On the Ephemerides of the Planets

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An **ephemeris** is an established astronomical term for the tabulated data representing the position of natural (not man-made) astronomical objects at a given time point in the sky.

Historically such information has been provided in the form of printed tables at regular time intervals. Analytically, we shall also define the ephemerides as **time dependent curves that are followed by the planets**.

For practical reasons one may use lower accuracy formulae for observation scheduling, telescope pointing and prediction of regular phenomena such as moon cycles up to planning and design of regular spacecraft missions.

The first rigorous scientific astronomical theory that gained world-wide exposure was Almagest by the Greek astronomer Ptolemy who worked in Alexandria during the second century BC. The Platonic idea of circular and the principle to "save appearances" were central to his theory. It is fascinating to point that the "save appearances" principle is the ancient origin of data fitting routines.

The idea of Ptolemy for geocentric world posed substantive challenges for the astronomers as the data fitting methods were of ever growing complexity. In his Almagest Ptolemy included easy-to-use tables, which aided the reader in calculating the future or past position of different celestial bodies.

This unwarranted complications led to the Copernicus idea for a heliocentric system. However, even after this scientific breakthrough, Copernicus' tables were still based on circular orbits and epicycles (circles rolling on other circles). In essence epicycles were the predecessors of the Fourier series used nowadays to represent planetary dynamics.

In the period 1609 to 1619 Johannes Kepler formulated and published the three laws named after him that describe the orbits of planets rotating around single star. Namely, the planets move in ellipses with the Sun in at one focus; the radius vector from the Sun to a planet sweeps out equal areas in equal times; the square of the orbital period of a planet is proportional to the cube of its semi-major axis.

The three laws of Kepler provide a first approximation for the planetary trajectories, i. e. simplest analytical ephemerides. The motion of every planet is fully defined by six orbital elements: semi-major axis, a, eccentricity, e, inclination, I, longitude of perihelia, ϖ, longitude of node, Ω, and mean longitude λ. Five of these elements are constants, which fix the Keplerian ellipse, while the mean longitude

$$\lambda \;=\; n\cdot(t - t_0) + \varpi$$

is linear in time function that defines the position of the planet on this ellipse. The constant t_0 is a time of perihelion passage. The so–called mean motion n is connected with the semi-major axes a and the relative mass μ of the planet via Kepler's Third low:

$$n := \sqrt{1+\mu}\, a^{-3/2}\,, \qquad \mu := \frac{\text{mass of the planet}}{\text{mass of the Sun}}.$$

Although Kepler did propose a "magnetic field" to explain the planetary dynamics, his laws were purely empirical.

The one of the most important works in the history of science, Principia, was published by Isaac Newton in 1687. Combining the Newton's inverse square low of gravitation and the three laws of motion made possible to derive the three laws of Kepler. Now, in heliocentric coordinates, the dynamics of the eight principal planets in the solar system is governed by a system of 24 second order ordinary differential equations (ODE's):

$$\frac{d^2\mathbf{r}_j}{dt^2} = -(1+\mu_j)\frac{\mathbf{r}_j}{|\mathbf{r}_j|^3} + \frac{\partial R_j}{\partial \mathbf{r}_j}\,, \qquad j = 1,\ldots,8\ .$$

Here the suffix j stays for Mercury, Venus, EM-Bary – the barycenter of Earth and Moon, Mars, Jupiter, Saturn, Uranus and Neptune correspondingly. For the sake of simplicity, the equations of motion have written in vector form: $\mathbf{r}_j$ denotes the position of the jth planet, and $|\mathbf{r}_j|$ – its the distance to the Sun.

The disturbing function

$$R_j := \sum_{s\neq j}\mu_s\left[\frac{1}{(\mathbf{r}_j\mathbf{r}_j+\mathbf{r}_s\mathbf{r}_s-2\mathbf{r}_j\mathbf{r}_s)^{1/2}} - \frac{\mathbf{r}_j\mathbf{r}_s}{|\mathbf{r}_s|^3}\right]\,,$$

where $\mathbf{r}_b\mathbf{r}_c$ is the Euclidean dot product, corresponds to the gravitation potentials of the planets.

Unlike the Keplerian theory, where all R_j are considered to vanish, now every planet is moving on a slowly changing ellipse and the mean longitudes λ_j are almost linear functions of the time. These **small perturbations** appear as a result of the relatively small but non–zero masses μ_j.

It is convenient to change the Newton' equations of motion with an equivalent system of 48 first order ODE's, called Lagrangian Planetary Equations (LPE). The 48 new variables are the orbital elements $(a_j, e_j, I_j, \varpi_j, \Omega_j, \lambda_j)$, $j = 1, 2, \ldots, 8$. In Cartesian coordinates,

$$x = (\cos\omega\cos\Omega - \sin\omega\sin\Omega\cos I)\, X - (\sin\omega\cos\Omega + \cos\omega\sin\Omega\cos I)\, Y,$$

$$y = (\cos\omega\sin\Omega + \sin\omega\cos\Omega\cos I)\, X - (\sin\omega\sin\Omega - \cos\omega\cos\Omega\cos I)\, Y,$$

$$z = \sin I\ \sin\omega\ X\ +\ \sin I\ \cos\omega\ Y\ ,$$

where $\omega := \varpi - \Omega$ is called argument of perihelia and

$$\begin{aligned} X &:= a\cdot\left(\cos l - e - \sum_{s=1}^{\infty} \frac{e^s}{s!} \frac{d^{s-1}\sin^{s+1} l}{dl^{s-1}}\right)\ , \\ Y &= a\sqrt{1-e^2}\cdot\left(\sin l + \sum_{s=1} \frac{e^s}{s!} \frac{d^{s-1}(\sin^s l\cos l)}{dl^{s-1}}\right)\ . \end{aligned}$$

The dynamics of motion is governed by a system of 48 first order ODE's (six equations for each planet), named Lagrangian Planetary Equations:

$$
\begin{aligned}
\frac{da}{dt} &= +\frac{2}{na}\frac{\partial R}{\partial \lambda} \,, \\
\frac{de}{dt} &= -\frac{(1-e^2)^{1/2}}{na^2 e}\frac{\partial R}{\partial \varpi} + \frac{1-e^2-(1-e^2)^{1/2}}{na^2 e}\frac{\partial R}{\partial \lambda} \,, \\
\frac{dI}{dt} &= -\frac{1}{na^2(1-e^2)^{1/2}\sin I}\frac{\partial R}{\partial \Omega} - \frac{\tan(I/2)}{na^2(1-e^2)^{1/2}}\Big(\frac{\partial R}{\partial \lambda} + \frac{\partial R}{\partial \varpi}\Big) \,, \\
\frac{d\Omega}{dt} &= +\frac{1}{na^2(1-e^2)^{1/2}\sin I}\frac{\partial R}{\partial I} \,, \\
\frac{d\varpi}{dt} &= \frac{(1-e^2)^{1/2}}{na^2 e}\frac{\partial R}{\partial e} + \frac{\tan(I/2)}{na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial I} \,, \\
\frac{d\lambda}{dt} &= n - \frac{2}{na}\frac{\partial R}{\partial a} - \frac{1-e^2-(1-e^2)^{1/2}}{na^2 e}\frac{\partial R}{\partial e} + \frac{\tan(I/2)}{na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial I} \,.
\end{aligned}
$$

For simplicity, all the running from 1 to 8 indexes j have been omitted and one should understand a_j instead of a, e_j instead of e, ... , R_j instead of R.

After these standard considerations we go to the problem in our talk. In order to find the first order perturbation in LPE, it is necessary to expand the Newtonian potential R in double Fourier series in mean longitudes. That is, for any pair of planets $(\mathbf{r}, \mathbf{r}')$ one should expand the inverse of their distance:

$$\begin{aligned}
\frac{1}{(\mathbf{rr} + \mathbf{r'r'} - 2\mathbf{rr'})^{1/2}} &= \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \\
&= \frac{constant}{\sqrt{1 - P(\lambda, \lambda')}} \\
&= \sum A_{k,k'} \cos\big(k\lambda + k'\lambda'\big) + \sum B_{k,k'} \sin\big(k\lambda + k'\lambda'\big)
\end{aligned}$$

where $(k, k') \in \mathbb{Z}^2$.

Here $P(\lambda, \lambda')$ is a double Fourier series, depending on the initial conditions for our planetary system. In fact, the sum of its Fourier coefficients is less then 1 but could be about 0.95.

In fact, the sum Fourier coefficients of P is less then 1 but could be about 0.95. That's why, the formula

$$\frac{1}{\sqrt{1-P}} \;=\; 1+\frac{1}{2}P+\frac{1\cdot 3}{2\cdot 4}P^2+\cdots+\frac{(2m-1)!!}{(2m)!!}\,P^m+\cdots$$

does not work. Indeed, P has usually about 100 impact harmonics. To calculate the valuable harmonics of $(1-P)^{-1/2}$, we need of (say) $m=1000$ in the above formula. It is impossible to be calculated.

The main result of this talk is the following calculation

$$
\begin{aligned}
\frac{1}{\sqrt{1-P}} &= \frac{1+P/2}{\sqrt{(1+P/2)^2\cdot(1-P)}} = \frac{Q_1}{\sqrt{1-P_1}} \\
&= \frac{(1+P_1/2)\cdot Q_1}{\sqrt{(1+P_1/2)^2\cdot(1-P_1)}} = \frac{Q_2}{\sqrt{1-P_2}} \\
&= \frac{(1+P_2/2)\cdot Q_2}{\sqrt{(1+P_2/2)^2\cdot(1-P_2)}} = \frac{Q_3}{\sqrt{1-P_3}} \\
&= \cdots\cdots\cdots\cdots \\
&= \frac{(1+P_5/2)\cdot Q_5}{\sqrt{(1+P_5/2)^2\cdot(1-P_5)}} = \frac{Q_6}{\sqrt{1-P_6}}
\end{aligned}
$$

It turns out that $|P_6| < 10^{-6}$ and could be truncated to zero. Thus the problem has been solved sufficiently well:

$$
\frac{1}{\sqrt{1-P}} = Q_6 \pm 10^{-6} .
$$

Recall that the mean longitudes are linear on the time t; it is elementary to integrate

$$
\int Q_6 dt
$$

and so to find some very satisfactory ephemerides for the planets.

Mercury

$$
\begin{aligned}
a &= 0.387100 + 10^{-7}\big[-10\cos(3\lambda_5) - 7\cos(2\lambda_1 - 5\lambda_2) + 7\sin(\lambda_1 - 2\lambda_2) \\
&\quad +7\cos(2\lambda_5) + 4\cos(2\lambda_1 - 2\lambda_2) - 2\sin(2\lambda_1 - 2\lambda_2) + \cdots \big] \\
e &= 0.205634 + 3{\cdot}10^{-8}\, t + 10^{-7}\big[-51\sin(\lambda_1 - 2\lambda_2) - 51\cos(2\lambda_5) \\
&\quad +31\sin(2\lambda_5) + 25\sin(2\lambda_1 - 5\lambda_2) - 15\sin(\lambda_5) - 15\sin(\lambda_1 - 3\lambda_2) + \cdots \big] \\
I &= 0.122259 + 10^{-7}\big[9\sin(2\lambda_5) - 3\cos(2\lambda_5) + 3\cos(2\lambda_1 - 5\lambda_2) + 2\cos(\lambda_1 - 2\lambda_2) \\
&\quad -2\sin(\lambda_1 - 4\lambda_3) - 2\sin(2\lambda_1 - 5\lambda_2) - 2\sin(\lambda_1 - 3\lambda_2) + \cdots \big] \\
\varpi &= 1.351886 + 0.0000041\, t + 10^{-6}\big[93\cos(2\lambda_5) - 32\sin(2\lambda_5) - 21\cos(\lambda_1 - 2\lambda_2) \\
&\quad +15\cos(\lambda_1 - 3\lambda_2) + 12\cos(2\lambda_1 - 5\lambda_2) + 11\cos(2\lambda_6) + \cdots \big] \\
\Omega &= 0.843535 - 0.0000036\, t + 10^{-7}\big[-40\cos(2\lambda_5) + 27\cos(2\lambda_1 - 5\lambda_2) \\
&\quad -27\sin(\lambda_5) + 13\cos(\lambda_1 - 2\lambda_2) - 13\sin(\lambda_1 - \lambda_2) + 13\cos(\lambda_1 - 4\lambda_3) + \cdots \big] \\
\lambda &= 4.1520838\, t + 4.402599 + 10^{-7}\big[85\cos(0.176608\, t - 7.075472) \\
&\quad -73\cos(0.168605\, t + 1.200662) - 33\cos(0.724453\, t + 5.125803) \\
&\quad +24\cos(0.901061\, t - 1.949669) - 26\sin(0.0843025\, t + 0.600331) + \cdots \big]
\end{aligned}
$$

Venus

$$
\begin{aligned}
a &= 0.723336 + 10^{-7}\big[-60\cos(3\lambda_5) + 43\cos(2\lambda_2 - 2\lambda_3) + 29\cos(2\lambda_2 - 2\lambda_5) \\
&\quad +27\cos(3\lambda_2 - 3\lambda_3) + 17\cos(\lambda_2 - \lambda_3) + 17\cos(4\lambda_2 - 4\lambda_3) + \cdots \big] \\
e &= 0.006764 - 7\cdot 10^{-8}\, t + 10^{-6}\big[35\cos(2\lambda_5) + 17\cos(3\lambda_2 - 5\lambda_3) \\
&\quad +17\cos(\lambda_2 - 2\lambda_3) + 17\sin(\lambda_2 - 3\lambda_3) - 17\cos(2\lambda_2 - 3\lambda_3) + \cdots \big] \\
I &= 0.059258 + 10^{-7}\big[-207\sin(8\lambda_2 - 13\lambda_3) - 6\sin(3\lambda_2 - 5\lambda_3) \\
&\quad +5\sin(2\lambda_5) - 4\cos(8\lambda_2 - 13\lambda_3) - \sin(5\lambda_2 - 8\lambda_3) - \sin(2\lambda_2 - 4\lambda_3) + \cdots \big] \\
\varpi &= 2.289114 + 6\cdot 10^{-8}\, t + 10^{-4}\big[62\cos(2\lambda_5) + 34\cos(3\lambda_2 - 5\lambda_3) \\
&\quad -25\cos(2\lambda_2 - 3\lambda_3) + 22\sin(2\lambda_2 - 3\lambda_3) + 19\cos(2\lambda_3 - \lambda_2) + \cdots \big] \\
\Omega &= 1.338327 - 0.0000077\, t + 10^{-6}\big[20\sin(3\lambda_2 - 5\lambda_3) - 16\sin(\lambda_2 - \lambda_3) \\
&\quad -10\sin(2\lambda_5) - 8\sin(2\lambda_2 - 2\lambda_3) + 6\cos(3\lambda_2 - 5\lambda_3) - \sin(\lambda_5) + \cdots \big] \\
\lambda &= 1.6255062\, t + 3.176132 + 10^{-6}\big[21\cos(0.168605\, t + 1.200662) \\
&\quad +16\cos(0.123446\, t - 0.761212) - 10\sin(0.625516\, t + 1.422696) \\
&\quad -10\cos(0.251035\, t + 1.091954) - 8\sin(1.251033\, t + 2.845392) + \cdots \big]
\end{aligned}
$$

EM–Bary

$$
\begin{aligned}
a &= 0.999995 + 10^{-6}\big[-14\cos(3\lambda_5) + 11\cos(2\lambda_3 - 2\lambda_5) + 8\cos(\lambda_2 - \lambda_3) \\
&\quad +5\cos(\lambda_3 - 3\lambda_5) - 4\cos(2\lambda_2 - 2\lambda_3) - 3\cos(4\lambda_5) + \cdots \big] \\
e &= 0.016697 - 10^{-7}\, t + 10^{-6}\big[18\sin(2\lambda_2 - 3\lambda_3) + 18\cos(2\lambda_5) \\
&\quad -18\sin(\lambda_3 - 2\lambda_5) - 12\sin(\lambda_5) + 6\sin(\lambda_3) - 6\sin(\lambda_2 - 2\lambda_3) + \cdots \big] \\
I &= 10^{-6} + 10^{-7}\big[-10\cos(8\lambda_2 - 13\lambda_3) + 9\sin(3\lambda_2 - 5\lambda_3) - 7\sin(2\lambda_5) \\
&\quad +5\sin(8\lambda_2 - 13\lambda_3) - 4\sin(8\lambda_2 - \lambda_3) + 4\sin(5\lambda_2 - 8\lambda_3) + \cdots \big] \\
\varpi &= 1.796852 + 0.0000088\, t + 10^{-4}\big[53\cos(2\lambda_5) - 20\cos(3\lambda_2 - 5\lambda_3) \\
&\quad +13\cos(8\lambda_3 - 5\lambda_2) + 12\cos(2\lambda_2 - 3\lambda_3) - 11\cos(\lambda_3 - 2\lambda_5) + \cdots \big] \\
\Omega &= 2.321956 - 10^{-8}\, t - 0.7\sin(3\lambda_2 - 5\lambda_3) + 0.3\sin(8\lambda_3 - 5\lambda_2) + \cdots \\
\lambda &= 0.9999991\, t + 1.753415 + 10^{-6}\big[39\cos(0.168605\, t + 1.200662) \\
&\quad +35\sin(0.625516\, t + 1.422696) - 21\cos(0.123446\, t - 0.761212) \\
&\quad +14\cos(0.127589\, t + 1.85317) - 12\sin(0.084302\, t + 0.600331 + \cdots \big]
\end{aligned}
$$

Mars

$$
\begin{aligned}
a &= 1.523713 + 10^{-6}\big[\,66\cos(2\lambda_4 - 2\lambda_5) + 42\cos(2\lambda_5) + 24\cos(\lambda_2 - 3\lambda_4) \\
&\quad -20\cos(\lambda_4 - 2\lambda_5) - 16\cos(3\lambda_5) + 16\cos(3\lambda_4 - 3\lambda_5) + \cdots\big] \\
e &= 0.093545 + 14\cdot 10^{-8}\,t + 10^{-5}\big[-17\cos(2\lambda_5) + 8\cos(\lambda_4 - 2\lambda_5) \\
&\quad +4\cos(3\lambda_4 - \lambda_2) - 4\cos(\lambda_5) - 4\cos(\lambda_4 + 2\lambda_5) + 3\sin(\lambda_4 - 2\lambda_5) + \cdots\big] \\
I &= 0.032282 + 10^{-7}\big[\,13\sin(2\lambda_5) - 3\sin(2\lambda_4 - 2\lambda_5) - 4\cos(2\lambda_4 - 2\lambda_5) \\
&\quad -4\sin(\lambda_4 - \lambda_5) - 3\sin(\lambda_5) - 3\sin(\lambda_4 - 3\lambda_5) + 2\sin(4\lambda_4 - 2\lambda_3) + \cdots\big] \\
\varpi &= -0.417864 + 0.0000123\,t + 10^{-5}\big[-80\sin(\lambda_4 - 2\lambda_5) - 77\cos(2\lambda_5) \\
&\quad -44\sin(\lambda_5) + 36\cos(\lambda_4 - 2\lambda_5) - 21\cos(\lambda_5) + 20\cos(3\lambda_4 - \lambda_5) + \cdots\big] \\
\Omega &= 0.864882 - 0.0000083\,t + 10^{-6}\big[\,33\sin(2\lambda_5) + 23\cos(\lambda_5) + 20\cos(2\lambda_5) \\
&\quad -18\sin(2\lambda_4 - \lambda_3) - 13\cos(2\lambda_4 - \lambda_3) - 13\sin(\lambda_4 - \lambda_5) + 10\sin(2\lambda_6) + \cdots\big] \\
\lambda &= 0.5316694\,t - 0.079512 + 10^{-6}\big[-45\sin(0.894746\,t - 1.359607) \\
&\quad +38\sin(0.063353\,t - 1.912383) + 30\sin(0.468322\,t + 1.832910) \\
&\quad -21\sin(0.447373\,t - 0.679803) - 20\sin(0.084302\,t + 0.600331) + \cdots\big]
\end{aligned}
$$

Jupiter

$$
\begin{aligned}
a &= 5.207181 + 10^{-4}\big[\,49\cos(2\lambda_5 - 5\lambda_6) + 7\cos(2\lambda_5 - 2\lambda_6) + 4\cos(\lambda_5 - 3\lambda_6) \\
&\quad -3\cos(\lambda_5 - 2\lambda_6) + 3\cos(3\lambda_5 - 3\lambda_6) - 2\cos(2\lambda_5 - 4\lambda_6) + \cdots\big] \\
e &= 0.045431 + 2\cdot 10^{-7}\,t + 10^{-4}\big[-23\cos(2\lambda_5 - 5\lambda_6) + 6\cos(\lambda_5 - 2\lambda_6) \\
&\quad -4\cos(\lambda_5 - 3\lambda_6) - 3\sin(2\lambda_5 - 5\lambda_6) - 2\sin(\lambda_5 - 2\lambda_6) - 2\cos(2\lambda_6) + \cdots\big] \\
I &= 0.022766 + 10^{-6}\big[\,21\sin(2\lambda_5 - 5\lambda_6) - 5\sin(3\lambda_5 - 7\lambda_6) - 4\sin(\lambda_5 - 3\lambda_6) \\
&\quad +2\sin(2\lambda_5 - 4\lambda_6) - 2\sin(\lambda_5 - \lambda_6) + \cos(\lambda_5 - 3\lambda_6) + \cdots\big] \\
\varpi &= 0.259603 + 0.0000049\,t + 10^{-3}\big[\,14\cos(2\lambda_5 - 5\lambda_6) - 13\sin(\lambda_5 - 2\lambda_6) \\
&\quad -3\cos(\lambda_5 - 2\lambda_6) + 2\cos(\lambda_5 - 3\lambda_6) - 2\sin(2\lambda_5 - 3\lambda_6) + \cdots\big] \\
\Omega &= 1.752718 + 0.0000050\,t + 10^{-5}\big[-68\cos(2\lambda_5 - 5\lambda_6) + 68\sin(2\lambda_5 - 5\lambda_6) \\
&\quad -14\sin(\lambda_5 - 3\lambda_6) + 14\sin(\lambda_5 - \lambda_6) - 11\sin(\lambda_5 - 2\lambda_6) + \cdots\big] \\
\lambda &= 0.0842977\,t + 0.602143 + 10^{-4}\big[\,20\cos(0.0011945\,t + 3.158668) \\
&\quad -2\sin(0.0163827\,t - 1.143401) + 2\cos(0.0175772\,t + 2.015267) \\
&\quad -2\sin(0.1006853\,t - 0.543070) - \sin(0.0503426\,t - 0.271535) + \cdots\big]
\end{aligned}
$$

Saturn

$$
\begin{aligned}
a &= 9.463096 + 10^{-3}\big[-43\cos(2\lambda_5 - 5\lambda_6) + 34\cos(\lambda_5 - \lambda_6) - 3\cos(2\lambda_5 - 2\lambda_6) \\
&\quad +3\cos(3\lambda_5 - 7\lambda_6) + 3\cos(\lambda_5 - 2\lambda_6) - 3\cos(\lambda_5 - 3\lambda_6) + \cdots\big] \\
e &= 0.052517 - 43\cdot 10^{-8}\,t + 10^{-4}\big[20\sin(\lambda_5) + 13\sin(\lambda_6) - 9\cos(\lambda_5 - \lambda_6) \\
&\quad -6\sin(\lambda_5 - 2\lambda_6) + 5\sin(2\lambda_5 - 5\lambda_6) + 5\sin(2\lambda_5 - 3\lambda_6) + \cdots\big] \\
I &= 0.043430 + 10^{-6}\big[-19\cos(\lambda_5 - \lambda_6) + 18\cos(2\lambda_5 - 5\lambda_6) + 10\sin(\lambda_5 - 3\lambda_6) \\
&\quad +6\sin(\lambda_5 + \lambda_6) - 4\sin(2\lambda_6) + 4\sin(2\lambda_5 - 4\lambda_6) + \cdots\big] \\
\varpi &= 1.202934 + 0.0000128\,t + 10^{-2}\big[-47\cos(2\lambda_5 - 5\lambda_6) - 4\cos(\lambda_5) \\
&\quad -3\cos(\lambda_5 + \lambda_6) + 2\cos(3\lambda_5 - 7\lambda_6) - 2\cos(\lambda_6) - 2\sin(2\lambda_5 - 5\lambda_6) + \cdots\big] \\
\Omega &= 1.985769 - 0.0000072\,t + 10^{-4}\big[17\cos(2\lambda_5 - 5\lambda_6) - 9\sin(2\lambda_5 - 5\lambda_6) \\
&\quad +2\sin(\lambda_5 - \lambda_6) + 2\sin(\lambda_5 - 2\lambda_6) + \sin(\lambda_5 - 3\lambda_6) - \cos(3\lambda_5 - 7\lambda_6) + \cdots\big] \\
\lambda &= 0.0340456\,t + 0.854562 + 10^{-4}\big[206\cos(0.001195\,t - 3.158668) \\
&\quad +62\sin(0.050343\,t - 0.271535) + 12\cos(0.015188\,t - 4.302069) \\
&\quad -10\cos(0.084302\,t + 0.600331) - 9\cos(0.018772\,t + 5.173935) + \cdots\big]
\end{aligned}
$$

Uranus

$$
\begin{aligned}
a &= 19.193517 + 10^{-3}\big[\,80\cos(\lambda_5-\lambda_7)+21\cos(\lambda_6-\lambda_7)-19\cos(\lambda_5-6\lambda_7)\\
&\quad -10\cos(\lambda_5-5\lambda_7)-7\cos(\lambda_5-4\lambda_7)+6\cos(\lambda_6-3\lambda_7)+\cdots\big]\\
e &= 0.055514-4\cdot 10^{-8}\,t+10^{-4}\big[-51\cos(\lambda_5-6\lambda_7)-27\cos(\lambda_5-5\lambda_7)\\
&\quad -27\cos(\lambda_5)-24\cos(\lambda_5+\lambda_7)-21\cos(\lambda_5+2\lambda_7)+21\cos(\lambda_7-2\lambda_8)+\cdots\big]\\
I &= 0.013472+10^{-6}\big[\,11\sin(\lambda_5-\lambda_7)+10\sin(\lambda_7-2\lambda_8)-9\sin(2\lambda_7-4\lambda_8)\\
&\quad +7\cos(\lambda_5-8\lambda_7)+7\sin(4\lambda_7-8\lambda_8)+6\cos(\lambda_5-\lambda_7)+\cdots\big]\\
\varpi &= 3.037667+0.0000022\,t+10^{-3}\big[-57\sin(\lambda_5)-43\sin(\lambda_7-2\lambda_8)\\
&\quad +24\sin(\lambda_5-2\lambda_7)-29\sin(\lambda_7)-18\cos(\lambda_5-6\lambda_7)-13\sin(\lambda_6)+\cdots\big]\\
\Omega &= 1.292003+0.0000021\,t+10^{-5}\big[-92\sin(\lambda_6-3\lambda_7)+79\cos(\lambda_5-\lambda_7)\\
&\quad -65\sin(\lambda_5+\lambda_7)+55\cos(\lambda_6-\lambda_7)-53\sin(\lambda_5-\lambda_7)+\cdots\big]\\
\lambda &= 0.0119291\,t+5.461451+10^{-4}\big[44\sin(0.0724059\,t-4.866705)\\
&\quad +16\sin(0.0220632\,t-4.595170)-11\sin(0.0843025\,t+0.6003311)\\
&\quad -9\sin(0.0118967\,t+5.467036)+5\sin(0.0002328\,t-7.391088)+\cdots\big]
\end{aligned}
$$

Neptune

$$
\begin{aligned}
a &= 30.069648 + 10^{-3}\big[\,148\cos(\lambda_5-\lambda_8) + 36\cos(\lambda_6-\lambda_8) + 8\cos(\lambda_7-2\lambda_8)\\
&\quad +7\cos(2\lambda_5-\lambda_8) + 6\cos(2\lambda_5) + 6\cos(2\lambda_5+\lambda_8) + \cdots\big]\\
e &= 0.005912 + 10^{-8}\,t + 10^{-4}\big[\,24\cos(\lambda_5) + 22\sin(\lambda_5) + 22\cos(\lambda_5+\lambda_8)\\
&\quad -22\cos(\lambda_6-6\lambda_8) + 21\cos(\lambda_5+2\lambda_8) + 20\cos(\lambda_5+3\lambda_8) + \cdots\big]\\
I &= 0.030913 + 10^{-6}\big[-15\sin(\lambda_5-\lambda_8) - 14\cos(\lambda_5-\lambda_8) + 13\sin(\lambda_5+\lambda_8)\\
&\quad +6\sin(2\lambda_7-4\lambda_8) - 4\sin(\lambda_6-\lambda_8) + 3\sin(\lambda_6-\lambda_8) + \cdots\big]\\
\varpi &= 0.951658 + 64\cdot 10^{-8}\,t + 10^{-2}\big[\,28\sin(\lambda_5) - 28\cos(\lambda_5) - 26\cos(\lambda_5+\lambda_8)\\
&\quad +25\cos(\lambda_6-6\lambda_8) - 23\cos(\lambda_5+3\lambda_8) - 22\cos(\lambda_5+4\lambda_8) + \cdots\big]\\
\Omega &= 2.299253 - 17\cdot 10^{-8}\,t + 10^{-5}\big[-47\cos(\lambda_5-\lambda_8) + 46\sin(\lambda_5-\lambda_8)\\
&\quad -36\sin(\lambda_5+\lambda_8) + 23\sin(\lambda_7-2\lambda_8) - 13\cos(\lambda_6-\lambda_8) + \cdots\big]\\
\lambda &= 0.0060812\,t - 0.966916 + 10^{-4}\big[50\sin(0.0782378\,t + 1.562357)\\
&\quad +13\sin(0.0278952\,t + 1.833892) - 12\cos(0.0843025\,t + 0.6003311)\\
&\quad +12\sin(0.0843025\,t + 0.6003311) - 11\cos(0.0903672\,t - 0.3616949) + \cdots\big]
\end{aligned}
$$

Example: Semi-major axis a_7 of Uranus.

According to the website of NASA (based on the paper of Standish et all), the linear approximation of the semi-major axis of Uranus

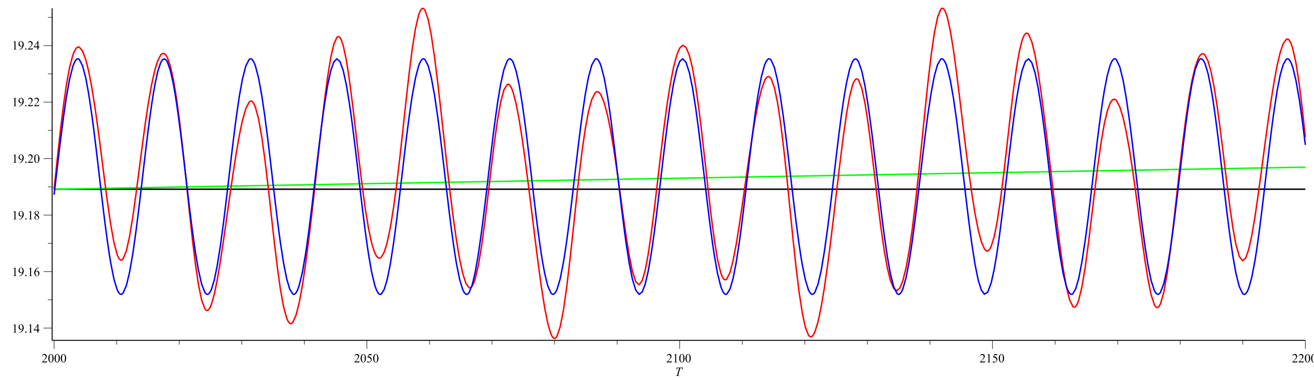
$$a_7(T) \;\approx\; 19.18916 - 0.0000196\,T$$

(measured in AU and T in years) could generate about 8 millions kilometers (about 0.3% relative) inaccuracy in real distance to the Sun. This is the less accurate approximation.

Our approximation reads

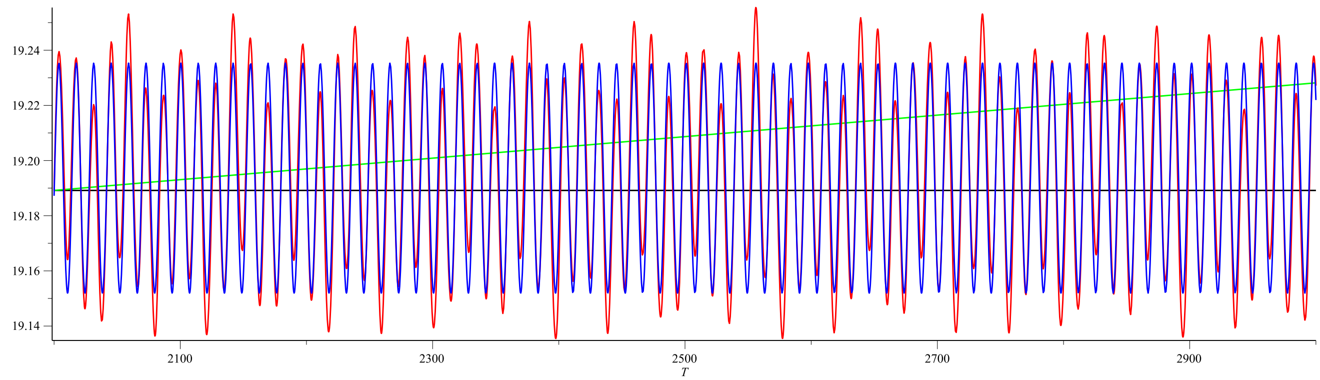
$$\begin{aligned}
&a_7(T) \approx 19.18916 + 10^{-5} \cdot \big[447 - 4169\cos(4.86671 - 0.454939\,T)\\
&-1356\cos(4.59517 - 0.138627\,T) + 252\cos(10.3337 - 0.380191\,T)\\
&+191\cos(6.42906 + 0.036643\,T) + 187\cos(10.0622 - 0.063878\,T)\\
&-177\cos(4.26637 - 0.984628\,T) - 160\cos(5.46704 + 0.074749\,T)\\
&-64\cos(9.19034 - 0.277255\,T) + \cdots\big]\ .
\end{aligned}$$

Even the shorter variant with the first harmonica (which comes from Jupiter), $a_7 \approx 19.19363 - 0.04169\cos(4.86671 - 0.454939\,T)$ gives about 10^{-3} relative inaccuracy, see the blue graph.



The red plot presents almost the exact graph of $a_7(T)$; green – the NASA approximation; black – Kepler's approximation.

If we take the first 10 harmonics, the relative inaccuracy would be about 10^{-4}.



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Don't worry! Be happy!