mathbb{C}[x]/I(C) \arrow[u, "I(C)=\{0\}"'] \arrow[r, "g_{V/I}"] \arrow[rrr, bend left, "g_{V/I,\mu}"] & U_g[h]/I(C) \arrow[r, "\rho"', "\text{rep}"] & gl(V^\mu) \arrow[r, "R"] & gl(V^\mu)
\end{tikzcd}
$$

Casimir $C_i(x)$ Casimir $g_\mu(C_i(x))$

$$f(x) = h(x) + r(x) \in \mathbb{C}[x]$$

$$[f(x)] = [r(x)]$$

$$r(x) = \sum_I a_I x_I \;\mapsto\; g_{V/I}([r(x)]) = \sum a_I [X_I] \;\mapsto\; \sum_I a_I e_I^{(\mu)} \;\mapsto\; \sum a_I e_I^{(\mu)}, \quad |I| < n_\mu$$

$$\begin{tikzcd} \mathbb{C}[x]/I(C) \arrow[r, "\varrho_{V/I}"] \arrow[rrr, bend left, "\varrho_{A/I,\mu}"] & U_{\mathfrak{g}}[\hbar]/I(C) \arrow[r, "\rho_{V/I,\mu}"] & \mathfrak{gl}(V^{\mu}) \arrow[r, "R_{\mu}"] & \mathfrak{gl}(V^{\mu}) \end{tikzcd}$$

Casimir $C_i(x)$ Casimir $\varrho_{\mu}(C_i(x))$

Def). $\varrho_{A/I,\mu} : A_{\mathfrak{g}}/I(C) \longrightarrow \mathfrak{gl}(V^{\mu})$

$$\varrho_{A/I,\mu} := R_{\mu} \circ \varrho_{\mu}^{\mathrm{Pre}} = R_{\mu} \circ \rho_{V/I,\mu} \circ \varrho_{V/I}$$

$\boxed{\text{Thm. } \varrho_{A/I,\mu} \text{ is a weak M.R.} \quad [\varrho_{A/I,\mu}([f]), \varrho_{A/I,\mu}([g])] = \hbar\, \varrho_{A/I,\mu}(\{[f],[g]\}) + \hbar^2 P}$

Thm.

$$\varrho_{A/I,\mu}([f]\cdot[g]) = \varrho_{A/I,\mu}([f])\, \varrho_{A/I,\mu}([g]) + \hbar P$$

4. Reducible rep.

Thm. [Kirillov-Kostant-Souriau] (cpt case)

G : compact connected Lie group

λ : dominant integral highest weight

- The Coadjoint orbit $\mathcal{O}_\lambda \subset \mathfrak{g}^*$ exists.
- The irred. rep $V_\lambda \leftarrow$ geom. quantization.

c.f.) $\mathcal{O}_\lambda$ is determined by all casimir op.

Traditionally understanding

$$\mathcal{O}_\lambda \xrightarrow[\text{quantize}]{} V_\lambda .$$

How can we interprete our results from this pt of view?

ex) $su(3)$

$[a,b]$: Dynkin idx

$\lambda = a\omega_1 + b\omega_2 = (\lambda_1, \lambda_2, \lambda_3)$

ω_1, ω_2: fund. weight

V_λ : (a,b) rep.

- $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$

$\mathcal{O}_\lambda = SU(3)/T^2$

$\lambda_1 = \lambda_2 \neq \lambda_3$

$\mathcal{O}_\lambda = \mathbb{C}P^2$

▷ Red. rep.

$V^{\mu} = \bigoplus_{a=1}^{m_\mu} V_\mu^a$, $P_a : V^\mu \to V_\mu^a$ projection

$\rho_\mu^a : \mathfrak{g} \to \mathrm{End}(V_\mu^a)$: irreducible rep.

$\rho_\mu : \sum_{a=1}^{m_\mu} \rho_\mu^a P_a$, base $h_a\, \rho_\mu^a(e_i) =: e_i^{(\mu,a)}$

$\hat{h}_\mu := h_1 \oplus h_2 \oplus \cdots \oplus h_m$ $r_i \in \mathbb{R}$

$[\rho_\mu(e_i), \rho_\mu(e_j)] = f_{ij}^{k}\, \rho_\mu(e_k)$

$[e_i^{(\mu)}, e_j^{(\mu)}] = \hat{h}_\mu\, e_j^{(\mu)}$

@ Casimir Op. (k-th)

not depend on "a"

By $C_i^{k,a}(e^{(\mu,a)}) = h_a^{k}\, \Lambda_i^{a,k}(V_\mu^a)\, \mathrm{Id}_a = \lambda_i^k$

we fix each h_a.

$\rightsquigarrow$ Everything extends to the case of red rep.

4-3

Def) $\varrho^R_{A/I,\mu} : A_{\cancel{\mathfrak{g}}}/I(C) \to \mathfrak{gl}(V^\mu) = \mathfrak{gl}\left(\bigoplus_{a=1}^{m_\mu} V^a_\mu\right)$

$$\varrho^R_{A/I,\mu} = R^R_\mu \circ \rho^R_{U/I,\mu} \circ \varrho^R_{U/I}$$

$$\mathbb{C}[x]/I(C) \xrightarrow{\varrho^R_{U/I}} \begin{array}{c} U_{\mathfrak{g}}[\hbar_1]/I(C) \\ \oplus \\ \vdots \\ \oplus \\ U_{\mathfrak{g}}[\hbar_{m_\mu}]/I(C) \end{array} \xrightarrow{\rho^R_{U/I,\mu}} \begin{array}{c} \mathfrak{gl}(V^\mu_1) \\ \oplus \\ \vdots \\ \oplus \\ \mathfrak{gl}(V^\mu_{m_\mu}) \end{array} \xrightarrow{R^R_\mu} \mathfrak{gl}(V^\mu)$$

$$[f] = [r_f] \mapsto \begin{array}{c} [\varrho_U(r_f)|_{\hbar=\hbar_1}] \\ \oplus \\ \vdots \\ \oplus \\ [\varrho_U(r_f)|_{\hbar=\hbar_{m_\mu}}] \end{array} \mapsto \begin{array}{c} \sum_J a^{(J)} e^{(\mu,1)}_{(J)} \\ \oplus \\ \vdots \\ \oplus \\ \sum_J a^{(J)} e^{(\mu,m_\mu)}_{(J)} \end{array} \mapsto \begin{array}{c} \sum_{|J|\le n^1_\mu} a^{(J)} e^{(\mu,1)}_{(J)} \\ \oplus \\ \vdots \\ \oplus \\ \sum_{|J|\le n^{m_\mu}_\mu} a^{(J)} e^{(\mu,m_\mu)}_{(J)} \end{array}$$

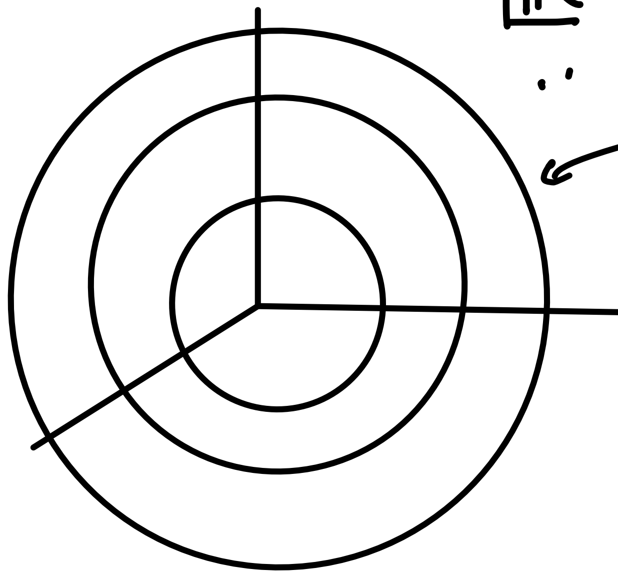
Then.

$$\boxed{\begin{aligned} &[\varrho^R_{A/I,\mu}([f]), \varrho^R_{A/I,\mu}([g])] = \hat{\hbar}(\mu)\,\varrho^R_{A/I,\mu}(\{[f],[g]\}) + O(\hbar^2) \\ &\varrho^R_{A/I,\mu}([f][g]) = \varrho^R_{A/I,\mu}([f]) \cdot \varrho^R_{A/I,\mu}([g]) + O(\hbar) \end{aligned}}$$

Singular foliation

$\mathfrak{su}(2)$ case :

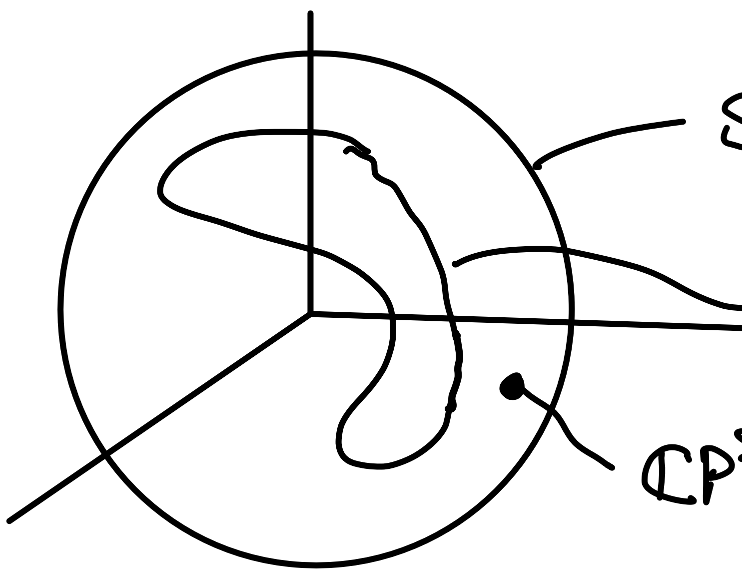
$\mathbb{R}^3$



$SU(2)/U(1) \sim S^2$

$\mathfrak{su}(3)$ case :

$x_3 \sim x_8$ $\quad$ $\mathbb{R}^8$



$S^7 \leftarrow$ 2nd Casimir

$+$

$SU(3)/T^2$ $\quad$ 3rd Casimir

$\mathbb{C}P^2 \ (\lambda_1 = \lambda_2)$

Ex.) Fuzzy $\mathbb{R}^3$

$$\mathcal{S}_{k,r} : A_{su(2)} \longrightarrow \mathrm{Mat}_k(\mathbb{C})$$

$$x_{a_1}\cdots x_{a_m} \longmapsto \frac{\hbar^m(k,r)}{m!} \sum_{\sigma\in S_m} J_{a_{\sigma(1)}}\cdots J_{a_{\sigma(m)}}$$

where J_a : a generator for k-dim irre. rep of $su(2)$

$$\delta^{ab} J_a J_b = \frac{1}{4}(k^2-1)\,\mathbb{1}_k ,$$

$\hbar(k,r)$ is determined by

$$\hbar^2(k,r)\,\frac{1}{4}(k^2-1) = r^2$$

Consider $\displaystyle \mathcal{S}^R_k := \bigoplus_{i=1}^{k^2} \mathcal{S}_{k,r_i}$. $r_1 = \sqrt{\frac{1}{k}}, r_2 = \sqrt{\frac{2}{k}}, \cdots, r_{k^2} = \sqrt{\frac{k^2}{k}} = \sqrt{k}$

When $k\to\infty$. $r_{k^2}\to\infty$, $r_1 \to 0$, $r_{i+1}-r_i \to 0$

(commutative lim $\hbar\to 0$)

$\mathbb{R}^3$ is densely foliated by S^2

5. Fuzzy S^7

$\mathfrak{g} = \mathfrak{su}(3)$.

▷ (p,q) rep.

fundmental weight vect.

- Highest weight $\mu = p\omega_1 + q\omega_2$, (p,q) : Dynkin label

$$\dim V_{(p,q)} = \frac{1}{2}(p+1)(q+1)(p+q+2)$$

$$C_2(p,q) = \frac{1}{3}(p^2+q^2+pq) + (p+q) \geq pq$$

$$C_3(p,q) = \frac{1}{18}(p-q)(2p+q+3)(p+2q+3)$$

Step 1. Fuzzy $\mathbb{R}^8$

Let's def. $\mathcal{Q}^r_{p,q} : \underbrace{A_{\mathfrak{su}(3)}}_{\simeq \mathbb{R}^8} \longrightarrow \mathfrak{gl}(V_{(p,q)})$, similarly.

$$\hbar_{(p,q)} := \frac{r}{\sqrt{C_2(p,q)}} \qquad r : \text{some fixed constant}.$$

Lem. Series of rep (P_n, g_n) s.t. $\lim_{n\to\infty} \frac{P_n}{g_n} = t \quad (0 \leq t)$

$$\lim_{n\to\infty} g_n(C_3(x)) = \left(\lim_{n\to\infty} \hbar^3 (P_n, g_n)\, C_3(P_n, g_n)\right) Id = r^3 R(t)\, Id$$

where $-\frac{\sqrt{3}}{3} \leq R(t) \leq \frac{\sqrt{3}}{3}$

- $R(0) = -\frac{\sqrt{3}}{3}$
- $R(\infty) = \frac{\sqrt{3}}{3}$
- $R(t)$: monotonic increse.

Lem.

$$\mathcal{O}_R = \left\{ x \in \mathbb{R}^8 \;\middle|\; \sum_{i=1}^{8} x_i^2 = r^2,\ \ C_3(x) = r^3 R \right\},$$

Let D be a dense subset of $\left[-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right]$. $\mathcal{D} := \coprod_{R\in D} \mathcal{O}_R$

Then, $\overline{\mathcal{D}} = S^7$

5-3

$$C^3 = \frac{1}{18}(2\sqrt{3}x_8^3 - 6\sqrt{3}x_1^2x_8 - 6\sqrt{3}x_2^2x_8 - 6\sqrt{3}x_3^2x_8 + 3\sqrt{3}x_4^2x_8 + 3\sqrt{3}x_5^2x_8 + 3\sqrt{3}x_6^2x_8 + 3\sqrt{3}x_7^2x_8 - 18x_2x_5x_6 + 18x_2x_4x_7 - 18x_1(x_4x_6 + x_5x_7) - 9x_3\left(x_4^2 + x_5^2 - x_6^2 - x_7^2\right)).$$

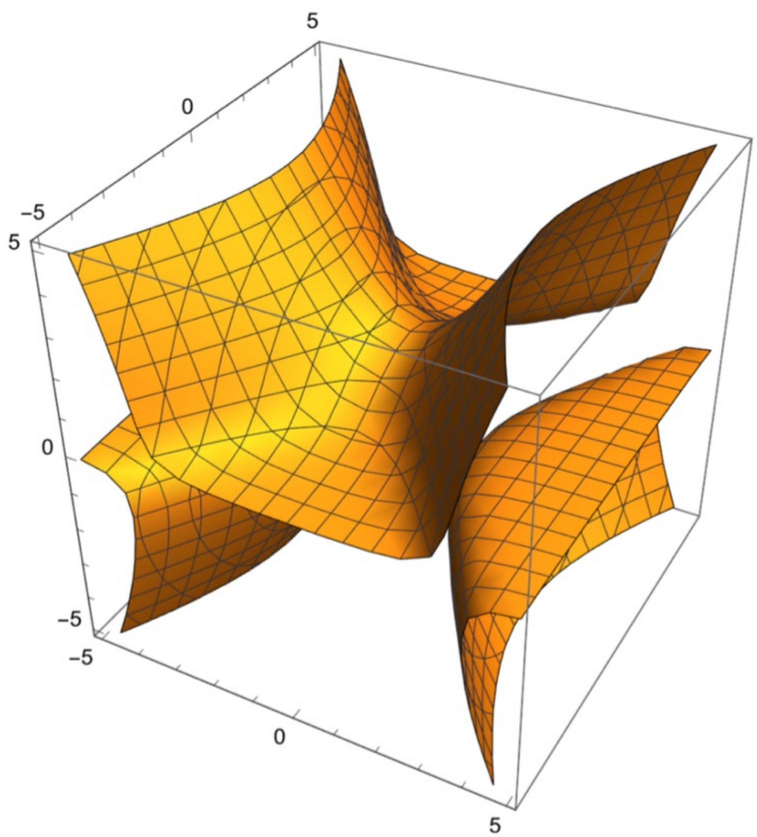


Figure 1: Variety with $C^3 = 1$. $(x_2 = x_3 = x_4 = x_5 = x_7 = 0)$

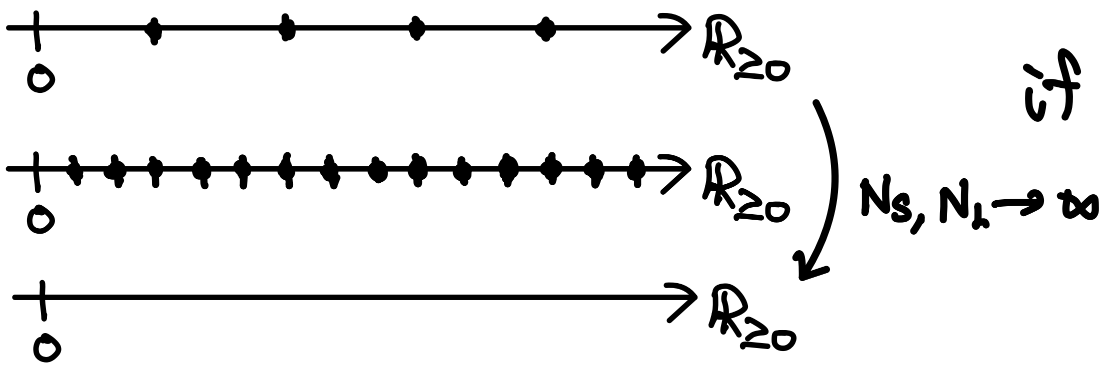
Step 2. $I(C)(x) = \left((C_2(x) - r) \right)$

$$g_{A/I,(p_n,q_n)}([f]) = g^{r}_{p_n,q_n}(r_f) \quad (\text{If } r_f \le n_{(p,q)})$$

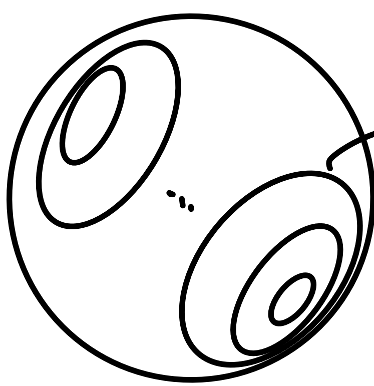
remainder $f = h + r_f$, $h \in I(C)(x)$

Step 3. $\{(p_n, q_n) \mid N_S^2 \le p_n^2 + q_n^2 \le N_L^2\}$

$t_n = \dfrac{p_n}{q_n} = 0, 1, \cdots, N_L^2, \infty \rightsquigarrow$ t_n is chosen so that $\hbar^3 C_3$ fills D densely if $N_S, N_L \to \infty$



Step 4. $$g^{R}_{A/I,(N_S^2,N_L^2)} := \bigoplus_{N_S^2 \le p_n^2 + q_n^2 \le N_L^2} g_{A/I,(p_n,q_n)}$$



$SU(3)/T^2 \xrightarrow{\hbar \to 0} S^7$

6. Summary

$$
\begin{tikzcd}
\mathbb{C}[x]/I(C) \arrow[r, "\rho^R_{V/I}"] \arrow[rrr, bend left, "\rho^R_{A/I,\mu}"] & \begin{array}{c} U_g[\hbar_1]/I(C) \\ \oplus \\ \vdots \\ \oplus \\ U_g[\hbar_{m_\mu}]/I(C) \end{array} \arrow[r, "\rho^R_{V/I,\mu}"] & \begin{array}{c} gl(V_1^\mu) \\ \oplus \\ \vdots \\ \oplus \\ gl(V^\mu_{m_\mu}) \end{array} \arrow[r, "R^R_\mu"] & gl(V^\mu)
\end{tikzcd}
$$

$\uparrow$ an Algebraic variety defined by inv. polynomial of g. (ex. S^7) $\longrightarrow$ (Fuzzy S^7)

$\uparrow$ A Irreducible rep. each rep $\sim$ Coadjoint orbit.

- In $\dim V^\mu \to \infty$
- $\hbar \to 0$ (commutative alg).
- The increasing family of subvarieties (Coadjoint orbits) becomes dense in the original variety.
- We can generalize S^7 case to $^\forall C_i(x) = \lambda_i$ (Casimir) in a compact semisimple Lie alg. case.

Thank you very much
for your kind attention !!