Investigation of Some Surfaces Related to Nonlinear Schrödinger Equation

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Investigation of motion of a vortex filament provides the crucial problems of mathematical physics and differential geometry. The work of Hasimoto in 1972 is one of the leading studies that guide us in solving these extremely important problems. If the position vector of vortex filament is denoted by $r = r(s, t)$, then the equation

$$r_t = r_s \times r_{ss} \tag{3.1}$$

is hold, (Hasimoto; 1972).

This equation is called the smoke ring or vortex filament equation. It can be considered that these vortex motions, which involve no change of form, correspond to travelling wave solutions of the Nonlinear Schrödinger (NLS) equation, (Rogers and Schief; 2002). These kind of soliton surfaces that are associated with the NLS equation are called NLS surfaces or Hasimoto surfaces (Hasimoto; 1972).

Hasimoto stated the complex function

$$q = \kappa e^{i \int \tau ds}$$

of the curvature κ and torsion τ of a space curve and obtained that if the curve evolves according to the vortex filament equation, then it gives the solution of the focussing cubic nonlinear Schrödinger (NLS) equation

$$iq_t + q_{ss} + vq \left| q \right|^2 = 0$$

In this study, we investigate differential geometric properties the soliton surface accociated with Nonlinear Schrödinger (NLS) equation, which is called NLS surface or Hasimoto surface, are investigated by using the Darboux frame. Firstly, a brief summary is presented to provide the necessary background. We give a proof of that the s parameter curves of NLS surface are the geodesics of the soliton surface and that this surface provides the NLS equation. Then, we discuss the geometric properties of NLS surface. We find Gaussian and mean curvature of NLS surface. Also, we obtain new results and the necessary and sufficient conditions for NLS surfaces to be flat or minimal surfaces. Finally, we investigate an example of NLS surface as application.

In this section, we give the necessary information about the regular curves on surfaces to understand the main subject of the study. Let $\alpha : I \longrightarrow M$ be a regular unit speed curve on the orientiable surface M. We may define Frenet frame $\{T, N, B\}$ at each points of the curve α where T is unit tangent vector, N is principal normal vector and B is binormal vector.

The Serret-Frenet formulas of the curve α are given by

$$\begin{aligned} T'(s) &= \kappa(s)N(s), \\ N'(s) &= -\kappa(s)T(s) + \tau(s)B(s), \\ B'(s) &= -\tau(s)N(s), \end{aligned}$$

where the functions κ and τ are called the curvature and the torsion of the curve α, respectively. On the other hand, we may also define another orthonormal frame fields on the curve α, which is called Darboux frame, since the curve α lies on the orientible surface M.

Darboux frame of (α, M) curve-surface couple includes unit tangent vector field T of α and unit normal vector field of the surface $n = N \circ \alpha$ on the curve α. To describe an orthonormal frame including these vector fields, there is only one way to choose last frame field as $g = n \times T$. This implies the following relations between Frenet and Darboux frames;

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} T \\ n \\ g \end{bmatrix}$$

and

$$\begin{bmatrix} T \\ n \\ g \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where β is the angle between the vector fields N and n.

The derivative formulas of Darboux frame can be given as follows:

$$\frac{d}{ds}\begin{bmatrix} T \\ n \\ g \end{bmatrix} = \begin{bmatrix} 0 & k_n & k_g \\ -k_n & 0 & -t_r \\ -k_g & t_r & 0 \end{bmatrix}\begin{bmatrix} T \\ n \\ g \end{bmatrix}$$

where

$$\begin{aligned} k_n(s) &= \kappa(s)\cos\beta(s), \\ k_g(s) &= -\kappa(s)\sin\beta(s), \\ t_r(s) &= -\beta'(s) - \tau(s). \end{aligned}$$

The functions k_g is called geodesic curvature, k_n is called normal curvature and t_r is called geodesic torsion of α.

In this section, NLS surface or Hasimoto surface are investigated by using the Darboux frame and discuss the geometric properties of NLS surface.

Theorem 3.1

Suppose $r = r(s,t)$ is a NLS surface such that $r = r(s,t)$ is a unit speed curve with normal vector field for all t. Then the following is satisfied;

$$\begin{bmatrix} T \\ n \\ g \end{bmatrix}_t = \begin{bmatrix} 0 & \alpha & \lambda \\ -\alpha & 0 & -\gamma \\ -\lambda & \gamma & 0 \end{bmatrix} \begin{bmatrix} T \\ n \\ g \end{bmatrix}$$

where $\{T, n, g\}$ is the Darboux frame of the s parameter curves of NLS surface. Here, α, λ and γ are smooth functions given by the following equalities

$$\alpha = k_{g_s} - k_n t_r, \tag{3.2}$$

$$\lambda = -k_{n_s} - k_g t_r, \tag{3.3}$$

$$\gamma = \frac{1}{k_g^2 + k_n^2} \left\{ \begin{array}{c} (k_g k_{g_s} + k_n k_{n_s})_s - (k_{n_s} + k_{g_s} t_r)^2 \\ -(k_{g_s} - k_n t_r)^2 + k_{g_t} k_n - k_{n_t} k_g \end{array} \right\}. \tag{3.4}$$

Proof. We need to find smooth functions α, λ and γ in terms of the curvatures k_g, k_n, t_r of the solution curve $r = r(s,t)$ of smoke ring equation for all t. Using compatibility conditions $T_{ts} = T_{st}$, we get

$$\alpha_s = k_{n_t} + \gamma k_g - \lambda t_r,$$
$$\lambda_s = k_{g_t} - \gamma k_n + \alpha t_r,$$

and using equality of $n_{ts} = n_{st}$, we

$$\alpha_s = k_{n_t} + \gamma k_g - \lambda t_r,$$
$$\gamma_s = \lambda k_n - \alpha k_g + t_{r_t}.$$

Similarly using equality of $g_{ts} = g_{st}$, we get following equalities

$$\gamma_s = \lambda k_n - \alpha k_g + t_{r_t},$$
$$\lambda_s = k_{g_t} - \gamma k_n + \alpha t_r.$$

Thus, by the above relations, the following equations are obtained

$$\alpha_s = k_{n_t} + \gamma k_g - \lambda t_r, \tag{3.5}$$
$$\gamma_s = \lambda k_n - \alpha k_g + t_{r_t}, \tag{3.6}$$
$$\lambda_s = k_{g_t} - \gamma k_n + \alpha t_r. \tag{3.7}$$

Again by compatibility condition of $r_{st} = r_{ts}$, we find the following equalities

$$\alpha = k_{g_s} - k_n t_r \text{ and } \lambda = -k_{n_s} - t_r k_g .$$

Now, we assume that the velocity of the curve is of the form

$$\gamma = \frac{1}{k_g^2 + k_n^2} \left\{ \begin{array}{c} (k_g k_{g_s} + k_n k_{n_s})_s - (k_{n_s} + k_{g_s} t_r)^2 \\ -(k_{g_s} - k_n t_r)^2 + k_{g_t} k_n - k_{n_t} k_g \end{array} \right\}.$$

For a solution of smoke ring equation, the velocity vector is obtained as follows:

$$r_t = r_s \times r_{ss} = k_g n - k_n g.$$

Theorem 3.2

If $r = r(s,t)$ is a NLS surface where $r = r(s,t)$ is a unit speed curve with normal vector field for all t, then the Gaussian curvature K and mean curvature H of $r = r(s,t)$ are

$$K = \frac{1}{k_g^2 + k_n^2}\left\{-\left(k_g^2 + k_n^2\right)\gamma + k_{g_t}k_n - k_{n_t}k_g - \frac{(\alpha k_n + \lambda k_g)}{k_g^2 + k_n^2}\right\},$$

$$H = \frac{1}{2\left(k_g^2 + k_n^2\right)\sqrt{k_g^2 + k_n^2}}\left\{\left(k_g^2 + k_n^2\right)\gamma - k_{g_t}k_n + k_{n_t}k_g - \left(k_g^2 + k_n^2\right)^2\right\}$$

respectively. Here, k_n is the normal curvature function and k_g is the geodesic curvature of the curve $r = r(s,t)$ for all t.

Proof. We have found the coefficients of first fundamental forms of surface $r = r(s,t)$ as $E = 1$, $F = 0$ and $G = k_g^2 + k_n^2$. Thus we say that $EG - F^2 = k_g^2 + k_n^2$. Normal vector field of the NLS surface is given by

$$N = \frac{r_s \times r_t}{\|r_s \times r_t\|} = \frac{-k_g g - k_n n}{\sqrt{k_g^2 + k_n^2}}.$$

After computations, one can easily obtain Gaussian curvature K and mean curvature H of the surface $r(s,t)$ are

$$K = \frac{1}{k_g^2 + k_n^2}\left\{-\left(k_g^2 + k_n^2\right)\gamma + k_{g_t} k_n - k_{n_t} k_g - \frac{(\alpha k_n + \lambda k_g)}{k_g^2 + k_n^2}\right\},$$

$$H = \frac{1}{2\left(k_g^2 + k_n^2\right)\sqrt{k_g^2 + k_n^2}}\left\{\left(k_g^2 + k_n^2\right)\gamma - k_{g_t} k_n + k_{n_t} k_g - \left(k_g^2 + k_n^2\right)^2\right\}$$

respectively.

Theorem 3.3

Suppose $r = r(s,t)$ is a NLS surface in $\mathbb{E}^3$, the s -parameter curves of the surface $r = r(s,t)$ are geodesics.

Proof. Suppose $r = r(s,t)$ is a NLS surface such that $r = r(s,t)$ is unit speed curve for all t. We know that

$$r_{ss} = k_n n + k_g g.$$

Thus, r_{ss} is parallel to surface normal which means s -parameter curves of the surface are geodesics.

Remark. Since the s -parameter curves of NLS surface are geodesics, this implies that $k_g = 0$.

Theorem 3.4

Suppose $r = r(s,t)$ is a NLS surface in $\mathbb{E}^3$. t -parameter curves of the surface $r = r(s,t)$ are geodesics if and only if $k_{n_s} = 0$.

Proof. By above remark, we have

$$r_t = k_g n - k_n g = -k_n g.$$

Thus, we obtain the tangent vector field of t -parameter curve as

$$\bar{t} = \frac{r_t}{\|r_t\|} = -g.$$

Then, the third vector field of Darboux frame of t -parameter curve is found

$$\overline{g} = -n \times \overline{t} = -n \times (-g) = -T.$$

The geodesic curvature of the t -parameter curve is obtained as

$$\overline{k_g} = \langle \overline{t}_t, \overline{g} \rangle = \langle \lambda T - \gamma n, -T \rangle = \lambda$$

where

$$\lambda = -k_{n_s} - k_g t_r = -k_{n_s}.$$

Therefore $t-$parameter curves of the surface $r = r(s,t)$ are geodesics if and only if $k_{n_s} = 0$.

Theorem 3.5

Suppose $r = r(s,t)$ *is a NLS surface in* $\mathbb{E}^3$, *NLS equation*

$$iq_t + q_{ss} + vq\,|q|^2 = 0$$

is provide with Hasimoto transformation $q = \pm k_n e^{i\int -t_r ds}$.

Proof. We know that Hasimoto transformation with Frenet Frame is obtained as follow

$$q = \kappa e^{i\sigma}$$

such that $\sigma = \int \tau ds$. Since the following equations are satisfied

$$\kappa^2 = k_n^2 + k_g^2$$
$$t_r(s) = -\beta'(s) - \tau(s),$$

and $k_g = 0$, then we obtain that Hasimoto transformation with Darboux frame as follows

$$q = \pm k_n e^{i \int -t_r ds}.$$

By taking derivative of q according to t parameter, we get

$$q_t = \left(k_{n_t} + ik_n t_{r_t}\right) e^{i\int -t_r ds}. \tag{3.8}$$

We also know that

$$k_{n_t} = \alpha_s + \lambda t_r \text{ and } t_{r_t} = \gamma_s - \lambda k_n$$

such that

$$\alpha = -k_n t_r,\ \lambda = -k_{n_s},\ \gamma = \frac{1}{k_n}\left(k_{n_{ss}} - k_n t_r^2\right)$$

by equations 3.5 and 3.6.

Therefore, we obtain

$$k_{n_t} = -2k_{n_s} t_r - k_n t_{r_s}$$

and

$$t_{r_t} = \left(\frac{k_{n_{ss}}}{k_n} - t_r^2 + \frac{k_n^2}{2} \right)_s$$

If we substitude above equations into Equation 3.8, then we obtain

$$q_t = \left(-2k_{n_s} t_r - k_n t_{r_s} + ik_n \left(\frac{k_{n_{ss}}}{k_n} - t_r^2 + \frac{k_n^2}{2} \right) \right) e^{i \int -t_r ds}.$$

Thus, we also get

$$q_{ss} = \left(k_{n_{ss}} + 2ik_{n_s} t_r + ik_n t_{r_s} - k_n t_r^2 \right) e^{i \int -t_r ds}$$

According to above findings, it is seen that the following NLS equation is satisfied

$$iq_t + q_{ss} + vq\left|q\right|^2 = 0.$$

The vortex filament equation can be rewriten in terms of Darboux frame as follows

$$r_t = k_g n - k_n g = -k_n g$$

such that

$$n = \cos\left(\int -t_r ds\right) N + \sin\left(\int -t_r ds\right) B,$$

$$g = -\sin\left(\int -t_r ds\right) N + \cos\left(\int -t_r ds\right) B.$$

Corollary 3.6

NLS surface $r = r(s,t)$ *is developable if and only if*

$$(k_n k_{n_s})_s = k_{n_s}^2 + k_n^2 t_r^2 - t_r.$$

Corollary 3.7

NLS surface $r = r(s,t)$ *is minimal surface if and only if*

$$(k_n k_{n_s})_s = 1 + k_{n_s}^2 + k_n^2 t_r^2.$$

Consider the NLS surface with the following parametrization

$$r(s,t)=\left(s-\frac{2}{3}\tanh(3s),-\frac{2}{3}\operatorname{sech}(3s)\cos(9t),-\frac{2}{3}\operatorname{sech}(3s)\sin(9t)\right).$$

Let us examine Darboux frame of the curves $r=r(s,t)$ for all $t\in\mathbb{R}$ and NLS surface. We find Darboux vector fields as follows:

$$T(s,t)=(2\tanh^2(3s)-1,2\operatorname{sech}(3s)\tanh(3s)\cos(9t),2\operatorname{sech}(3s)\tanh(3s)\sin(9t$$

$$n(s,t)=(2\tanh(3s)\operatorname{sech}(3s),(1\text{-}2\operatorname{sech}^2(3s))\cos(9t),(1\text{-}2\operatorname{sech}^2(3s))\sin(9t)),$$

$$g(s,t)=(0,\sin(9t),-\cos(9t)).$$

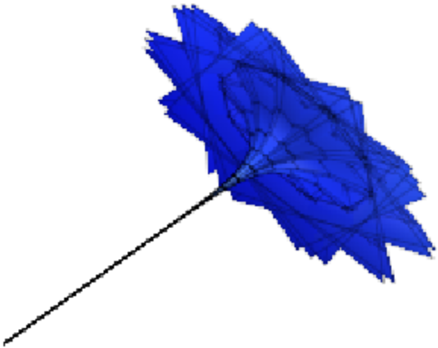
Then we obtain Darboux curvatures as

$$
\begin{aligned}
k_n(s,t) &= -6\,\mathrm{sech}(3s),\\
k_g(s,t) &= 0,\\
t_r(s,t) &= 0.
\end{aligned}
$$

Furthermore, we get Gaussian and Mean curvature of the NLS surface

$$
\begin{aligned}
K(s,t) &= -9(\tanh^2(3s) - \mathrm{sech}^2(3s))\\
H(s,t) &= -\frac{9}{2}\,\mathrm{sech}(3s) + \frac{3}{4\,\mathrm{sech}(3s)}.
\end{aligned}
$$

The following figure illustrate NLS surface



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