Prequantization of super symplectic manifolds

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Super geometry

The idea : Create a manifold with local coordinates $(x_1, \ldots, x_p, \xi_1, \ldots, \xi_q)$, the x_i commuting, the ξ_j anti-commuting: $\xi_i \xi_j = -\xi_j \xi_i$.

How to realize : Replace $\mathbf{R}$ by a graded commutative ring $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ and take $x_i \in \mathcal{A}_0$ and $\xi_j \in \mathcal{A}_1$.

Basic example : $\mathcal{A} = \bigwedge E = \Big(\bigoplus_{k=0}^{\infty} \bigwedge^{2k} E \Big) \oplus \Big(\bigoplus_{k=0}^{\infty} \bigwedge^{2k+1} E \Big)$ with E an infinite dimensional vector space.

Smooth functions : In the x_i just ordinary smooth functions of real variables, in the ξ_i just polynomials of dergee at moost one in each ξ_i separately:

$$C^{\infty}(\mathcal{A}_0^p \times \mathcal{A}_1^q) \cong C^{\infty}(\mathbf{R}^p) \otimes \bigwedge \mathbf{R}^q \ .$$

Super geometry

Problems with the definition of a derivative : In the super context we can't speak of a difference quotient because of nilpotent elements, nor can we speak of limits because the natural topology on $\mathcal{A}$ is not separable.

The idea beahind an intrinsic definition of smooth functions :
Let $f : \mathbf{R} \to \mathbf{R}$ be of class C^1, then the function $g : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$g(x, y) = \int_0^1 f'\big(sx + (1-s)y\big)\, ds$$

is continuous and satisfies $\forall x, y \in \mathbf{R} \ : \ f(x) - f(y) = g(x, y) \cdot (x - y)$.

Proposition : Let $f : \mathbf{R} \to \mathbf{R}$ be any function. If there exists a continuous function $g : \mathbf{R}^2 \to \mathbf{R}$ satisfying

$$\forall x, y \in \mathbf{R} \ : \ f(x) - f(y) = g(x, y) \cdot (x - y) \ ,$$

then f is of class C^1 with $f'(x) = g(x, x)$.

Super symplectic geometry

An example : On $M = \mathcal{A}_0^2 \times \mathcal{A}_1^2 \ni (x, y, \xi, \eta)$ consider the 2-form

$$\omega = dx \wedge dy + d\xi \wedge d\eta + dx \wedge d\xi \ .$$

Consider the vector fields

$$X = 2y \, \frac{\partial}{\partial x} - 2y \, \frac{\partial}{\partial \eta} \qquad \text{and} \qquad Y = -\xi \, \frac{\partial}{\partial \xi} + \eta \, \frac{\partial}{\partial \eta} + \xi \, \frac{\partial}{\partial y} \ .$$

They satisfy $\iota(X)\omega = d(y^2)$ and $\iota(Y)\omega = d(\eta\xi)$.

Their commutator is given as $[X, Y] = -2\xi \, \frac{\partial}{\partial x} - 2y \, \frac{\partial}{\partial \eta} - 2\xi \, \frac{\partial}{\partial \eta}$.

But $\iota([X, Y])\omega = d(y\xi) + 2\xi \, d\xi$ is not even closed.

Super symplectic geometry

Definition : A closed 2-form $\omega = \omega_0 + \omega_1$ on M is said to be symplectic if

$$\ker(\omega_0 : T_m \to T_m^*) \cap \ker(\omega_1 : T_m \to T_m^*) = \{0\} \ .$$

Remark : If a closed 2-form satisfies $\ker(\omega : T_m \to T_m^*) = \{0\}$, then it is symplectic.

Definition : A vector field X on a symplectic manifold (M, ω) is locally/globally hamiltonian if

$$\iota(X)\omega_0 \qquad \text{and} \qquad \iota(X)\omega_1$$

are closed/exact.

Definition : The Poisson algebra $\mathcal{P}$ of a symplectic manifold is given by

$$\mathcal{P} = \{(f_0, f_1) \in C^\infty(M)^2 \mid \exists X : \iota(X)\omega_0 = -df_0 \text{ and } \iota(X)\omega_1 = -df_1\,\} \ .$$

Super symplectic geometry

A trick : To avoid the separation in even and odd parts, consider all forms (and thus functions) as forms in a vector space of dimension $1|1$ for which ω_0 and ω_1 are the "coefficients" with respect to a basis.

Proposition : The commutator of two locally hamiltonian vector fields is globally hamiltonian.

Definition : For an element $f = (f_0, f_1) \in \mathcal{P}$, the unique vector field X satisfying $\iota(X)\omega_\alpha = -df_\alpha$ is called the hamiltonian vector field of f and denoted as X_f. The Poisson bracket $\{f, g\}$ of two elements $f, g \in \mathcal{P}$ is defined as $\{f, g\}_\alpha = X_f g_\alpha$.

Proposition :
— The Poisson bracket satisfies the conditions of a super Lie algebra structure.
— The map $f \mapsto X_f$ is an even homomorphism of (super) Lie algebras.

Super symplectic geometry

Definition : A momentum map for a symmetry group G with Lie algebra $\mathfrak{g}$ of a symplectic manifold (M, ω) is a map $J : M \to \mathfrak{g}^*$ satisfying the condition

$$\forall v \in \mathfrak{g} \ : \ \iota(v^M)\omega_\alpha = -d\langle v | J_\alpha \rangle \ .$$

It is said to be strongly hamiltonian if the map $\mathfrak{g} \to \mathcal{P}$, $v \mapsto \langle v | J \rangle$ is a Lie algebra morphism.

Proposition : Let G be a Lie group with Lie algebra $\mathfrak{g}$, let $\mu_o \in \mathfrak{g}^*$ be a fixed dual element, let $\mathcal{O}_{\mu_o}$ be its Coadjoint orbit and let ω^{KKS} be the Kirillov-Kostant-Souriau 2-form on $\mathcal{O}_{\mu_o}$ defined by

$$-\iota(v^*)\iota(w^*)\omega_\mu^{KKS} = \langle \ [v, w] \ | \mu \rangle \ .$$

Then ω^{KKS} is symplectic but not necessarily non-degenerate and the identity map $J : \mathcal{O}_{\mu_o} \to \mathfrak{g}^*$, $J(\mu) = \mu$ is a strongly hamiltonian momentum map.

Non-super prequantization

The stage : a symplectic manifold (M, ω).

Prequantization according to Kostant : a complex line bundle $\pi : L \to M$ over M with a connection ∇ whose curvature is the symplectic form:

$$\operatorname{curv}(\nabla) = \frac{-i\,\omega}{\hbar} \ .$$

Prequantization according to Souriau : a principal $\mathbf{S}^1$-bundle $\pi : Y \to M$ over M equipped with a 1-form α satisfying three conditions:

(i) α is invariant under the $\mathbf{S}^1$-action;

(ii) $\mathrm{d}\alpha = \pi^*\omega$;

(iii) $\int_{\mathbf{S}^1\text{-orbit}} \alpha = 2\pi\hbar$.

Non-super prequantization

Relation : L is the $\mathbf{C}$-line bundle associated to the principal $\mathbf{S}^1$-bundle Y by the tautological representation of $\mathbf{S}^1 \subset \mathbf{C}$ on $\mathbf{C}$.

Condition for existence : $\omega/\hbar$ represents an integral class in cohomology.

An equivalent condition : $\mathrm{Per}(\omega) \subset 2\pi\hbar\mathbf{Z}$, with

$$\mathrm{Per}(\omega) = \{ \int_\gamma \omega \mid \gamma \text{ a 2-cycle on } M \} \; .$$

Connection 1-forms for principal $\mathbf{S}^1$-bundles

Proposition : Let $d > 0$ be a fixed positive real number, let M be a manifold, let θ be a 1-form on M, let $G = \mathbf{R}/d\mathbf{Z}$ be the real line modulo d and let x be a coordinate (modulo d) on G. Then ∂_x is a left-invariant vector field on G and

$$\alpha = \big(\theta + dx\big) \otimes \partial_x$$

is a connection 1-form on the pricipal G-bundle $M \times G \to M$. Its curvature is given by

$$\operatorname{curv}(\alpha) = \mathrm{d}\theta \otimes \partial_x \ .$$

Proposition : Let ω be a closed 2-form on M whose group of periods is contained in $d\mathbf{Z}$. Then there exists a principal $\mathbf{R}/d\mathbf{Z}$-bundle with connection 1-form α whose curvature is $\omega \otimes \partial_x$.

The classical part of Prequantization

The stage : A symplectic manifold (M, ω) such that the group of periods of ω is discrete. This is a necessary and sufficient condition for the existence of a principal $(\mathcal{A}_0/d\mathbf{Z}) \times \mathcal{A}_1$-bundle $\pi : Y \to M$ with a 1-form α satisfying the following three conditions:

(i) α is invariant under the $(\mathcal{A}_0/d\mathbf{Z}) \times \mathcal{A}_1$-action;

(ii) $\mathrm{d}\alpha = \pi^*\omega$;

(iii) $\int_{\mathcal{A}_0/d\mathbf{Z}\text{-orbit}} \alpha = d$ is a non-zero constant.

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Remark 1 : Condition (iii) implies that the group of periods is included in $d\mathbf{Z}$.

Remark 2 : If e_0, e_1 is the appropriate basis for the Lie algebra of $(\mathcal{A}_0/d\mathbf{Z}) \times \mathcal{A}_1$, then $\alpha_0 \otimes e_0 + \alpha_1 \otimes e_1$ is a connection 1-form with curvature $\omega_0 \otimes e_0 + \omega_1 \otimes e_1$.

Remark 3 : All quotients of Y by $\mathbf{Z}/n\mathbf{Z} \subset \mathbf{S}^1$ satisfy the same conditions!

Remark 4 : The bundle Y is not necessarily unique.

The classical part of Prequantization

Proposition : The bundle $\pi : Y \to M$ has the following property.

For each $f \in \mathcal{P}$ there exists a unique vector field η_f on Y preserving α and projecting to the Hamiltonian vector field X_f of H on M. It (thus) satisfies

$$\mathcal{L}(\eta_f)\alpha = 0 \qquad , \qquad \iota(\eta_f)\alpha = \pi^* f \qquad \text{and} \qquad \pi_* \eta_f = X_f \ .$$

This correspondence is a Lie algebra *isomorphism* form the Poisson algebra $\mathcal{P}$ to the α-preserving vector fields on Y.

What about line bundle prequantization?

There is no representation of $(\mathcal{A}_0/d\mathbf{Z}) \times \mathcal{A}_1$ on a finite dimensional vector space that is not trivial on the $\mathcal{A}_1$-part. This means that the best we can do is:

Look for a "line" bundle $L \to M$ with a connection ∇ whose curvature is $-i\omega_0/\hbar$.

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Answer : I don’t know!

Remark : The bundle $\pi : Y \to M$ is not just a mathematical construction, it also has physical content.

- The space $Q = \mathbf{R}^{3N}/\mathfrak{S}_N$ represents the configuration space of N identical particles in $\mathbf{R}^3$. For $M = T^*Q$ there exist two (inequivalent) choices for the principal $\mathbf{S}^1$-bundle $\pi : Y \to M$, corresponding to the two characters for the permutation group $\mathfrak{S}_N$: the identity and the signature.

- Suppose we have an infinitely thin and infinitely long solenoid along the z-axis in $\mathbf{R}^3$ through which a current passes. This produces a magnetic field inside the solenoid, but no field outside. Assuming that a charged particle is not allowed in the solenoid, we thus have a configuration space $Q = \mathbf{R}^3 \backslash z$-axis. The inequivalent principal $\mathbf{S}^1$-bundles Y over the phase space $M = T^*Q$ are classified by a circle $\mathbf{R}/g\mathbf{Z}$. The Aharonov-Bohm experiment shows that, depending upon the current through the solenoid, there exists inequivalent physical situations, classified by a circle $\mathbf{R}/g\mathbf{Z}$ with the same g as above.