Outer Billiards in the Spaces of Oriented Geodesics of the Three Dimensional Space Forms

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Let M_κ be the three-dimensional space form of constant curvature $\kappa = 0, 1, -1$, that is, Euclidean space $\mathbf{R}^3$, the sphere S^3, or hyperbolic space H^3. Let S be a smooth, closed, strictly convex surface in M_κ. We define an outer billiard map B on the four dimensional space $\mathcal{G}_\kappa$ of oriented complete geodesics of M_κ, for which the billiard table is the subset of $\mathcal{G}_\kappa$ consisting of all oriented geodesics not intersecting S. We show that B is a diffeomorphism when S is quadratically convex.

For $\kappa = 1, -1$, $\mathcal{G}_\kappa$ has a Kähler structure associated with the Killing form of $\operatorname{Iso}(M_\kappa)$. We prove that B is a symplectomorphism with respect to its fundamental form and that B can be obtained as an analogue to the construction of Tabachnikov of the outer billiard in $\mathbf{R}^{2n}$ defined in terms of the standard symplectic structure. We show that B does not preserve the fundamental symplectic form on $\mathcal{G}_\kappa$ associated with the cross product on M_κ, for $\kappa = 0, 1, -1$.

We initiate the dynamical study of this outer billiard in the hyperbolic case by introducing and discussing a notion of holonomy for periodic points.

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