Geometry, Integrability and Quantization
8-13 June 2007
Bulgarian Akademy of Sciences

$sl(2, \mathbb{R})$ symmetry and solvable multiboson systems

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T. Goliński, M. Horowski, A. Odzijewicz, A. Sliżewska,
$sl(2, \mathbb{R})$ symmetry and solvable multiboson systems,
J. Math. Phys. 48, 023508 (2007)

Let
$\mathbf{a}$, $\mathbf{a}^*$ be the standard annihilation and creation operators satisfying $[\mathbf{a}^*, \mathbf{a}] = \mathbb{1}$ and acting in the Hilbert space $\mathcal{H}$.

For any fixed $l \in \mathbb{N}$, we define the *multiboson representation* of $sl(2, \mathbb{R})$ as the triple of operators

$$\mathbf{A}_0 := \alpha_0(\mathbf{n}), \qquad \mathbf{A}_- := \alpha_-(\mathbf{n})\, \mathbf{a}^l, \qquad \mathbf{A}_+ := (\mathbf{a}^*)^l \alpha_-(\mathbf{n}),$$

where $\mathbf{n} := \mathbf{a}^*\mathbf{a}$,
defined on dense subset of $\mathcal{H}$ and satisfying $sl(2, \mathbb{R})$ commutation relations:

$$[\mathbf{A}_-, \mathbf{A}_+] = \mathbf{A}_0, \qquad [\mathbf{A}_0, \mathbf{A}_\pm] = \pm 2\mathbf{A}_\pm$$

with symmetricity conditions:

$$\mathbf{A}_0 \subset \mathbf{A}_0^*, \qquad \mathbf{A}_- \subset \mathbf{A}_+^*, \qquad \mathbf{A}_+ \subset \mathbf{A}_-^*.$$

The previous relations imply that the functions α_0, α_- are real-valued and satisfy the following difference equations

$$\begin{aligned}
\Big(\alpha_0(n) - \alpha_0(n-l) - 2\Big)\alpha_-(n-l) &= \quad 0 \quad \text{for } n \geqslant l, && (1)\\
(n+1)_l\, \alpha_-^2(n) - (n-l+1)_l\, \alpha_-^2(n-l) &= \alpha_0(n) \quad \text{for } n \geqslant l, && (2)\\
(n+1)_l\, \alpha_-^2(n) &= \alpha_0(n) \quad \text{for } 0 \leqslant n < l, && (3)
\end{aligned}$$

where $(n)_l = n(n+1)\dots(n+l-1)$ and one has applied the identities

$$(\mathbf{a}^*)^l \mathbf{a}^l = (\mathbf{n} - l + 1)_l \qquad \mathbf{a}^l (\mathbf{a}^*)^l = (\mathbf{n} + 1)_l.$$

The solution to the above system of difference equations is of the form {sl-sol}

$$\alpha_0(n) = 2\left[\frac{n}{l}\right] + \alpha_0(n \bmod l),$$

$$\alpha_-(n) = \sqrt{\frac{1}{(n+1)_l}\left(\left[\frac{n}{l}\right] + \alpha_0(n \bmod l)\right)\left(\left[\frac{n}{l}\right] + 1\right)},$$

where $[x]$ is the integer part of x.

In order to express the operators $\mathbf{A}_0$, $\mathbf{A}_-$, $\mathbf{A}_+$ explicitly in terms of the creation and annihilation operators let us define the bounded operator

$$\mathbf{R} := \frac{l-1}{2} + \sum_{m=1}^{l-1} \frac{\exp(-\frac{2\pi i m}{l}\mathbf{n})}{\exp(\frac{2\pi i m}{l}) - 1}$$

for $l > 1$ and $\mathbf{R} := 0$ for $l = 1$. This operator acts on elements of the basis by

$$\mathbf{R}\,|n\rangle = n \bmod l \;\; |n\rangle$$

and commutes with operators $\mathbf{A}_0$, $\mathbf{A}_-$, $\mathbf{A}_+$.

Thus one has

$$\frac{1}{l}\,(\mathbf{n}-\mathbf{R})\,|n\rangle = \left[\frac{n}{l}\right]|n\rangle\,.$$

Finally the multiboson representation of $sl(2,\mathbb{R})$ is given in terms of $\mathbf{a}$ and $\mathbf{a}^*$ by {sl-oper}

$$\mathbf{A}_0 = \frac{2}{l}\,(\mathbf{n}-\mathbf{R}) + \alpha_0(\mathbf{R}),$$

$$\mathbf{A}_- = \sqrt{\frac{1}{(\mathbf{n}+1)_l}\left(\frac{1}{l}(\mathbf{n}-\mathbf{R})+\alpha_0(\mathbf{R})\right)\left(\frac{1}{l}(\mathbf{n}-\mathbf{R})+1\right)}\;\mathbf{a}^l,$$

$$\mathbf{A}_+ = (\mathbf{a}^*)^l\sqrt{\frac{1}{(\mathbf{n}+1)_l}\left(\frac{1}{l}(\mathbf{n}-\mathbf{R})+\alpha_0(\mathbf{R})\right)\left(\frac{1}{l}(\mathbf{n}-\mathbf{R})+1\right)},$$

where α_0 is an arbitrary positive function on $\{0,\ldots,l-1\}$

The above formulae show that the Hilbert space $\mathcal{H}$ splits

$$\mathcal{H} = \bigoplus_{r=0}^{l-1} \mathcal{H}_r$$

onto invariant subspaces

$$\mathcal{H}_r := \mathrm{span}\{\, |k\rangle_r := |kl + r\rangle \ \mid k \in \mathbb{N} \cup \{0\}\},$$

which are eigenspaces of $\mathbf{R}$ corresponding to the eigenvalue r.

Remarks:

$|k\rangle_r$ are eigenvectors of $\mathbf{A}_0$

$$\mathbf{A}_0 \, |k\rangle_r = (2k + \alpha_0(r)) \; |k\rangle_r$$

and $\mathbf{A}_-$, $\mathbf{A}_+$ act on $|k\rangle_r$ as weighted shift operators:

$$\mathbf{A}_- \, |k\rangle_r = \sqrt{k(k + \alpha_0(r) - 1)} \; |k-1\rangle_r \, ,$$

$$\mathbf{A}_+ \, |k\rangle_r = \sqrt{(k + \alpha_0(r))(k+1)} \; |k+1\rangle_r \, .$$

Bogoliubov-like transformations

We consider the group $\mathfrak{B} := \mathbb{R}^\times \rtimes \mathbb{Z}_2$, where $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$, $\mathbb{Z}_2 = \{-1, 1\}$, with the group operation defined by

$$(a, \sigma) \cdot (b, \tau) := (ab^\sigma, \sigma\tau).$$

which acts on the generators of $\mathcal{A}$ in the following way

$$\mathfrak{b}_{a,\sigma}(\mathbf{A}_0) := \frac{1 + a^2}{2a}\mathbf{A}_0 + \sigma\frac{1 - a^2}{2a}(\mathbf{A}_- + \mathbf{A}_+),$$

$$\mathfrak{b}_{a,\sigma}(\mathbf{A}_-) := \frac{1 - a^2}{4a}\mathbf{A}_0 + \sigma\frac{(1 - a)^2}{4a}\mathbf{A}_+ + \sigma\frac{(1 + a)^2}{4a}\mathbf{A}_-,$$

$$\mathfrak{b}_{a,\sigma}(\mathbf{A}_+) := \frac{1 - a^2}{4a}\mathbf{A}_0 + \sigma\frac{(1 + a)^2}{4a}\mathbf{A}_+ + \sigma\frac{(1 - a)^2}{4a}\mathbf{A}_-.$$

There exists
the unitary representation of the subgroup $\mathbb{R}_+ \rtimes \mathbb{Z}_2 \subset \mathfrak{B}$

$$\mathbb{U}_{a,\sigma} \left|n\right\rangle_r := \left|n; a, \sigma\right\rangle_r, \qquad (a,\sigma) \in \mathbb{R}_+ \rtimes \mathbb{Z}_2,$$

where

$$\left|n; a, \sigma\right\rangle_r := \begin{cases} \sigma^n \sqrt{\frac{n!}{(\alpha_0(r))_n c^n}} \sum_{k=0}^{\infty} M_k(n; \alpha_0(r), c) \ \left|k\right\rangle_r & \text{for } a^\sigma < 1 \\ \sigma^n \sqrt{\frac{n!}{(\alpha_0(r))_n c^n}} \sum_{k=0}^{\infty} (-1)^k M_k(n; \alpha_0(r), c) \ \left|k\right\rangle_r & \text{for } a^\sigma > 1 \\ \sigma^n \left|n\right\rangle_r & \text{for } a = 1 \end{cases}$$

$c = \left(\frac{a-1}{a+1}\right)^2$ and $M_k(n, \gamma, c)$ — Meixner polynomials

such that

$$\mathfrak{b}_{a,\sigma}(\mathbf{X}) = \mathbb{U}_{a,\sigma} \mathbf{X} \mathbb{U}^*_{a,\sigma}$$

Integrable one-mode Hamiltonians

Let us take the quantum system described by arbitrary self-adjoint operator belonging to the multiboson algebra $\mathcal{A}$:

$$\mathbf{H}_{\mu\nu} := \frac{\mu+\nu}{2}\mathbf{A}_0 + \frac{\mu-\nu}{2}(\mathbf{A}_- + \mathbf{A}_+) =$$

$$= \frac{\mu+\nu}{2}\alpha_0(\mathbf{n}) + \frac{\mu-\nu}{2}(\alpha_-(\mathbf{n})\,\mathbf{a}^l + (\mathbf{a}^*)^l\alpha_-(\mathbf{n}))$$

where $(\mu,\nu) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

Let us observe that

$$\mathbf{H}_{\mu\nu} \left|k\right\rangle_r = b_{k-1} \ \left|k-1\right\rangle_r + a_k \ \left|k\right\rangle_r + b_k \ \left|k+1\right\rangle_r ,$$

where

$$a_k = \frac{\mu+\nu}{2}(2k+\alpha_0(r))$$

$$b_k = \frac{\mu-\nu}{2}\sqrt{(k+\alpha_0(r))(k+1)}.$$

If $\mu \neq \nu$ the formula of $\mathbf{H}_{\mu\nu}$ is directly related to three term recurrence relation

$$xP_k(x) = b_{k-1}\, P_{k-1}(x) + a_k\, P_k(x) + b_k\, P_{k+1}(x)$$

which is valid for any orthonormal polynomials family $\{P_n\}_{n=0}^{\infty}$. Since $\sum \frac{1}{b_k}$ is divergent then there exists the unique measure $d\omega$ on $\mathbb{R}$ such the map F given by

$$\mathcal{H}_r \ni |k\rangle_r \longmapsto F(|k\rangle_r) := P_k \in L^2(\mathbb{R}, d\omega)$$

is the isomorphism of Hilbert spaces with the property that

$$F \circ \mathbf{H}_{\mu\nu|\mathcal{H}_r} \circ F^{-1} = \widehat{x},$$

where $\widehat{x}$ is the operator of multiplication by x in $L^2(\mathbb{R}, d\omega)$. Thus we gather that the spectrum of $\mathbf{H}_{\mu\nu}$ is the support of measure

$d\omega$. It means that by finding the measure $d\omega$ and polynomials P_n we obtain the evolution flow

$$R \ni t \longrightarrow e^{it\mathbf{H}_{\mu\nu}} = F^{-1} \circ e^{it\widehat{x}} \circ F \in \mathsf{Aut}(\mathcal{H}_r)$$

of quantum system described by the Hamiltonian $\mathbf{H}_{\mu\nu}$.

From definition of Bogoliubov group $\mathfrak{B}$ it follows that transformations $\mathfrak{b}_{a,\sigma}$ preserve the family of operators $\mathbf{H}_{\mu\nu}$ and the labels (μ, ν) transform as follows

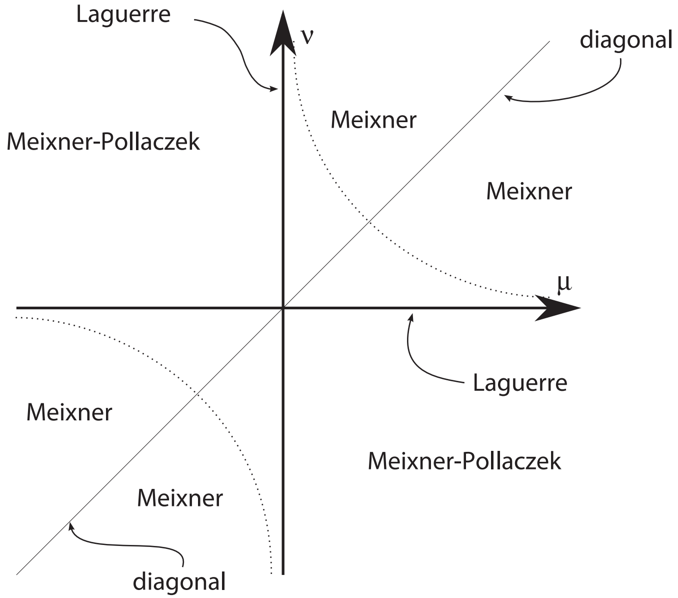
$$(a, 1) : (\mu, \nu) \mapsto (a^{-1}\mu, a\nu), \qquad (a, -1) : (\mu, \nu) \mapsto (a\nu, a^{-1}\mu).$$

This defines the action of the group $\mathfrak{B}$ in the set $\mathbb{R}^2 \setminus \{(0, 0)\}$ of labels. Orbits of $\mathfrak{B}$ are pairs of hiperbolae indexed by one real parameter $c \in \mathbb{R}$

$$\mathcal{O}_c := \mathfrak{B} \cdot (c, 1) = \{(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\} \mid xy = c\}.$$

We can restrict our considerations to each component $\mathcal{H}_r$ of decomposition of $\mathcal{H}$ separately since they are invariant under the action of $\mathbf{H}_{\mu\nu}$.
Due to the implementation formula $\mathfrak{b}_{a,\sigma}(\mathbf{X}) = \mathbb{U}_{a,\sigma}\mathbf{X}\mathbb{U}^*_{a,\sigma}$ it is sufficient to find spectral decomposition for the one operator from each orbit $\mathcal{O}_c$, e.g. $\mathbf{H}_{\sqrt{c},\sqrt{c}}$, $\mathbf{H}_{\sqrt{c},-\sqrt{c}}$ and $\mathbf{H}_{1,0}$. Taking into account scaling by constant we can further without loosing generality restrict ourselves to three Hamiltonians $\mathbf{H}_{1,1}$, $\mathbf{H}_{1,-1}$ and $\mathbf{H}_{1,0}$.

μ, ν	polynomials	spectrum of $\mathbf{H}_{\vert\mathcal{H}}$
$\nu = 0, \mu > 0$	Laguerre	$\mathbb{R}_+ \cup \{0\}$
$\nu = 0, \mu < 0$	Laguerre	$\mathbb{R}_- \cup \{0\}$
$\mu = 0, \nu > 0$	Laguerre	$\mathbb{R}_+ \cup \{0\}$
$\mu = 0, \nu < 0$	Laguerre	$\mathbb{R}_- \cup \{0\}$
$\mu > 0,\ \nu < 0$	Meixner-Pollaczek	$\mathbb{R}$
$\mu < 0,\ \nu > 0$	Meixner-Pollaczek	$\mathbb{R}$
$\mu, \nu > 0$	Meixner	$\{2\sqrt{\mu\nu}\, n + \frac{1}{2}\sqrt{\mu\nu} \mid n = 0, 1, 2, \ldots\}$
$\mu, \nu < 0$	Meixner	$\{-2\sqrt{\mu\nu}\, n - \frac{1}{2}\sqrt{\mu\nu} \mid n = 0, 1, 2, \ldots\}$



Orthogonal polynomials assigned to $\mathbf{H}_{\mu\nu}$

Example: $\nu < \mu < 0$ - Meixner orthonormal polynomials

$$P_n(x) = (-1)^n M_n\left(\frac{-x}{2\sqrt{\mu\nu}} - \frac{1}{4}; \frac{1}{2}, c\right)$$

where $c = \frac{\mu+\nu+2\sqrt{\mu\nu}}{\mu+\nu-2\sqrt{\mu\nu}}$

$$d\omega(x) = \sum_{n=0}^{\infty} \delta(x + \frac{1}{2}\sqrt{\mu\nu} + 2\sqrt{\mu\nu}\ n)\frac{(\frac{1}{2})_n}{n!}c^n\,dx$$

eigenvectors of $\mathbf{H}_{|\mathcal{H}}$

$$|E_n\rangle = \sqrt{\frac{n!}{(\frac{1}{2})_n c^n}} \sum_{k=0}^{\infty} (-1)^k M_k(n; \frac{1}{2}, c)\ |k\rangle$$

Coherent state representation

Due to the decomposition $\mathcal{H} = \bigoplus_{r=0}^{l-1} \mathcal{H}_r$ into irreducible representations, it is sufficient to restrict our considerations to each $\mathcal{H}_r$ separately. We consider coherent states as eigenstates of $\mathbf{A}_-$

$$\mathbf{A}_- \left|\zeta\right\rangle_r := \zeta \left|\zeta\right\rangle_r .$$

The coherent states $\left|\zeta\right\rangle_r \in \mathcal{H}_r$ are given by the series

$$\left|\zeta\right\rangle_r = \sum_{k=0}^{\infty} \frac{\zeta^k}{\sqrt{k!(\alpha_0(r))_k}} \left|k\right\rangle_r$$

which converges for any $\zeta \in \mathbb{C}$ and belongs to domain

$$\mathcal{D}_1 := \left\{ \sum_{n=0}^{\infty} v_n \left|n\right\rangle \in \mathcal{H} \;\middle|\; \sum_{n=0}^{\infty} n^2 \left|v_n\right|^2 < \infty \right\}.$$

The notion of the coherent states allows us to construct the anti-unitary embedding

$$\mathcal{H}_r \ni |\psi\rangle \longmapsto I_r(\psi)(\zeta) := \langle\psi|\zeta\rangle_r \in L^2\mathcal{O}(\mathbb{C}, d\mu_r)$$

of $\mathcal{H}_r$ into the Hilbert space $L^2\mathcal{O}(\mathbb{C}, d\mu_r)$ of holomorphic functions on $\mathbb{C}$, which are square integrable with respect to the measure

$$d\mu_r(\zeta,\overline{\zeta}) := \frac{\rho^{\alpha_0(r)} K_{\alpha_0(r)}(2\rho)}{2\pi\Gamma(\alpha_0(r))}\rho\, d\rho\, d\varphi,$$

where $\zeta = \rho e^{i\varphi}$ and $K_{\alpha_0(r)}$ is the modified Bessel function of the second kind.

The space $L^2\mathcal{O}(\mathbb{C}, d\mu_r)$ has the reproducing kernel

$$\mathcal{K}(\overline{\eta}, \zeta) := \langle \eta | \zeta \rangle = {}_0F_1\left(\begin{matrix} - \\ \alpha_0(r) \end{matrix} \middle| \, \overline{\eta}\zeta \right),$$

i.e. for any $f \in L^2\mathcal{O}(\mathbb{C}, d\mu_r)$ one has

$$\int_{\mathbb{C}} \mathcal{K}(\overline{\eta}, \zeta) f(\eta) d\mu_r(\eta, \overline{\eta}) = f(\zeta).$$

The isomorphism I_r gives the realization of the operators $\mathbf{A}_0$, $\mathbf{A}_+$, $\mathbf{A}_-$ as the differential operators

$$I_r \circ \mathbf{A}_0 \circ I_r{}^{-1} = 2\zeta \frac{d}{d\zeta} + \alpha_0(r), \tag{7}$$

$$I_r \circ \mathbf{A}_+ \circ I_r{}^{-1} = \zeta, \tag{8}$$

$$I_r \circ \mathbf{A}_- \circ I_r{}^{-1} = \left(\alpha_0(r) + \zeta \frac{d}{d\zeta} \right) \frac{d}{d\zeta} \tag{9}$$

acting in $L^2\mathcal{O}(\mathbb{C}, d\mu_r)$. In order to describe them as the generators of the discrete series $\alpha_0(r) = 2, 3, \ldots$ representation of the group $SL(2, \mathbb{R})$, let us consider a unitary integral transform

$$\mathcal{P} : L^2\mathcal{O}(\mathbb{C}, d\mu_r) \longrightarrow L^2\mathcal{O}(\mathbb{D}, d\nu_r),$$

where $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ and

$$d\nu_r(z, \overline{z}) := \frac{\alpha_0(r) - 1}{\pi} \left(1 - |z|^2\right)^{\alpha_0(r)-2} d^2z,$$

given by

$$\mathcal{P}f(z) := \int\limits_{\mathbb{C}} e^{z\overline{\zeta}} f(\zeta) d\mu_r(z, \overline{z}).$$

Using the above formula we find that in the space $L^2\mathcal{O}(\mathbb{D}, d\nu_r)$

the operators (7)-(9) are given by

$$\begin{aligned} \mathcal{P} \circ I_r \circ \mathbf{A}_0 \circ I_r^{-1} \circ \mathcal{P}^{-1} &= 2z\frac{d}{dz} + \alpha_0(r) \\ \mathcal{P} \circ I_r \circ \mathbf{A}_+ \circ I_r^{-1} \circ \mathcal{P}^{-1} &= z^2\frac{d}{dz} + \alpha_0(r)z \\ \mathcal{P} \circ I_r \circ \mathbf{A}_- \circ I_r^{-1} \circ \mathcal{P}^{-1} &= \frac{d}{dz} \end{aligned} \tag{10}$$

and they are the generators of the discrete series representation

$$U_g^{\alpha_0(r)}\varphi(z) = (bz + \overline{a})^{-\alpha_0(r)}\varphi\left(\frac{az + \overline{b}}{bz + \overline{a}}\right)$$

of the group $SL(2, \mathbb{R})$ in $L^2\mathcal{O}(\mathbb{D}, d\nu_r)$. Here we have identified $SL(2, \mathbb{R})$ with $SU(1, 1)$ using the isomorphism

$$SL(2, \mathbb{R}) \ni g \longleftrightarrow \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} := \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} g \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \in SU(1, 1).$$

Ending let us remark that in $L^2\mathcal{O}(\mathbb{C}, d\mu_r)$ the Hamiltonian $\mathbf{H}_{\mu\nu}$ is represented as a second order differential operator

$$I_r \circ \mathbf{H}_{\mu\nu} \circ I_r^{-1} = \frac{\mu+\nu}{2}\left(2\zeta\frac{d}{d\zeta} + \alpha_0(r)\right) + \frac{\mu-\nu}{2}\left(\zeta + \left(\alpha_0(r) + \zeta\frac{d}{d\zeta}\right)\frac{d}{d\zeta}\right)$$

and in $L^2\mathcal{O}(\mathbb{D}, d\nu_r)$ as a first order differential operator

$$\mathcal{P} \circ I_r \circ \mathbf{H}_{\mu\nu} \circ I_r^{-1} \circ \mathcal{P}^{-1} = \frac{\mu+\nu}{2}\left(2z\frac{d}{dz} + \alpha_0(r)\right) + \frac{\mu-\nu}{2}\left((z^2+1)\frac{d}{dz} + \alpha_0(r)z\right).$$

Two-mode Algebra

$$\mathbf{A}_0 := \alpha_0(\mathbf{n}_0, \mathbf{n}_1) \qquad \mathbf{A}_- := \alpha_-(\mathbf{n}_0, \mathbf{n}_1)\, {\mathbf{a}_0}^{l_0} \qquad \mathbf{A}_+ := (\mathbf{a}_0^*)^{l_0} \alpha_-(\mathbf{n}_0, \mathbf{n}_1)$$

$$\mathbf{B}_0 := \beta_0(\mathbf{n}_0, \mathbf{n}_1) \qquad \mathbf{B}_- := \beta_-(\mathbf{n}_0, \mathbf{n}_1)\, {\mathbf{a}_1}^{l_1} \qquad \mathbf{B}_+ := (\mathbf{a}_1^*)^{l_1} \beta_-(\mathbf{n}_0, \mathbf{n}_1)$$

$l_0, l_1 \in \mathbb{N}$

$sl(2, \mathbb{R})$ commutation relations

$$[\mathbf{A}_-, \mathbf{A}_+] = \mathbf{A}_0, \qquad [\mathbf{A}_0, \mathbf{A}_\pm] = \pm 2\mathbf{A}_\pm$$

$$[\mathbf{B}_-, \mathbf{B}_+] = \mathbf{B}_0, \qquad [\mathbf{B}_0, \mathbf{B}_\pm] = \pm 2\mathbf{B}_\pm$$

symmetricity conditions

$$\mathbf{A}_0 \subset \mathbf{A}_0^*, \qquad \mathbf{A}_- \subset \mathbf{A}_+^*, \qquad \mathbf{A}_+ \subset \mathbf{A}_-^*$$

$$\mathbf{B}_0 \subset \mathbf{B}_0^*, \qquad \mathbf{B}_- \subset \mathbf{B}_+^*, \qquad \mathbf{B}_+ \subset \mathbf{B}_-^*$$

Solutions to these relations

$$\mathbf{A}_0 = \frac{2}{l_0}(\mathbf{n}_0 - \mathbf{R}_0) + \alpha_0(\mathbf{R}_0, \mathbf{R}_1)$$

$$\mathbf{A}_- = \sqrt{\frac{1}{(\mathbf{n}_0 + 1)_{l_0}} \left(\frac{1}{l_0}(\mathbf{n}_0 - \mathbf{R}_0) + \alpha_0(\mathbf{R}_0, \mathbf{R}_1)\right) \left(\frac{1}{l_0}(\mathbf{n}_0 - \mathbf{R}_0) + 1\right)}\, \mathbf{a}_0{}^{l_0}$$

$$\mathbf{B}_0 = \frac{2}{l_1}(\mathbf{n}_1 - \mathbf{R}_1) + \beta_0(\mathbf{R}_0, \mathbf{R}_1)$$

$$\mathbf{B}_- = \sqrt{\frac{1}{(\mathbf{n}_1 + 1)_{l_1}} \left(\frac{1}{l_1}(\mathbf{n}_1 - \mathbf{R}_1) + \beta_0(\mathbf{R}_0, \mathbf{R}_1)\right) \left(\frac{1}{l_1}(\mathbf{n}_1 - \mathbf{R}_1) + 1\right)}\, \mathbf{a}_1{}^{l_1}$$

Remainder operator

$$\mathbf{R}_j := \frac{l_j - 1}{2} + \sum_{m=1}^{l_j-1} \frac{\exp(-\frac{2\pi i m}{l_j}\mathbf{n}_j)}{\exp(\frac{2\pi i m}{l_j}) - 1}$$

$$\mathbf{R}_j \left|n_1, n_2\right\rangle = n_j \bmod l_j \ \left|n_1, n_2\right\rangle$$

α_0, β_0 are arbitrary positive functions on $\{0, \ldots, l_0 - 1\} \times \{0, \ldots, l_1 - 1\}$

$$[\mathbf{R}_j, \mathbf{H}] = 0$$

$$\mathcal{H} \otimes \mathcal{H} = \bigoplus_{r_0=0}^{l_0-1} \bigoplus_{r_1=0}^{l_1-1} \mathcal{H}_{r_0,r_1},$$

subspaces invariant with respect to $\mathbf{H}$

$$\mathcal{H}_{r_0,r_1} := \mathrm{span}\left\{ \ |k_0, k_1\rangle_{r_0,r_1} := |k_0 l_0 + r_0, k_1 l_1 + r_1\rangle \ \mid k_0, k_1 \in \mathbb{N} \cup \{0\} \right\}$$

Two-mode systems

$$
\begin{aligned}
\mathbf{H} &= \frac{(a^2+b^2)}{4ab}\mathbf{A}_0\mathbf{B}_0 - \\
&- \sigma\tau\frac{(a-b)^2}{4ab}(\mathbf{A}_+\mathbf{B}_+ + \mathbf{A}_-\mathbf{B}_-) - \sigma\frac{a^2-b^2}{4ab}(\mathbf{A}_+\mathbf{B}_0 + \mathbf{A}_-\mathbf{B}_0) + \\
&+ \tau\frac{a^2-b^2}{4ab}(\mathbf{A}_0\mathbf{B}_- + \mathbf{A}_0\mathbf{B}_+) - \sigma\tau\frac{(a+b)^2}{4ab}(\mathbf{A}_+\mathbf{B}_- + \mathbf{A}_-\mathbf{B}_+)
\end{aligned}
$$

acting in two mode bosonic Hilbert space $\mathcal{H}\otimes\mathcal{H}$ with the orthonormal basis

$$\{|n_0, n_1\rangle\}_{n_0,n_1=0}^{\infty}$$