On the Kaup-Kupershmidt equation. Completeness relations for the squared soutions

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1. Introduction

- Kaup-Kupershmidt equation (KKE)

$$\partial_t f = \partial^5_{x^5} f + 10 f \partial^3_{x^3} f + 25 \partial_x f \partial^2_{x^2} f + 20 f^2 \partial_x f.$$

- S-integrability of KKE (existence of a Lax representation)

$$\begin{aligned}
\partial_t \mathcal{L} &= [\mathcal{L}, \mathcal{A}], \\
\mathcal{L} &= \partial^3_{x^3} + 2 f \partial_x + \partial_x f, \\
\mathcal{A} &= 9 \partial^5_{x^5} + 30 f \partial^3_{x^3} + 45 \partial_x f \partial^2_{x^2} \\
&+ (20 f^2 + 35 \partial^2_{x^2} f) \partial_x + 10 \partial^3_{x^3} f + 20 f \partial_x f.
\end{aligned}$$

- Factorization of $\mathcal{L}$ operator.

  Scalar third order differental operator $\Rightarrow$ matrix first order differential operator

$$\mathcal{L} \to L = i\partial_x + Q - \lambda J,$$

  where

$$Q = \begin{pmatrix} \tilde{Q} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tilde{Q} \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

  Lax operator assoc. with $\mathfrak{sl}(3, \mathbb{C})$ with a $\mathbb{Z}_3$ reduction.

2. Some necessary facts about Lie algebras and the Inverse scattering method

2.1. The $\mathfrak{sl}(r+1, \mathbb{C})$ algebra

- Definition of $\mathfrak{sl}(r+1, \mathbb{C})$

$$A \in \mathfrak{sl}(r+1, \mathbb{C}) \;\Leftrightarrow\; \operatorname{tr} A = 0.$$

- Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ — maximal commutative subalgebra. For $\mathfrak{sl}(r+1, \mathbb{C})$ the Cartan basis of $\mathfrak{h}$ reads

$$H_k = E_{kk} - \frac{1}{r+1} \sum_{j}^{r+1} E_{jj}, \qquad k = 1, \ldots, r.$$

- Root system Δ of $\mathfrak{g}$ contains all covectors α that

$$[H_k, E_\alpha] = \alpha(H_k) E_\alpha,$$

where $\alpha \in \mathfrak{h}^*$ is a root and $E_\alpha \in \mathfrak{g}$ is the corresponding root vector. For $\mathfrak{sl}(r+1)$ the roots can be presented by

$$e_i - e_j, \qquad i \neq j, \qquad i, j = 1, \ldots, r+1,$$

where $\{e_k\}_{k=1}^{r}$ is an orthonormal basis in the Euclidean space $\mathbb{E}^r$. The set of all simple roots is formed by

$$\alpha_j = e_j - e_{j+1},$$

while the maximal root is

$$\alpha_{\max} = e_1 - e_{r+1}.$$

- The Weyl basis of $\mathfrak{sl}(r+1)$ reads

$$E_{e_j - e_j} = E_{ij}, \quad (E_{ij})_{mn} = \delta_{im}\delta_{jn}.$$

We require that H_j and E_α are normalized with respect to Killing form, i.e.

$$\begin{aligned} \langle H_j, H_k \rangle &\equiv \frac{1}{2}\mathrm{tr}\,(H_j H_k) = \delta_{jk}, \\ \langle E_\alpha, E_\beta \rangle &= \delta_{\alpha,-\beta}. \end{aligned}$$

- The second Casimir operator P which has the important property

$$P(A \otimes B) = (B \otimes A)P, \qquad \forall\, A, B \in SL(r+1)$$

looks as follows

$$P = \sum_{j=1}^{r} H_j \otimes H_j + \sum_{\alpha \in \Delta} E_\alpha \otimes E_{-\alpha}.$$

2.2. Spectral problem for L operator

- Generalized Zakharov-Shabat system

$$L\psi = (i\partial_x + Q - \lambda J)\psi = 0,$$

where $Q(x), J \in \mathfrak{g}$ (without loss of generality J can be chosen as a Cartan element while $Q(x)$ is a linear combinations the Weyl generators of $\mathfrak{g}$) and $\psi(x,\lambda) \in G$. In the simplest case of zero boundary conditions, i.e. $\lim_{x\to\pm\infty} Q(x) = 0$ the continuous part of the spectrum of L fills up the $\mathbb{R}$-axis of the complex λ-plane.

- The scattering (data) matrix

$$T(\lambda) = \hat{\psi}_+(x,\lambda)\psi_-(x,\lambda), \qquad \lambda \in \mathbb{R},$$

where $\psi_\pm(x,\lambda)$ are Jost solutions to fulfill

$$\lim_{x\to\pm\infty} \psi_\pm(x,\lambda) e^{i\lambda J x} = \mathbb{1}.$$

There exist fundamental solutions $\chi^+(x,\lambda)$ and $\chi^-(x,\lambda)$ which possess analitical properties in $\mathbb{C}_+$ and $\mathbb{C}_-$. $\chi^+(x,\lambda)$ and $\chi^-(x,\lambda)$

are interrelated via

$$\chi^+(x,\lambda) = \chi^-(x,\lambda)G(\lambda), \qquad \lambda \in \mathbb{R},$$

$$G(\lambda) = \begin{cases} \hat{S}^-(\lambda)S^+(\lambda), \\ \hat{D}^-(\lambda)\hat{T}^+(\lambda)T^-(\lambda)D(\lambda). \end{cases}$$

$S^\pm(\lambda)$, $T^\pm(\lambda)$ and $D^\pm(\lambda)$ are factors in the Gauss decomposition of $T(\lambda)$

$$T(\lambda) = \begin{cases} T^-(\lambda)D^+(\lambda)\hat{S}^+(\lambda), \\ T^+(\lambda)D^-(\lambda)\hat{S}^-(\lambda). \end{cases}$$

2.3. Algebraic reductions

- Reduction group

  Let G_R be a discrete group acting on the set fundamental solutions $\{\psi(x,\lambda)\}$ as follows

$$C\psi(x,\kappa(\lambda))C^{-1} = \tilde{\psi}(x,\lambda).$$

The requirement of G_R-invariance of the lin. problem yields to the following conditions

$$CQ(x)C^{-1} = Q(x), \qquad \kappa(\lambda)CJC^{-1} = \lambda J.$$

- Coxeter type reduction

$$\kappa : \lambda \to \omega\lambda, \qquad \omega = e^{2i\pi/h},$$
$$C = \exp(\sum_k \omega^k H_k).$$

Consequently $Q(x)$ and J have the form

$$Q = \sum_{k=1}^{r} Q_k H_k, \qquad J = \sum_{\alpha\in A} E_\alpha,$$

where A stands for the set of all admissible roots (simple + minimal roots) of $\mathfrak{g}$. Thus KKE can be related to a $\mathbb{Z}_3$-reduced L operator assoc. with the $\mathfrak{sl}(3)$ algebra.

2.4. Spectral problem for L and Coxeter type reductions

The presence of a Coxeter reduction affects the spectral properties of L — its continuous spectrum consists of a bunch of $2h$ rays l_ν closing equal angles π/h. From now on we shall consider L operator assoc. with the $\mathfrak{sl}(3)$ algebra with a $\mathbb{Z}_3$ reduction. Each ray l_ν is connected with a $\mathfrak{sl}(2)$ subalgebra— the algebra $\{E_\alpha, E_{-\alpha}, H_\alpha\}$ generated by the root α to fulfill the condition

$$\delta_\nu \equiv \{\alpha \in \Delta |\ \mathrm{im}\lambda\alpha(J) = 0,\ \forall\,\lambda \in l_\nu\}, \quad \nu = 1, \dots, 6.$$

The following relations between these root subsystems hold true

$$\delta_\nu = \delta_{\nu+3}, \qquad \Delta = \bigcup_{\nu=1}^{3} \delta_\nu.$$

The λ-plane splits into 6 domains Ω_ν separ. by l_ν. One can introduce ordering in Ω_ν

$$\Delta_\nu^+ \equiv \{\alpha \in \Delta |\ \mathrm{Im}\,\lambda\alpha(J) > 0,\ \forall\,\lambda \in \Omega_\nu\},$$
$$\Delta_\nu^- \equiv \{\alpha \in \Delta |\ \mathrm{Im}\,\lambda\alpha(J) < 0,\ \forall\,\lambda \in \Omega_\nu\}$$

We shall use the notation $\delta_\nu^\pm = \Delta_\nu^\pm \cap \delta_\nu$ as well. One can easily see that the following symmetries hold

$$\Delta_{\nu+3}^\pm = \Delta_\nu^\mp, \qquad \delta_{\nu+3}^\pm = \delta_\nu^\mp.$$

In each Ω_ν exists a FAS $\chi^\nu(x,\lambda)$ in such a way that

$$\chi^\nu(x,\lambda) = \chi^{\nu-1}(x,\lambda) G^\nu(\lambda), \qquad \lambda \in l_\nu,$$

$$G^\nu(\lambda) = \left\{ \begin{array}{c} \hat{S}_\nu^-(\lambda) S_\nu^+(\lambda) \\ \hat{D}_\nu^-(\lambda) \hat{T}_\nu^+(\lambda) T_\nu^-(\lambda) D_\nu^+(\lambda). \end{array} \right.$$

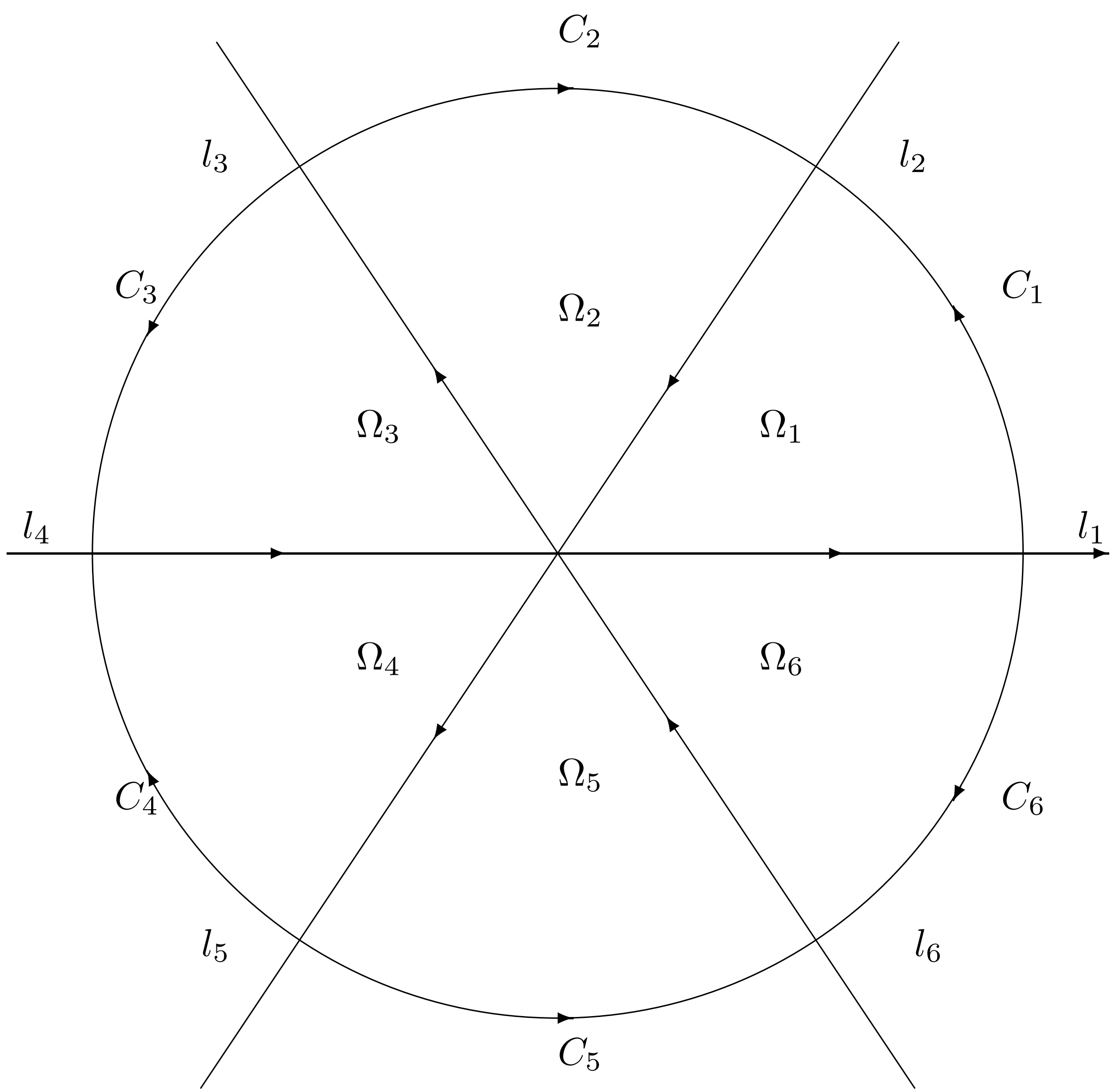

Figure 1: The contours of integration $\gamma_\nu = l_{\nu-1} \cup C_{\nu-1} \cup l_\nu$.

3. Completeness relations for squared solutions of a $\mathbb{Z}_3$ reduced scattering problem related to the algebra $\mathfrak{sl}(3,\mathbb{C})$

- Squared solutions (eigenfunctions) are introduced by

$$e_\alpha^{(\nu)}(x,\lambda) = P_J(\chi^\nu E_\alpha \hat{\chi}^\nu(x,\lambda)),$$
$$h_j^{(\nu)}(x,\lambda) = P_J(\chi^\nu H_j \hat{\chi}^\nu(x,\lambda)),$$

where P_J stands for the projection which maps onto $\mathfrak{sl}(3)/\mathfrak{h}$. They originate from the Wronskian relations, for example

$$(\hat{\chi}^\nu J \chi^\nu(x,\lambda) - J)|_{-\infty}^{\infty} = \int_{-\infty}^{\infty} dx \hat{\chi}^\nu [J, Q(x)] \chi^\nu(x,\lambda)$$

The next theorem represents our main result:

Theorem 1 *The squared solutions form a complete set with the following completeness relations*

$$\delta(x-y)\Pi = \frac{1}{2\pi}\sum_{\nu=1}^{6}(-1)^{\nu+1}\int_{l_\nu} d\lambda \left(G^{(\nu)}_{\beta_\nu}(x,y,\lambda) \right.$$

$$\left. -G^{(\nu-1)}_{-\beta_\nu}(x,y,\lambda) \right) - i\sum_{\nu=1}^{6}\sum_{n_\nu} \underset{\lambda=\lambda_{n_\nu}}{Res}\ G^{(\nu)}(x,y,\lambda).$$

where

$$\begin{aligned} G^{(\nu)}_{\beta_\nu}(x,y,\lambda) &= e^{(\nu)}_{\beta_\nu}(x,\lambda)\otimes e^{(\nu)}_{-\beta_\nu}(y,\lambda), \\ \Pi &= \sum_{\alpha\in\Delta^+}\frac{E_\alpha\wedge E_{-\alpha}}{\alpha(J)}. \end{aligned}$$

Proof: The completeness relations can be derived starting from the expression

$$\mathcal{J}(x,y) = \sum_{\nu=1}^{6}(-1)^{\nu+1}\oint_{\gamma_\nu} G^{(\nu)}(x,y,\lambda)d\lambda.$$

The Green functions $G^{(\nu)}(x,y,\lambda)$ have the form

$$G^{(\nu)}(x,y,\lambda) = \theta(y-x) \sum_{\alpha\in\Delta_\nu^+} e_\alpha^{(\nu)}(x,\lambda) \otimes e_{-\alpha}^{(\nu)}(y,\lambda)$$
$$-\theta(x-y)\left[\sum_{\alpha\in\Delta_\nu^-} e_\alpha^{(\nu)}(x,\lambda)\otimes e_{-\alpha}^{(\nu)}(y,\lambda) + \sum_{j=1}^{2} h_j^{(\nu)}(x,\lambda)\otimes h_j^{(\nu)}(y,\lambda)\right]$$

The proof can be done in three steps

1. **Lemma 1** *The following equality holds for* $\lambda \in l_\nu$

$$\sum_{\alpha\in\Delta} e_\alpha^{(\nu-1)}(x,\lambda)\otimes e_{-\alpha}^{(\nu-1)}(y,\lambda) + \sum_{j=1,2} h_j^{(\nu-1)}(x,\lambda)\otimes h_j^{(\nu-1)}(y,\lambda)$$
$$= \sum_{\alpha\in\Delta} e_\alpha^{(\nu)}(x,\lambda)\otimes e_{-\alpha}^{(\nu)}(y,\lambda) + \sum_{j=1,2} h_j^{(\nu)}(x,\lambda)\otimes h_j^{(\nu)}(y,\lambda).$$

According to Cauchy's residue theorem we have

$$\mathcal{J}(x,y) = 2\pi i \sum_{\nu=1}^{6}\sum_{n_\nu} \mathrm{Res}_{\lambda=\lambda_{n_\nu}} G^{(\nu)}(x,y,\lambda).$$

In the simplest case when $e_\alpha^{(\nu)}(x,\lambda)$ and $h_j^{(\nu)}(x,\lambda)$ possess only a single simple pole $\lambda_\nu \in \Omega_\nu$, i.e.

$$e_\alpha^{(\nu)}(x,\lambda) \approx \frac{e_{\alpha,n_\nu}^{(\nu)}(x)}{\lambda-\lambda_{n_\nu}} + \dot{e}_\alpha^{(\nu)}(x) + O(\lambda-\lambda_{n_\nu})$$

$$h_j^{(\nu)}(x,\lambda) \approx \frac{h_j^{(\nu)}(x)}{\lambda-\lambda_{n_\nu}} + \dot{h}_j^{(\nu)}(x) + O(\lambda-\lambda_{n_\nu}).$$

2. **Lemma 2** *Residues of* $G^{(\nu)}(x,y,\lambda)$ *are given by*

$$\mathrm{Res}_{\lambda=\lambda_{n_\nu}} G^{(\nu)}(x,y,\lambda) = \dot{e}_{\beta_\nu,n}^{(\nu)}(x)\otimes e_{-\beta_\nu,n}^{(\nu)}(y) + e_{\beta_\nu,n}^{(\nu)}(x)\otimes \dot{e}_{-\beta_\nu,n}^{(\nu)}(y).$$

Taking into account the orientation of the contours γ_ν we have

$$\mathcal{J}(x,y) = \sum_{\nu=1}^{6}(-1)^{\nu+1}\int_{l_\nu}(G^{(\nu)}(x,y,\lambda) - G^{(\nu-1)}(x,y,\lambda))d\lambda$$
$$+ \sum_{\nu=1}^{6}(-1)^{\nu+1}\int_{C_\nu} G^{(\nu)}(x,y,\lambda)d\lambda.$$

3. **Lemma 3** *In the integrals along the rays contribute only terms related to the roots that belong to* δ_ν^+ *and* δ_ν^- *respectively, i.e.*

$$\begin{aligned} &G^{(\nu)}(x,y,\lambda) - G^{(\nu-1)}(x,y,\lambda) \\ &\quad = e^{(\nu)}_{\beta_\nu}(x,\lambda) \otimes e^{(\nu)}_{-\beta_\nu}(y,\lambda) - e^{(\nu-1)}_{-\beta_\nu}(x,\lambda) \otimes e^{(\nu-1)}_{\beta_\nu}(y,\lambda). \end{aligned}$$

Asymptotically $G^{(\nu)}(x,y,\lambda)$ is an entire function hence we are allowed to deform the arcs C_ν into $\bar{l}_\nu \cup \bar{l}_{\nu+1}$. Thus for the integrals along the arcs C_ν one obtains

$$\sum_{\nu=1}^{6} (-1)^{\nu+1} \int_{C_\nu} G^{(\nu)}(x,y,\lambda) d\lambda = 2\pi\delta(x-y) \sum_{\alpha\in\Delta^+} \frac{(E_\alpha \wedge E_{-\alpha})}{\alpha(J)}.$$

- Importance of the completeness relations of the squared solutions.

  Completeness of the squared solutions $\Leftrightarrow$ a basis in a functional space.

$$X(x) = \frac{1}{2\pi}\sum_{\nu=1}^{6}(-1)^{\nu+1}\int_{l_\nu} \mathrm{d}\lambda \left(X_{\beta_\nu} e_{\beta_\nu}^{(\nu)}(x,\lambda) - X_{-\beta_\nu} e_{-\beta_\nu}^{(\nu-1)}(x,\lambda)\right) - i\sum_{\nu=1}^{6} X_\nu .$$