Algebraic CMC surfaces and Weierstrass-$\wp$ functions

joint with R. Dey and A. Paul

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Definitions and Notations

- $\mathbb{E}^3$ and $\mathbb{L}^3$ denotes the 3-dimensional Euclidean space $(\mathbb{R}^3, \langle , \rangle_E := dx^2 + dy^2 + dz^2)$ and the 3-dimensional Lorentz-Minkowski space $(\mathbb{R}^3, \langle , \rangle_L := dx^2 + dy^2 - dz^2)$ respectively.

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- A surface S in $\mathbb{R}^3$ is called *regular algebraic* if $S = \{(x, y, z) \in \mathbb{R}^3 | P(x, y, z) = 0\}$ and $\nabla P(x, y, z) \neq 0$ for every $(x, y, z) \in S$, where $P(x, y, z)$ denotes a polynomial. Degree of the polynomial P is called the *degree* of the surface S.

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- In particular, regular algebraic surfaces are properly embedded (without self-intersections). This follows from regular value theorem.
- A surface S in $\mathbb{E}^3$ is called a minimal surface if its mean curvature function $H \equiv 0$ and called a CMC surface if $H \equiv constant (\neq 0)$.

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- A vector $v \in \mathbb{L}^3$ is said to be a *spacelike* (respectively *timelike*, *lightlike*) if $\langle v, v\rangle_L > 0$ or $v = 0$ (respectively $\langle v, v\rangle_L < 0$, $\langle v, v\rangle_L = 0$ and $v \neq 0$).

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- A spacelike surface S in $\mathbb{L}^3$ is called a maximal surface if its mean curvature function $H \equiv 0$ and called a CMC surface if $H = constant(\neq 0)$.

History of algebraic CMC surfaces in $\mathbb{E}^3$

- When $H = 0$, there are many examples of minimal surfaces which are algebraic such as plane (degree 1), Enneper surface $\left((\frac{y^2-x^2}{2z} + \frac{2z^2}{9} + \frac{2}{3})^3 - (\frac{y^2-x^2}{4z} - \frac{1}{4}(x^2 + y^2 + \frac{8z^2}{9}) + \frac{2}{9})^2 = 0)) \right)$, Henneberg surface (degree 15).

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- It is interesting to note that there are no regular algebraic minimal surfaces in $\mathbb{E}^3$ except plane.
- In 2013, Oscar M Perdomo[1] showed that the only algebraic constant mean curvature (CMC) surfaces in $\mathbb{E}^3$ of order less than 4 are the planes, the spheres and the cylinders.

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- When $H \equiv constant$ (not zero), Barbosa and do Carmo[2] have shown that the spheres and the right circular cylinders are the only examples of regular algebraic CMC surfaces in $\mathbb{E}^3$ (proof uses "structure theorem" for CMC surfaces).

[2] *On regular algebraic surfaces of* $\mathbb{R}^3$ *with constant mean curvature*; J. Differential Geom. 102 (2016), no. 2, 173–178.

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- Interestingly, spheres and cylinders belongs to the class of surfaces of revolution (this class includes other examples of CMC surfaces, namely unduloids and nodoids. They are also called *Delaunay surfaces*).

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- Interestingly, spheres and cylinders belongs to the class of surfaces of revolution (this class includes other examples of CMC surfaces, namely unduloids and nodoids. They are also called *Delaunay surfaces*).
- Again, in 2019, Perdomo and Tkachev[3] showed that there exist no degree 3 algebraic hypersurfaces in the Euclidean space , $n \geq 3$, with nonzero constant mean curvature.

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- Recently, in 2022, Barreto, Fontenele,and Hartmann[4] have shown that there are no regular algebraic hypersurfaces with non-zero constant mean curvature in $\mathbb{E}^{n+1}$, $n \geq 2$, defined by polynomials of odd degree. They also prove that the hyperspheres and the round cylinders are the only regular algebraic hypersurfaces with non-constant mean curvature in $\mathbb{E}^{n+1}$, $n \geq 2$, defined by polynomials of degree less than or equal to 3.

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- The study of general algebraic CMC surfaces in the Euclidean spaces (other ambient spaces like $\mathbb{L}^3$, etc.) is a basic question that has very little progress.

[4] *On regular algebraic hypersurfaces with non-zero constant mean curvature in Euclidean spaces*; Proc. Roy. Soc. Edinburgh Sect. A 152 (2022), no. 4, 1081–1088.

Although there exists a complete classification of regular algebraic CMC surfaces in $\mathbb{E}^3$. Not much is known on the classification of general algebraic CMC hypersurfaces in $\mathbb{E}^n$. However, there exists a classification of semialgebraic CMC surfaces in $\mathbb{E}^3$ with singularities[5].

[5] Sampaio, J. E.; *Globally subanalytic CMC surfaces in $\mathbb{E}^3$ with singularities*; Proc. Roy. Soc. Edinburgh Sect. A 151 (2021), no. 1, 407–424.

Algebraic CMC surfaces in $\mathbb{L}^3$

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- Now, it would be natural to investigate all regular algebraic spacelike CMC surfaces in $\mathbb{L}^3$ as they serve as the Lorentzian analogues of CMC surfaces in $\mathbb{E}^3$.
- However, unlike the Euclidean setting, where a structure theorem for such surfaces is available, no analogous classification exists for spacelike CMC surfaces in $\mathbb{L}^3$.
- Also, as noted before, the regular algebraic CMC surfaces in $\mathbb{E}^3$ belongs to the class of surfaces of revolution. This motivated us to study the algebraicity of spacelike CMC surfaces of revolution (about a timelike axis and a spacelike axis) in $\mathbb{L}^3$. Infact, we showed that in this class, only spacelike cylinders and standard hyperboloids are algebraic.

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- For every elliptic curve whose discriminant is non-zero, there exists a Weierstrass-$\wp$ function.

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- For a given surface of revolution, one can associate an elliptic curve (this is done using Kenmotsu's parametrization)
- For every elliptic curve whose discriminant is non-zero, there exists a Weierstrass-$\wp$ function.
- Weierstrass-$\wp$ function being *transcendental* can not satisfy any algebraic relation.

Kenmotsu's parametrization for Delaunay surfaces in $\mathbb{E}^3$

[6] *Surfaces of revolution with prescribed mean curvature*. Tohoku Math. J. (2) 32 (1980), no. 1, 147–153.

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- Let $\alpha(s) := (x(s),\ y(s))$, where $s \in I$, I an interval, and $y(s) > 0$ for all $s \in I$. Then a parametrization of surface of revolution (about x-axis) generated by the curve $\alpha(s)$ is given by $(x(s), y(s)cos\theta, y(s)sin\theta)$, where θ denotes the rotational parameter.

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- For CMC surfaces of revolution in $\mathbb{E}^3$, Kenmotsu obtains

$$\begin{aligned} x(s) &= \int_0^s \frac{1 + B\sin 2Ht}{\sqrt{1 + B^2 + 2B\sin 2Ht}}dt \quad \text{and} \\ y(s) &= \frac{1}{2H}\sqrt{1 + B^2 + 2B\sin 2Hs}, \end{aligned} \tag{0.1}$$

where $B \in [0, \infty)$ and H denotes the constant mean curvature[6].

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Parametrization for Delaunay surfaces in $\mathbb{L}^3$ (x-axis)

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- Let $\alpha(s) := (x(s),\ 0,\ z(s))$ be a spacelike curve, where $s \in I$, and $x(s) > 0$ for all $s \in I$. Then a parametrization of the corresponding spacelike surface of revolution (about x-axis) generated by the curve $\alpha(s)$ is given by $(x(s), z(s)\sinh\theta, z(s)\cosh\theta)$.

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proof

- We begin with the parametrization of a spacelike surface of revolution about z-axis (timelike) and we assume that the generating curve is arc-length parametrized. Then, its first and second fundamental forms are given by $I = ds^2 + x(s)^2 d\theta^2$ and $II = (x'(s)z''(s) - z'(s)x''(s))ds^2 + x(s)z'(s)d\theta^2$ respectively. We also obtain the following ODE satisfied by H

$$2Hx(s) + z'(s) + x(s)(x'(s)z''(s) - z'(s)x''(s)) = 0. \qquad (0.4)$$

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- Again, by multiplying the ODE (0.4) by $z'(s)$ and $x'(s)$ and using the fact that $x'(s)^2 - z'(s)^2 = 1$, we obtain the following two simultaneous ODEs, given by

- $$2Hx(s)z'(s) + (x(s)x'(s))' - 1 = 0. \tag{0.5}$$

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  $$2Hx(s)x'(s) + (z'(s)x(s))' = 0. \tag{0.6}$$

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- Before proceeding further, let us recall a split-complex number denoted by $t = x + ky$, where $x,\ y$ are two real numbers. The conjugate of t is denoted by $t^* = x - ky$. Since $k^2 = 1$, the absolute value of t is given by $\mid t \mid = \sqrt{x^2 - y^2}$. Also, the analogue of Euler's formula in split-complex number system has the form

  $$e^{k\theta} = \cosh\theta + k\sinh\theta.$$

- Now, by setting $W(s) = x(s)x'(s) + kx(s)z'(s)$, we see that the ODEs (0.5) and (0.6) is equivalent to the following first order linear (split-complex) ODE:

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- The equivalence of the equations (0.5) and (0.6) with the equation (0.7) can be seen as follows:

$$\begin{aligned}
&W'(s) + 2kHW(s) - 1 = 0 \\
&\Leftrightarrow \left(x(s)x'(s) + kx(s)z'(s)\right)' + 2kH\left(x(s)x'(s) + kx(s)z'(s)\right) - 1 = 0 \\
&\Leftrightarrow \left(2Hx(s)z'(s) + (x(s)x'(s))' - 1\right) + k\left(2Hx(s)x'(s) + (z'(s)x(s))'\right) = \\
&\Leftrightarrow 2Hx(s)z'(s) + (x(s)x'(s))' - 1 = 0, \text{ and } 2Hx(s)x'(s) + (z'(s)x(s))
\end{aligned}$$

When $H = 0$, the solution of the ODE (0.7) gives us the elliptic catenary $x = c \sinh \frac{z}{c}$, and when rotated about z-axis gives us the elliptic catenoid $x^2 + y^2 = c^2 \sinh^2 \frac{z}{c}$, which is a maximal surface and it is obvious that it is not algebraic. When $H \neq 0$, the solutions of (0.7) are given by (0.3).

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Lemma

The generating curves (namely undulary and nodary, see (0.3)*) for a family of spacelike non-zero CMC surfaces of revolution in* $\mathbb{L}^3$ *(about z-axis) identify a family of elliptic curves.*

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The generating curves (namely undulary and nodary, see (0.3)) for a family of spacelike non-zero CMC surfaces of revolution in $\mathbb{L}^3$ (about z-axis) identify a family of elliptic curves.

From (0.3), we have $z(s) = \int_0^s \frac{1 - B\cosh 2Ht}{\sqrt{2B\cosh 2Ht - B^2 - 1}} dt$. By doing change of variables twice, we get

$$z(s) = \frac{1}{2H} \int_c^{\sinh(2Hs)+c} \frac{Bw - (1 + Bc)}{\sqrt{l + mw + nw^3}} dw, \qquad (0.8)$$

where

$$l = (B^2 + 1) + 2Bc - (B^2 + 1)c^2 - 2Bc^3;$$

$$m = 2c(B^2 + 1) + 6Bc^2 - 2B; \; n = 2B; \text{ and } c = \frac{-B^2 - 1}{6B}.$$

Let

$$v^2 = l + mw + nw^3 \tag{0.10}$$

By applying the transformation $w \to \sqrt[3]{\frac{4}{n}}\tilde{w}$ in equation (0.10), we get

$$v^2 = 4\tilde{w}^3 - (-m\sqrt[3]{\frac{4}{n}})\tilde{w} - (-l). \tag{0.11}$$

This represents a family of nonsingular elliptic curves parametrized by B, where $B \neq 0, 1$.

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Spacelike unduloids and nodoids are not algebraic

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We know that the discriminant of an elliptic curve (0.11) is given by $(-m\sqrt[3]{\frac{4}{n}})^3 - 27(-l)^2$, when evaluated this gives us a degree 8 polynomial $2(B^2-1)^4$ and we note that

$$(-m\sqrt[3]{\frac{4}{n}})^3 - 27(-l)^2 \neq 0; \text{ except for } B = 1. \tag{0.12}$$

This shows that the elliptic curves which appears in previous Lemma are all nonsingular.

- Therefore, by a known result[9], there is a lattice L such that

$$g_2(L) = -m\sqrt[3]{\frac{4}{n}}; \quad g_3(L) = -l \tag{0.13}$$

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- Therefore, by a known result[9], there is a lattice L such that

$$g_2(L) = -m\sqrt[3]{\frac{4}{n}}; \quad g_3(L) = -l \tag{0.13}$$

- and corresponding to this, there will be a Weierstrass-$\wp$ function, which will satisfy

$$(\wp'(\tilde{z}))^2 = 4(\wp(\tilde{z}))^3 - g_2\wp(\tilde{z}) - g_3, \tag{0.14}$$

where $\tilde{z} \in \mathbb{C}$.

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where $\tilde{z} \in \mathbb{C}$.

- Note that we can always choose a neighbourhood of $\tilde{z}$, sufficiently small, such that in this neighbourhood $\wp'(\tilde{z}) \neq 0$, and hence we can differentiate the above equation to obtain

$$2\wp''(\tilde{z}) = 12\wp^2(\tilde{z}) - g_2. \tag{0.15}$$

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Now again, by replacing w by $\wp(\tilde{z})$, v by $\wp'(\tilde{z})$ and then using certain properties of Weierstrass-$\wp$ function in (0.8), we get

$$z(t) = \frac{p}{2H}(t - t_0) + \frac{q}{2H}\int_{t_0}^{t} \wp(t)dt \tag{0.16}$$

and

$$x(t)^2 = c_1 + c_2\wp(t) \tag{0.17}$$

where $p = 1 + Bc$, $q = -B$, $c_1 = \frac{-(B^2+2Bc+1)}{4H^2}$ and $c_2 = \frac{2B}{4H^2}$.

- Next, we recall any surface of revolution about z-axis is given by zeroes of a function of the form $x^2 + y^2 - r(z)$.

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- **Claim**: $r(z)$ can not be a polynomial.

  On the contrary, if we assume $r(z)$ is a polynomial.

  Differentiating equation (0.18) w.r.t z we get

$$\frac{dr}{dz} = R_1(\wp)\wp'. \tag{0.19}$$

  where $R_1(\wp)$ is a rational function of $\wp$. Again

$$\frac{d^2r}{dz^2} = R_2(\wp), \tag{0.20}$$

- where $R_2(\wp)$ is again a rational function of $\wp$; continuing in this way and by frequently using the above relations, the derivatives at the odd stage and even stage will have the following form

$$\frac{d^{2k-1}r}{dz^{2k-1}} = R_{2k-1}(\wp)\wp', \tag{0.21}$$

$$\frac{d^{2k}r}{dz^{2k}} = R_{2k}(\wp) \tag{0.22}$$

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- Observe that on the left hand side of the above equations, we are repeatedly taking the derivative of a polynomial $r(z)$ (by assumption); hence, after a while, there exists a natural number (say m) such that $\frac{d^m r}{dz^m} = 0$, which can be either even or odd depending on the degree of $r(z)$ (m will be odd if the degree is even and vice-versa).

- If m is even, i.e., $m = 2k$

$$\frac{d^{2k} r}{dz^{2k}} = 0 \Longrightarrow R_{2k}(\wp) = 0. \tag{0.23}$$

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- Now observe that we can always select a neighbourhood around the point z_0 so there will be no zeroes of $\wp'$ in that neighbourhood. Then equation (0.24) will imply $R_{2k-1}(\wp) = 0$.
- Thus, we arrive at a contradiction. Hence, we have shown that the spacelike CMC surfaces of revolution (unduloids and nodoids) about a timelike axis cannot be algebraic

Next, using exactly similar steps as before we obtain following theorems:

Theorem (R. Dey, A. Paul and S-)

The spacelike CMC surfaces of revolution in $\mathbb{L}^3$ *(whose axis of revolution is a spacelike axis) for* $B \in (0, \infty) \setminus \{1\}$*, cannot be algebraic except spacelike cylinders and standard hyperboloids.*

Theorem (R. Dey, A. Paul and S-)

The non-zero CMC surfaces of revolution in $\mathbb{E}^3$*, corresponding to* $0 < B < 1$ *(unduloids) and* $B > 1$ *(nodoids), are not algebraic.*

Thank you!