Azarbijan University of Tarbiat Moallem

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One Dimensional Quasi-Exactly Solvable Differential Equation

- **Introduction to Generalized Master Function Approach and L (Rodriger's operator)**

- **Recursion Relations and Factorization Method**

- **Corresponding Schrodinger Equation L (Rodriger's operator)**

- **An Example**

By generalizing master function of order up to two[9] to polynomial of order up to *k* together with the non-negative weight function *W*(*x*), defined at interval (*a; b*) such that

$$\frac{1}{W(x)}\frac{d}{dx}\left(A(x)W(x)\right)$$

be a polynomial of degree at most (k-1), we can define the operator

$$L = \frac{1}{W(x)}\frac{d}{dx}\left(A(x)W(x)\frac{d}{dx}\right) + B(x)$$

where B(x) is a polynomial of order up to (k - 2). The interval (a, b) is chosen so that, we have A(a)W (a) =A(b)W (b) = 0.

It is straightforward to show that the above defined operator L is a self-adjoint linear operator which at most, maps a given polynomial of order m to another polynomial of order (m + k - 2). Now, by an appropriate choose of B(x) and weight function W (x), the operator L can have an invariant subspace of polynomials of order up to n. Then by choosing the set of orthogonal polynomials $\{\phi_0, \phi_1, ..., \phi_n\}$ defined in the interval (a, b) with respect to the weight function W(x):

$$\int_a^b \phi_m(x)\phi_n(x)W(x)dx = 0 \quad , \qquad for \quad m \neq n,$$

As the base, the matrix elements of the operator *L* on this base will have the following block diagonal form:

$$L_{ij} = 0 \qquad , if \quad \{i \leq n \quad and \quad j \geq n+1\} \quad or \quad \{i \geq n+1 \quad and \quad j \leq n\}.$$

Since, according to the well known theorem of orthogonal polynomials,

$\phi_n(x)$ is orthogonal to any polynomial of order up to n - 1, therefore,

for matrix L we get:

$$L = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$$

where M is an (n + 1) x (n + 1) matrix with matrix elements

$$M_{ij} = \int_a^b dx W(x)\phi_1(x)L(x)\phi_j(x), \qquad i, j = 0, 1, 2, ..., n,$$

and N is an infinite matrix element defined as above with i, j = n + 1.

The block diagonal form of the operator L indicates that by diagonalizing the $(n+1) \times (n+1)$ matrix M, we can find $(n + 1)$ eigenvalues of the operator L together with the related eigenfunctions as linear functions of orthogonal polynomials

$$\{\phi_0, \phi_1, ..., \phi_n\}$$

In order to determine the appropriate B(x) and W (x) for given generalized master function A(x), we Taylor expand those functions:

$$A(x) = \sum_{i=0}^{k} \frac{A^{(i)}(0)}{i!} x^i, \qquad \text{where} \quad A^{(i)}(0) = \frac{d^i A(x)}{dx^i}|_{x=0},$$

$$\frac{(A(x)W(x))'}{W(x)} = \sum_{i=0}^{k-1} \frac{\left(\frac{(AW)'}{W}\right)^{(i)}(0)}{i!} x^i, \qquad \text{where} \quad \left(\frac{(AW)'}{W}\right)^{(i)}(0) = \frac{d^i \left(\frac{(A(x)W(x))'}{W(x)}\right)}{dx^i}|_{x=0},$$

$$B(x) = \sum_{i=0}^{k-2} \frac{B^{(i)}(0)}{i!} x^i, \qquad \text{where} \quad B^{(i)}(0) = \frac{d^i B(x)}{dx^i}|_{x=0}.$$

Then, the existence of invariant subspace of the polynomials of order n of the operator L leads to the following linear equations between the coefficients of above Taylor expansion:

$$-\frac{A^{(i+2)}}{(i+2)!} l(l-1) - \frac{(\frac{(AW)'}{W})^{(i+1)}}{(i+1)!} l + \frac{B^{(i)}}{i!} = 0, \qquad \text{(a)}$$

Where

$$\begin{cases} l = n, & and & i = 1, \quad 2, \quad \dots, \quad k-2 \\ l = n-1, & and & i = 2, \quad 3, \quad \dots, \quad k-2 \\ \dots & \dots & \dots \quad \dots \quad \dots \quad \dots \\ l = n-k+4, & and & i = k-3, \quad k-2 \\ l = n-k+3, & and & i = k-3 \end{cases}$$

The number of above equations, for a given value of k, is (k-1)(k-2)/2. If we are to determine only the unknown function B(x) without having any further constraint on the weight function W(x), then the above (k-1)(k-2)/2 equations should be satisfied with (k-2) coeficients of Taylor expansion of B as the only unknowns, since $B^{(0)}$ can be absorbed in the eigen-spectrum operator L. Therefore, we left with (k-2) unknowns to be determined, where the compatibility of equations (2-9) require k = 3 at most. On the other hand ,if we add the coeficients of Taylor expansions of A(x) an W(x) to our list of unknowns, (to be determined by solving equations

$$-\frac{A^{(i+2)}}{(i+2)!}l(l-1) - \frac{\left(\frac{(AW)'}{W}\right)^{(i+1)}}{(i+1)!}l + \frac{B^{(i)}}{i!} = 0,$$

then their compatibility conditions require that:

$$3(k-1) \geq \frac{(k-1)(k-2)}{2},$$

or $\mathbf{k \leq 8}$, where further investigations show that we can have at most *k* = 4, since for $\mathbf{k \geq 5}$ the coeficients $\mathbf{A^{(k)}(0)}$ and $\left(\frac{\mathbf{(A(x)W(x))'}}{\mathbf{W(x)}}\right)^{\mathbf{(k-1)}}\mathbf{(0)}$ will vanish. Below we summarize the above-mentioned discussion for *k* = 3 and *k* = 4, separately.

Case a: k=3

In this case, B(x) is a first order polynomial where $\mathrm{B}^{(1)}$ can be determined by solving equation:

$$-\frac{A^{(i+2)}}{(i+2)!}l(l-1) - \frac{(\frac{(AW)'}{W})^{(i+1)}}{(i+1)!}l + \frac{B^{(i)}}{i!} = 0,$$

$$B^{(1)} = \frac{n}{2}\left(\frac{A^{(3)}(0)}{3}(n-1) + \left(\frac{(AW)'}{W}\right)^{(2)}\right),$$

which is the only unknown in this case.

Case b: k=4

Again, solving the equation (a) leads to:

$$B^{(1)} = \frac{n}{2}\left(\frac{A^{(3)}(0)}{3}(n-1) + \left(\left(\frac{(AW)'}{W}\right)^{(2)}\right)\right),$$

$$B^{(2)} = -\frac{A^{(4)}}{12}n(n-1),$$

and

$$\left(\frac{(AW)'}{W}\right)^{(3)} = -\frac{A^{(4)}}{2}(n-1).$$

Here, besides having constraint over second order polynomial B(x), we have to put further constraints on the weight function W(x) given in the last equation.

Definitely, we can determine n+1 eigen-spectrum of the operator L, simply by diagonalizing the (n+1)x(n+1) matrix M, since it is a self-adjoint operator in Hilbert space of polynomials and it has a block diagonal form given in

$$L = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$$

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$$-\frac{A^{(i+2)}}{(i+2)!}l(l-1) - \frac{(\frac{(AW)'}{W})^{(i+1)}}{(i+1)!}l + \frac{B^{(i)}}{i!} = 0,$$

$$B^{(1)} = \frac{n}{2}\left(\frac{A \ldots}{\ldots}\right.$$

$$B^{(2)} = -\frac{A^{(4)}}{12}n(n-1),$$

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$$L = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$$

As we are going to see in the end of this section, we can determine its eigen-spectrum analytically, using some recursion relation.

❖ Recursion Relation

Now we show that the eigen-functions of the operator *L* are a generating function for a new set of polynomials $P_m(E)$ where the eigen-function equation of the operator *L* leads to the recursion relation between these polynomials. Quasi-exact solvable constraints

$$-\frac{A^{(i+2)}}{(i+2)!}l(l-1)-\frac{(\frac{(AW)'}{W})^{(i+1)}}{(i+1)!}l+\frac{B^{(i)}}{i!}=0, \qquad N \geq 0$$

will lead to their factorization, that is, $P_{n+N+1}(E) = P_{n+1}(E)Q_N$ for where roots of polynomials $P_{n+1}(E)$ turn out to be the eigen-values of the operator *L*. To achieve these results, first we expand $\tilde{A}(x)$, the eigen-function of *L*, as:

$$\psi(x)=\sum_{m=0}^{\infty}P_m(E)x^m,$$

where eigen-function equation:

$$L\psi(x)=E\psi(x),$$

can be expressed as

$$-A(x)\sum_{m=2}^{\infty} m(m-1)P_m(E)x^{m-2} - \left(\frac{(AW)'}{W}\right)\int_{m=1}^{\infty} mP_m(E)x^{m-1}$$
$$+B(x)\sum_{m=0}^{\infty} P_m(E)x^m = E\sum_{m=0}^{\infty} P_m(E)x^m,$$

which leads to the following recursion relations for the coeficients $P_m(E)$:

$$\left(A^{(1)}(m+1)(m+2) + \left(\frac{(AW)'}{W}\right)^{(0)}(m+2)\right)P_{m+2}(E)$$
$$+\left(\frac{A^{(2)}}{2!}m(m+1) + \left(\frac{(AW)'}{W}\right)^{(1)}(m+1) + E\right)P_{m+1}(E)$$
$$+\left(\frac{A^{(3)}}{3!}m(m-1) + \frac{\left(\frac{(AW)'}{W}\right)^{(2)}}{2!}m - B^{(1)}\right)P_m(E)$$
$$+\left(\frac{A^{(4)}}{4!}(m-1)(m-2) + \frac{\left(\frac{(AW)'}{W}\right)^{(3)}}{3!}m - \frac{B^{(2)}}{2!}\right)P_{m-1}(E) = 0.$$

Below we investigate recursion relations which are obtained for $k = 3$ (cubic $A(x)$) and $k = 4$ (quadratic $A(x)$), separately.

Cubic (A):

In this case the 4-term general recursion relation reduces to the following 3-term recursion relation:

$$\left(A^{(1)}(m+1)(m+2)+\left(\frac{(AW)'}{W}\right)^{(0)}(m+2)\right)P_{m+2}(E)$$
$$+\left(\frac{A^{(2)}}{2!}m(m+1)+\left(\frac{(AW)'}{W}\right)^{(1)}(m+1)+E\right)P_{m+1}(E) \qquad \text{(I)}$$
$$+\left(\frac{A^{(3)}}{3!}m(m-1)+\frac{\left(\frac{(AW)'}{W}\right)^{(2)}}{2!}m-B^{(1)}\right)P_m(E)=0.$$

In order to have finite eigen-spectrum, that is, quasi-exactly differential equation, the above recursion relation should be truncated for some value of $m = n$, which is obviously possible by an appropriate choice of:

$$B^{(1)}=\frac{n}{2}\left(\frac{A^{(3)}(0)}{3}(n-1)+\left(\frac{(AW)'}{W}\right)^{(2)}\right), \qquad \text{(II)}$$

which is in agreement with the result of previous subsection given in

$$B^{(1)}=\frac{n}{2}\left(\frac{A^{(3)}(0)}{3}(n-1)+\left(\frac{(AW)'}{W}\right)^{(2)}\right),$$

Using the recursion relation (I), with $B(1)$ given in (II),we get a factorization of polynomial $P_{n+N+1}(E)$ for $N \geq 0$ in terms of $P_{n+1}(E)$ as follows:

$$P_{n+N+1}(E) = P_{n+1}(E)Q_N(E) \qquad N \geq 0,$$

where , by choosing the eigen-value E as roots of polynomials $P_{n+1}(E)$, all polynomials of order higher than n will vanish.

By using Eq $\psi(x) = \sum_{m=0}^{\infty} P_m(E)x^m$ we obtain eigen function

$$\psi_i(x) = \sum_{m=0}^{n} P_m(E_i)x^m, \qquad i = 0, 1, \ldots, n.$$

where E_i are roots of polynomial $P_{n+1}(E)$.

The above eigen-functions are polynomials of order n, hence they have at most n roots in the interval $(a; b)$, where, according to the well-known oscillation and comparison theorem of second-order linear differential equation [18], these numbers order the eigen-values according to the number of roots of corresponding eigen-functions. Therefore, we can say that the eigen-values thus obtained are the first $n + 1$ eigen-values of the operator L. Using the recursion relation (I), we can evaluate the polynomials $P_m(E)$ in term of $P_0(E)$, where we have chosen $P_0(E) = 1$. We have evaluated the first five polynomials appeared in Appendix (II).

Quadratic (A):

Again in order to truncate the recursion relation and to factorize polynomials $P_{n+N+1}(E)$ in terms of $P_{n+1}(E)$, we should have:

$$B^{(1)} = \frac{n}{2}\left(\frac{A^{(3)}(0)}{3}(n-1) + \left(\frac{(AW)'}{W}\right)^{(2)}\right), \quad \text{(III)}$$

and

$$\frac{B^{(2)}}{2!} = \frac{A^{(4)}}{4!}(n-1)(n-2) + \frac{\left(\frac{(AW)'}{W}\right)^{(3)}}{3!}n,$$

$$\frac{B^{(2)}}{2!} = \frac{A^{(4)}}{4!}n(n-1) + \frac{\left(\frac{(AW)'}{W}\right)^{(3)}}{3!}(n+1).$$

Solving the above equations we get:

$$B^{(2)} = -\frac{A^{(4)}}{12}n(n-1), \quad \text{(IV)}$$

and

$$\left(\frac{(AW)'}{W}\right)^{(3)} = -\frac{A^{(4)}}{2}(n-1). \quad \text{(V)}$$

The equations (III) , (IV) and (V) are the same equations which are required in the reduction of the operator L to its block diagonal form.

Again roots of polynomials P_{n+1} will correspond to n+1 eigen-values of the differential operator L with eigen-functions which can be expressed in term of $P_m(E_i)$ for $m \leq n$, where polynomials $P_m(E)$ can be obtained from recursion relation by choosing $P_0 = 1$ and $P_{-1} = 0$.

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$$+\left(\frac{A^{(4)}}{4!}(m-1)(m-2)+\frac{\left(\frac{(AW)'}{W}\right)^{(3)}}{3!}m-\frac{B^{(2)}}{2!}\right)P_{m-1}(E)=0.$$

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and

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$$\frac{B^{(2)}}{2!} = \frac{A^{(4)}}{4!}n(n-1) + \frac{\left(\frac{(AW)'}{W}\right)^{(3)}}{3!}(n+1).$$

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The equations (III) , (IV) and (V) are the same equations which are required in the reduction of the operator *L* to its block diagonal form.

Again roots of polynomials P_{n+1} will correspond to n+1 eigen-values of the differential operator *L* with eigen-functions which can be expressed in term of $P_m(E_i)$ for $m \leq n$, where polynomials $P_m(E)$ can be obtained from recursion relation by choosing $P_0 = 1$ and $P_{-1} = 0$.

Quasi-exactly potential associated with generalized master function

As in [7, 8], writing :

$$\psi(t) = A^{1/4}(x)W^{1/2}(x)\phi(x),$$

with a change of variable $\frac{dx}{dt} = \sqrt{A(x)}$, the eigen-value equation of the operator *L* reduces to the Schrodinger equation:

$$H(t)\psi(t) = E\psi(t) \qquad \text{(I)}$$

with the same eigen-value *E* and $\tilde{A}(t)$ given (I), in terms of eigen function of *L*, where $H(t) = -\frac{d^2}{dt^2} + V(t)$ is the similarity transformation of *L*(*x*) defined as:

$$H(t) = A^{1/4}(x)W^{1/2}(x)L(x)A^{-1/4}(x)W^{-1/2}(x)$$

$$V(t) = -\frac{3}{16}\frac{\dot{A}^2(t)}{A^2(t)} - \frac{1}{4}\frac{\dot{W}^2(t)}{W^2(t)} + \frac{1}{4}\frac{\dot{A}(t)\dot{W}(t)}{A(t)W(t)} + \frac{1}{4}\frac{\ddot{A}(t)}{A(t)} + \frac{1}{2}\frac{\ddot{W}(t)}{W(t)} + B(t)$$

$$V(x) = \frac{\ddot{A}^2(x)}{4} - \frac{\dot{A}^2(x)}{16\,A(x)} - \frac{A(x)\,\dot{W}(x)^2}{4\,W^2(x)} + \frac{A(x)\ddot{W}(x)}{2\,W(x)} + \frac{\dot{A}(x)\dot{W}(x)}{2\,W(x)} + B(x)$$

It is also straightforward to show that:

$$\int dt \phi(t) H(t) \psi(t) = \int_a^b dx W(x) \psi(x) L(x) \psi(x)$$

Hence block diagonalization of *L* leads to block-diagonalization of *H*.

elliptic quasi-exactly solvable potential

The starting point to find elliptic quasi-exactly solvable potential is generalized master function *A*(*x*), as mentioned before. Therefore , the selection of master function *A* which leads to elliptic potential, is very important. Considering the relation $\frac{dx}{dt}=\sqrt{A(x)}$; we select the master function so that *x* comes into the form of elliptic Jacobi functions. The weight function *W*(*x*) related to the given master function *A*(*x*) of order 3 and 4 can be obtained somehow that $\frac{1}{W}\frac{d}{dx}(AW)$ be of order 2 and 3, respectively.

After determining *B*1 and *B*2 from equations (III) and (IV), the function *B*(*x*) can be obtained easily :

$$B(x) = B_1 x + \frac{1}{2!} B_2 x^2.$$

Now, we can determine operator *L* and potential *V* (*t*) by knowing *A*, *W*, and *B*.

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$$B(x) = B_1 x + \frac{1}{2!} B_2 x^2.$$

Now, we can determine operator *L* and potential *V* (*t*) by knowing *A*, *W*, and *B*.

The interval $(a; b)$ for x is chosen so that, we have $A(a)W(a)=A(b)W(b) = 0$, and the interval of the parameters α, β, γ and δ is also chosen so that $(A(x)W(x))$ have not any singularity and also $A(a)W(a)=A(b)W(b)=0$ and equations

$$\left(\frac{(AW)'}{W}\right)^{(3)} = -\frac{A^{(4)}}{2}(n-1)$$

are conserved.

Below, we introduce all of the possible generalized master functions $A(x)$ of order 3 and 4 with some of their relative weight functions $W(x)$ and B and operator L and also Jacobi potential $V(t)$ obtained from them in the interval in which x and parameters α, β, γ and δ are defined.

Qubic Generalized Master Function

$A=4x(1-x)(1-k^2x)\,,\quad x=sn^2(t,k)$	$A=4x(1+x)(1+(1-k^2)x),\quad x=\dfrac{sn^2(t,k)}{cn^2(t,k)}$
$A=4x(1-x)(1-k^2+k^2x),\ x=cn^2(t,k)$	$A=4x(1+k^2x)(1+(k^2-1)x),\ x=\dfrac{sn^2(t,k)}{dn^2(t,k)}$
$A=4x(1-x)(k^2-1+x),\ x=dn^2(t,k)$	$A=4x(k^2+x)(x+k^2-1)\,,\ x=\dfrac{dn^2(t,k)}{sn^2(t,k)}$
$A=4x(x-1)(x-k^2)\,,x=\dfrac{1}{sn^2(t,k)}$	$A=4x(x-1)((k^2-1)x+1)\,,\ x=\dfrac{1}{dn^2(t,k)}$
$A=4x(1+x)(1-k^2+x)\,,\ x=\dfrac{cn^2(t,k)}{sn^2(t,k)}$	$A=4x(x-k^2)(x-1)\,,\ x=\dfrac{dn^2(t,k)}{cn^2(t,k)}$
$A=4x(x-1)((1-k^2)x+k^2),\ x=\dfrac{1}{cn^2(k,t)}$	$A=4x(k^2x-1)(x-1),\ x=\dfrac{cn^2(t,k)}{dn^2(t,k)}$

Quadratic Generalized Master Function

$A=(x^2-k^2)(x^2-1),\ \ x=\dfrac{dn(t,k)}{cn(t,k)}$	$A=(x^2-1)(1-k^2-x^2),\ \ x=dn(t,k)$
$A=-(1+k^2x^2)((1-k^2)x^2-1),\ \ x=\dfrac{sn(t,k)}{dn(t,k)}$	$A=(x^2-1)(x^2-k^2),\ \ x=\dfrac{1}{sn(t,k)}$
$A=(k^2+x^2)(k^2-1+x^2),\ \ x=\dfrac{dn(t,k)}{sn(t,k)}$	$A=(x^2-1)((1-k^2)x^2+k^2),\ \ x=\dfrac{1}{cn(t,k)}$
$A=(k^2x^2-1)(x^2-1),\ \ x=\dfrac{cn(t,k)}{dn(t,k)}$	$A=(1-x^2)((1-k^2)x^2-1),\ \ x=\dfrac{1}{dn(t,k)}$
$A=(1+x^2)(1-k^2+x^2),\ \ x=\dfrac{cn(t,k)}{sn(t,k)}$	$A=(1-x^2)(1-k^2+k^2x^2),\ \ x=cn(t,k)$
$A=(1+x^2)(1+(1-k^2)x^2),\ \ x=\dfrac{sn(t,k)}{cn(t,k)}$	$A=(1-x^2)(1-k^2x^2),\ \ x=sn(t,k)$

Examples

For expressing the utilize of the proposed potential two examples are followed first we consider the Hamiltonian of the spin system which describe the biaxial paramagnetic in a magnetic field *b* orthogonal to the anisotropy axes that appear in reference [12]:

$$H = k'^2 S_z^2 - k^2 S_y^2 + bk' S_x$$

where $0 \leq k, k' \leq 1$ are the moduli of elliptic functions k and $k' = (1-k^2)^{\frac{1}{2}}$
The solution of eigenvalue problem

$$H \mid \psi >= E \mid \psi >$$

for such a Hamiltonian leads to effective potential in this form:

$$V = [\frac{1}{4}b^2 - k^2 S(S+1)]cn^2 u + b(S + \frac{1}{2})sn\ u\ dn\ u \qquad \text{(I)}$$

This potential can be obtained from our potentials. Consider the potential *V* ($x = \frac{dn(t,k)}{sn(t,k)}$).

Its master function is $A=(k^2+x^2)(k^2-1+x^2),\ x=\frac{dn(t,k)}{sn(t,k)}$ and weight function

Is: $W=(k^2+x^2)^{\alpha}(\sqrt{1-k^2}-x)^{\beta}(\sqrt{1+k^2}+x)^{\gamma}e^{\arctan(\frac{x}{k})}$

By the restrictions $\gamma=\beta=-\frac{1}{2},\ n=S=\frac{b}{k},k=k$

the potential *V* (*x* = $\frac{dn(t,k)}{sn(t,k)}$) leads to the form Eq. (I)

$$V(\frac{dn}{sn})=(\frac{1}{4}b^2-n(n+1)k^2)cn^2+b(n+\frac{1}{2})sn\ dn-\frac{7}{2}b^2k^2+\frac{5}{2}b^2k^4+10n^2k^2$$

$$-6n^2k^4+14nk^4-10nk^6-4nk^2-4n^2+b^2$$

This form of the potential is most convenient since it immediately yields the Lame equation in zero magnetic field (*b* = 0).As a second example we obtain Lame potential. Consider the generalized master function $A(x)=4x(1-x)(1-k^2x)$ where corresponding weight function is $W=x^{\alpha}(1-x)^{\beta}(1-k^2x)^{\gamma}$ If we restrict ourselves to the case that the parameters $\alpha,\ \beta,\ \gamma=-1/2$

Its master function is $A = (k^2 + x^2)(k^2 - 1 + x^2), \; x = \frac{dn(t,k)}{sn(t,k)}$ and weight function

Is: $W = (k^2 + x^2)^{\alpha}(\sqrt{1-k^2} - x)^{\beta}(\sqrt{1+k^2} + x)^{\gamma} e^{\arctan(\frac{x}{k})}$

By the restrictions $\gamma = \beta = -\frac{1}{2}, \; n = S = \frac{b}{k}, k = k$

the potential *V* (*x* = $\frac{dn(t,k)}{sn(t,k)}$) leads to the form Eq. (I)

$$V(\frac{dn}{sn}) = (\frac{1}{4}b^2 \ldots$$

$$V = [\frac{1}{4}b^2 - k^2 S(S+1)]cn^2 u + b(S + \frac{1}{2}) sn\ u\ dn\ u$$

This form of the potential is most convenient since it immediately yields the Lame equation in zero magnetic field (*b* = 0).As a second example we obtain Lame potential. Consider the generalized master function $A(x) = 4x(1-x)(1-k^2x)$ where corresponding weight function is $W = x^{\alpha}(1-x)^{\beta}(1-k^2x)^{\gamma}$ If we restrict ourselves to the case that the parameters $\alpha, \beta, \gamma = -1/2$

Its master function is $A = (k^2 + x^2)(k^2 - 1 + x^2),\ x = \dfrac{dn(t,k)}{sn(t,k)}$ and weight function

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By the restrictions $\gamma = \beta = -\frac{1}{2},\ n = S = \frac{b}{k}, k = k$

the potential V ($x = \dfrac{dn(t,k)}{sn(t,k)}$) leads to the form Eq. (I)

$$V(\frac{dn}{sn}) = (\frac{1}{4}b^2 - n(n+1)k^2)cn^2 + b(n+\frac{1}{2})sn\ dn - \frac{7}{2}b^2k^2 + \frac{5}{2}b^2k^4 + 10n^2k^2$$

$$-6n^2k^4 + 14nk^4 - 10nk^6 - 4nk^2 - 4n^2 + b^2$$

This form of the potential is most convenient since it immediately yields the Lame equation in zero magnetic field (b = 0).As a second example we obtain Lame potential. Consider the generalized master function $A(x) = 4x(1-x)(1-k^2x)$ where corresponding weight function is $W = x^{\alpha}(1-x)^{\beta}(1-k^2x)^{\gamma}$ If we restrict ourselves to the case that the parameters $\alpha, \beta, \gamma = -1/2$

The relative potential to generalized master function $A(x)$ reduces to:

$$V = 2n(2n+1)k^2 sn^2$$

which is Lame potential. At the following we obtain low laying eigen-values and eigen-states for this potential. In order to find eigen-values and eigen-states for $n = 1$, first we obtain from $P_2 = 0$ the eigen-values E_1, and E_2 as below:

$$P_2 = \frac{E^2}{24} - \frac{(k^2+1)\,E}{6} + \frac{k^2}{2}$$

$$E_1 = 2k^2 + 2 - 2\sqrt{k^4 - k^2 + 1}$$

$$E_2 = 2k^2 + 2 - 2\sqrt{k^4 - k^2 + 1}$$

$$\psi_i(x) = \sum_{m=0}^{n} P_m(E_i) x^m$$

Now from

we can obtain the eigen-states and as below: $\psi_1(x)$ & $\psi_2(x)$

$$\psi_1 = 1 + 2\left(k^2 + 1 + \sqrt{k^4 - k^2 + 1}\right) sn^2$$

$$\psi_2 = 1 + 2\left(k^2 + 1 - \sqrt{k^4 - k^2 + 1}\right) sn^2$$

Similarly for $n = 2$ with $P_3 = 0$ we obtain E_1, E_2, E_3 and relative eigen states as:

$$P_3 = -\frac{1}{720}\,E^3 + \frac{(1+k^2)E^2}{36} - \frac{k^2(4\,k^2+21)\,E}{45} + \frac{8\,k^2\,(k^2+1)}{9}$$

$$E_1 = -\frac{20}{3} - \frac{20}{3}\,k^2$$

$$E_2 = 10/3 + 10/3\,k^2 + 2\,\sqrt{9\,k^4 - 4\,k^2 + 9}$$

$$E_3 = 10/3 + 10/3\,k^2 - 2\,\sqrt{9\,k^4 - 4\,k^2 + 9}$$

$$\psi_1 = 1 + 10/3\,\left(1 + k^2\right)\,sn^2 + 1/27\,\left(80\,k^4 + 205\,k^2 + 80\right)\,sn^4$$

$$\psi_2 = -2/3 - 5/3\,k^2 - \sqrt{9\,k^4 - 4\,k^2 + 9}sn^2$$
$$+ 1/27\,\left(6\,\sqrt{9\,k^4 - 4\,k^2 + 9}\left(1 + k^2\right) + 38\,k^4 + 22\,k^2 + 38\right)\,sn^4$$

$$\psi_3 = -2/3 - 5/3\,k^2 - \sqrt{9\,k^4 - 4\,k^2 + 9}sn^2$$
$$+ 1/27\,\left(-6\,\sqrt{9\,k^4 - 4\,k^2 + 9}\left(1 + k^2\right) + 38\,k^4 + 22\,k^2 + 38\right)\,sn^4$$

Appendix A: Jacobian Elliptic Functions

Jacobian elliptic functions are similar to trigonometric functions and they can be defined as the inversion of Legendre's elliptic integral of the first kind [13]. Therefore, *sn*(*u,k*) is defined as:

$$u = \int_0^{sn(u)} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \qquad \text{(A-I)}$$

then the functions *cn*(*u,k*) and *dn*(*u,k*) are defined by

$$cn(u,k) = \sqrt{1-(sn(u,k))^2}, dn(u,k) = \sqrt{1-k^2(sn(u,k))^2} \qquad \text{(A-II)}$$

The above relations can also be represented by equations

$$sn^2(u,k) + cn^2(u,k) = 1$$

$$dn^2(u,k) + k^2 sn^2(u,k) = 1$$

By differentiating (*A-I*) and using (*A-II*) we obtain

$$\frac{d}{du} sn(u,k) = cn(u,k) dn(u,k)$$

Similarly by differentiating (*A-II*) we have

$$\frac{d}{du}(cn(u,k)) = -sn(u,k)dn(u,k)$$

$$\frac{d}{du}(dn(u,k)) = -k^2 sn(u,k)cn(u,k)$$

Appendix B: the first fore polynomial $P_n(E)$for $k = 3$

The first four polynomials $P_n(E)$, for $k = 3$.

To abbreviate, we set $F^{(i)} = (\frac{(AW)'}{W})^{(i)}$

$$P_0 = 1 \quad , \quad P_1 = -\frac{E}{F^0}$$

$$P_2 = 1/2\,\frac{B^1F^0 + EF^1 + E^2}{F^0(A^1 + F^0)}$$

$$P_3 = -(2EB^{(1)}A^{(1)} + A^{(2)}E^2 + 2F^{(1)}B^{(0)}F^{(0)}$$
$$+ A^{(2)}B^{(1)}F^{(0)} + A^{(2)}EF^{(1)} + 3EB^{(1)}F^{(0)} + E^3$$
$$+ 2EF^{(1)^2} + 3F^{(1)}E^2 - EF^{(2)}A^{(1)} - EF^{(2)}F^{(0)})$$
$$/(6F^{(0)}2A^{(1)^2} + 3A^{(1)}F^{(0)} + F^{(0)^2})$$

$$P[4] := (-A^{(3)}EF^{(1)}F^{(0)} + 4A^{(2)}E^3 + 6F^{(1)}E^3 + 6EF^{(1)^3} + 11F^{(1)^2}E^2$$
$$+ 3B^{(1)^2}F^{(0)^2} - 2A^{(3)}E^2A^{(1)} + 3A^{(2)^2}B^{(1)}F^{(0)} + 3A^{(2)^2}EF^{(1)} + 6F^{(1)^2}B^{(1)}F^{(0)}$$
$$+ 8E^2B^{(1)}A^{(1)} + 6E^2B^{(1)}F^{(0)} - 7E^2F^{(2)}A^{(1)} - 4E^2F^{(2)}F^{(0)} + 9A^{(2)}EF^{(1)^2}$$
$$+ 13A^{(2)}F^{(1)}E^2 - 3F^{(2)}B^{(1)}F^{(0)^2} + 6A^{(1)}B^{(1)^2}F^{(0)} - 2A^{(3)}EF^{(1)}A^{(1)}$$
$$+ 6A^{(2)}EB^{(1)}A^{(1)} + 9A^{(2)}F^{(1)}B^{(1)}F^{(0)} + 10A^{(2)}EB^{(1)}F^{(0)} - 3A^{(2)}EF^{(2)}A^{(1)}$$
$$- 3A^{(2)}EF^{(2)}F^{(0)} + 12F^{(1)}EB^{(1)}A^{(1)} + 14F^{(1)}EB^{(1)}F^{(0)} - 9F^{(1)}EF^{(2)}A^{(1)}$$
$$- 6F^{(1)}EF^{(2)}F^{(0)} - 6A^{(1)}F^{(2)}B^{(1)}F^{(0)} - 2A^{(1)}A^{(3)}B^{(1)}F^{(0)} - A^3 * B^1F^{(0)^2}$$
$$- A^{(3)}E^2F^{(0)} + 3A^{(2)^2}E^2 + E^4)/(24F^{(0)}(6A^{(1)^3} + 11A^{1^2}F^{(0)} + 6A^{(1)}F^{(0)^2} + F^{(0)^3}))$$

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