Constant mean curvature surfaces in Minkowski 3-space via loop groups

David Brander

Now: Department of Mathematics
Kobe University

(From August 2008: Danish Technical University)

Geometry, Integrability and Quantization - Varna 2008

Outline

CMC Surfaces in Euclidean Space

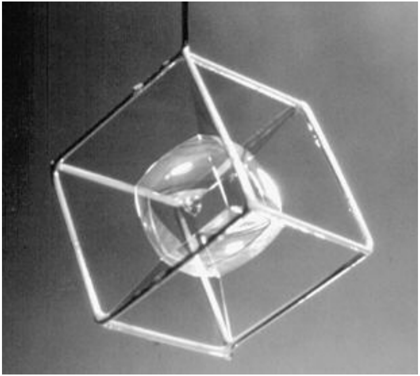
Outline

CMC Surfaces in Euclidean Space

CMC surfaces in Minkowski 3-Space

- The loop group construction

Constant Mean Curvature Surfaces in Euclidean 3-space



- Soap films are CMC surfaces.
- Air pressure on both sides of surface the same $\leftrightarrow$ mean curvature $H = 0$, *minimal surface*

Minimal Surfaces: $H = 0$

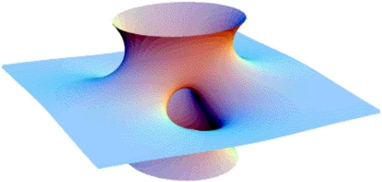


Figure: Costa's surface

- Gauss map of a minimal surface is *holomorphic*.

Minimal Surfaces: $H = 0$

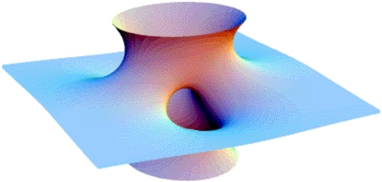


Figure: Costa's surface

- Gauss map of a minimal surface is *holomorphic*.
- **Weierstrass representation:** pair of holomorphic functions $\leftrightarrow$ minimal surface

CMC $H \neq 0$ Surfaces

Figure: A constant non-zero mean curvature surface

- Gauss map is a **harmonic** (not holomorphic) map into

$$S^2 = SU(2)/K,$$

$$K = \{\text{diagonal matrices}\}.$$

CMC $H \neq 0$ Surfaces

Figure: A constant non-zero mean curvature surface

- Gauss map is a **harmonic** (not holomorphic) map into

$$S^2 = SU(2)/K,$$

$$K = \{\text{diagonal matrices}\}.$$

- Loop group frame F_λ.

CMC $H \neq 0$ Surfaces

Figure: A constant non-zero mean curvature surface

- Gauss map is a **harmonic** (not holomorphic) map into
$$S^2 = SU(2)/K,$$
$K = \{\text{diagonal matrices}\}$.
- Loop group frame F_λ.
- Can recover f from the loop group map F_λ via a simple formula.

Loop Group Methods

- $\Lambda G^{\mathbb{C}} = \{\gamma : \mathbb{S}^1 \to G^{\mathbb{C}} \mid \gamma \text{ smooth}\}$
- $F_\lambda : M \to \Lambda G^{\mathbb{C}}$ is of *connection order* (a, b) if

$$F_\lambda^{-1} \mathrm{d} F_\lambda = \sum_a^b a_i \lambda^i .$$

Loop Group Methods

- $\Lambda G^{\mathbb{C}} = \{\gamma : \mathbb{S}^1 \to G^{\mathbb{C}} \mid \gamma \text{ smooth}\}$
- $F_\lambda : M \to \Lambda G^{\mathbb{C}}$ is of *connection order* (a, b) if

$$F_\lambda^{-1} \mathrm{d} F_\lambda = \sum_a^b a_i \lambda^i.$$

- **Example:** flat surfaces in S^3.

$$F_\lambda^{-1} \mathrm{d} F_\lambda = \begin{pmatrix} \omega & \lambda\beta & \lambda\theta \\ -\lambda\beta^t & 0 & 0 \\ -\lambda\theta^t & 0 & 0 \end{pmatrix} = a_0 + a_1 \lambda,$$

order $(0, 1)$.

Loop Group Methods

$F_\lambda : M \to \Lambda G^{\mathbb{C}}$ is of *connection order* (a, b) if

$$F_\lambda^{-1} \mathrm{d} F_\lambda = \sum_a^b a_i \lambda^i .$$

1. **AKS theory:**

2. **KDPW Method:**

3. **Dressing:**

Loop Group Methods

$F_\lambda : M \to \Lambda G^{\mathbb{C}}$ is of *connection order* (a, b) if

$$F_\lambda^{-1} \mathrm{d} F_\lambda = \sum_a^b a_i \lambda^i .$$

1. **AKS theory:** Constructs order $(0, b)$ maps, $b > 0$, by solving ODE's.
   Related to inverse scattering.
2. **KDPW Method:**
3. **Dressing:**

Loop Group Methods

$F_\lambda : M \to \Lambda G^{\mathbb{C}}$ is of *connection order* (a, b) if

$$F_\lambda^{-1} \mathrm{d} F_\lambda = \sum_a^b a_i \lambda^i .$$

1. **AKS theory:** Constructs order $(0, b)$ maps, $b > 0$, by solving ODE's.
   Related to inverse scattering.
2. **KDPW Method:** Constructs order (a, b) maps, $a < 0 < b$, from a pair of $(a, 0)$ and $(0, b)$ maps.
3. **Dressing:**

Loop Group Methods

$F_\lambda : M \to \Lambda G^{\mathbb{C}}$ is of *connection order* (a, b) if

$$F_\lambda^{-1} \mathrm{d} F_\lambda = \sum_a^b a_i \lambda^i.$$

1. **AKS theory:** Constructs order $(0, b)$ maps, $b > 0$, by solving ODE's.
   Related to inverse scattering.
2. **KDPW Method:** Constructs order (a, b) maps, $a < 0 < b$, from a pair of $(a, 0)$ and $(0, b)$ maps.
3. **Dressing:** Any kind of connection order (a, b) maps.
   Produces families of new solutions from a given solution.

Krichever-Dorfmeister-Pedit-Wu (KDPW) Method

- Need **Birkhoff factorization**:

$$\Lambda G^{\mathbb{C}} \quad \text{“}=\text{”} \quad \Lambda^+ G^{\mathbb{C}} \cdot \Lambda^- G^{\mathbb{C}},$$

where $\Lambda^\pm G^{\mathbb{C}}$ consists of loops which extend holomorphically to $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \mathbb{D}$ resp.

Krichever-Dorfmeister-Pedit-Wu (KDPW) Method

- Need **Birkhoff factorization**:

$$\Lambda G^{\mathbb{C}} \quad \text{``}=\text{''} \quad \Lambda^+ G^{\mathbb{C}} \cdot \Lambda^- G^{\mathbb{C}},$$

  where $\Lambda^\pm G^{\mathbb{C}}$ consists of loops which extend holomorphically to $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \mathbb{D}$ resp.
- If F_λ is of order (a, b), $a < 0 < b$, decompose

$$F = F_+ G_- = F_- G_+.$$

Krichever-Dorfmeister-Pedit-Wu (KDPW) Method

- Need **Birkhoff factorization**:

$$\Lambda G^{\mathbb{C}} \quad \text{``}=\text{''} \quad \Lambda^+ G^{\mathbb{C}} \cdot \Lambda^- G^{\mathbb{C}},$$

  where $\Lambda^{\pm} G^{\mathbb{C}}$ consists of loops which extend holomorphically to $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \mathbb{D}$ resp.
- If F_λ is of order (a, b), $a < 0 < b$, decompose

$$F = F_+ G_- = F_- G_+.$$

- Then F_+ is of order $(0, b)$ and F_- is of order $(a, 0)$:

Krichever-Dorfmeister-Pedit-Wu (KDPW) Method

- Need **Birkhoff factorization**:

$$\Lambda G^{\mathbb{C}} \text{ ``$=$'' } \Lambda^+ G^{\mathbb{C}} \cdot \Lambda^- G^{\mathbb{C}},$$

  where $\Lambda^{\pm} G^{\mathbb{C}}$ consists of loops which extend holomorphically to $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \mathbb{D}$ resp.
- If F_λ is of order (a, b), $a < 0 < b$, decompose

$$F = F_+ G_- = F_- G_+.$$

- Then F_+ is of order $(0, b)$ and F_- is of order $(a, 0)$:

$$\begin{aligned} F_+^{-1} \mathrm{d}F_+ &= G_-(F^{-1}\mathrm{d}F)G_-^{-1} + G_- \mathrm{d}G_-^{-1} \\ &= G_-(\sum_a^b a_i \lambda^i)G_-^{-1} + G_- \mathrm{d}G_-^{-1} \\ &= \qquad . \end{aligned}$$

Krichever-Dorfmeister-Pedit-Wu (KDPW) Method

- Need **Birkhoff factorization**:

$$\Lambda G^{\mathbb{C}} \ \ \text{“}=\text{”} \ \ \Lambda^+ G^{\mathbb{C}} \cdot \Lambda^- G^{\mathbb{C}},$$

  where $\Lambda^{\pm} G^{\mathbb{C}}$ consists of loops which extend holomorphically to $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \mathbb{D}$ resp.
- If F_λ is of order (a, b), $a < 0 < b$, decompose

$$F = F_+ G_- = F_- G_+.$$

- Then F_+ is of order $(0, b)$ and F_- is of order $(a, 0)$:

$$\begin{aligned} F_+^{-1} dF_+ &= G_-(F^{-1}dF)G_-^{-1} + G_- dG_-^{-1} \\ &= G_-(\sum_a^b a_i \lambda^i) G_-^{-1} + G_- dG_-^{-1} \\ &= c_0 + \ldots + c_b \lambda^b. \end{aligned}$$

KDPW Method

- Conversely, given order $(0, b)$ and $(a, 0)$ maps, F_+ and F_-, we can construct an order (a, b) map F.
- After a normalization, both directions unique:

$$F \longleftrightarrow \left\{ \begin{array}{c} F_+ \\ F_- \end{array} \right\}$$

$$(a, b) \qquad \begin{array}{c} (0, b) \\ (a, 0) \end{array}$$

Specific Case

Harmonic Maps into Symmetric Spaces

- G/K symmetric space, $K = G_\sigma$.
- On $\Lambda G^{\mathbb{C}}$, define involution $\hat{\sigma}$:

$$(\hat{\sigma}\gamma)(\lambda) := \sigma(\gamma(-\lambda)).$$

Specific Case

Harmonic Maps into Symmetric Spaces

- G/K symmetric space, $K = G_\sigma$.
- On $\Lambda G^{\mathbb{C}}$, define involution $\hat{\sigma}$:
- Fixed point subgroup $\Lambda G_{\hat{\sigma}} \subset \Lambda G^{\mathbb{C}}_{\hat{\sigma}} \subset \Lambda G^{\mathbb{C}}$.

- $F_\lambda(z)$ a connection order $(-1, 1)$ map, $\mathbb{C} \to \Lambda G_{\hat{\sigma}}$.
- **KDPW:**

- $F_\lambda(z)$ a connection order $(-1, 1)$ map, $\mathbb{C} \to \Lambda G_{\hat{\sigma}}$.
- **KDPW:** $F \leftrightarrow \{F_+, F_-\}$
- In this case, F_+ determined by F_-, so

$$F \leftrightarrow F_-$$

- $F_\lambda(z)$ a connection order $(-1, 1)$ map, $\mathbb{C} \to \Lambda G_{\hat{\sigma}}$.
- **KDPW:**

$$F \;\leftrightarrow\; F_-$$

- **Fix** $\lambda \in S^1$**:** then $F_\lambda : \mathbb{C} \to G$.

- $F_\lambda(z)$ a connection order $(-1,1)$ map, $\mathbb{C} \to \Lambda G_{\hat{\sigma}}$.
- **KDPW:**

$$F \;\leftrightarrow\; F_-$$

- **Fix** $\lambda \in S^1$**:** then $F_\lambda : \mathbb{C} \to G$.
- **Fact:** Projection of F, to G/K, is a *harmonic* map $\mathbb{C} \to G/K$ **if and only if** F_- is *holomorphic* in z:

- $F_\lambda(z)$ a connection order $(-1,1)$ map, $\mathbb{C} \to \Lambda G_{\hat{\sigma}}$.
- **KDPW:**

$$F \leftrightarrow F_-$$

- **Fix** $\lambda \in S^1$**:** then $F_\lambda : \mathbb{C} \to G$.
- **Fact:** Projection of F, to G/K, is a *harmonic* map $\mathbb{C} \to G/K$ **if and only if** F_- is *holomorphic* in z:

$$\textit{order } (-1,1) \qquad F \leftrightarrow F_- \qquad \textit{order } (-1,-1)$$

harmonic *holomorphic*

"Weierstrass Representation" for CMC $H \neq 0$ Surfaces

- $a(z)$, $b(z)$ arbitrary holomorphic. Set

$$\alpha = \begin{pmatrix} 0 & a(z) \\ b(z) & 0 \end{pmatrix} \lambda^{-1} dz.$$

"Weierstrass Representation" for CMC $H \neq 0$ Surfaces

- $a(z)$, $b(z)$ arbitrary holomorphic. Set

$$\alpha = \begin{pmatrix} 0 & a(z) \\ b(z) & 0 \end{pmatrix} \lambda^{-1} dz.$$

- Automatically, $d\alpha + \alpha \wedge \alpha = 0$. Integrate to get $F_- : \Sigma \to \Lambda G$, connection order $(-1, -1)$.

"Weierstrass Representation" for CMC $H \neq 0$ Surfaces

- $a(z)$, $b(z)$ arbitrary holomorphic. Set

$$\alpha = \begin{pmatrix} 0 & a(z) \\ b(z) & 0 \end{pmatrix} \lambda^{-1} dz.$$

- Automatically, $d\alpha + \alpha \wedge \alpha = 0$. Integrate to get $F_- : \Sigma \to \Lambda G$, connection order $(-1, -1)$.
- Apply KDPW correspondence to get F, frame for harmonic map.

“Weierstrass Representation” for CMC $H \neq 0$ Surfaces

- $a(z)$, $b(z)$ arbitrary holomorphic. Set

$$\alpha = \begin{pmatrix} 0 & a(z) \\ b(z) & 0 \end{pmatrix} \lambda^{-1} dz.$$

- Automatically, $d\alpha + \alpha \wedge \alpha = 0$. Integrate to get $F_- : \Sigma \to \Lambda G$, connection order $(-1, -1)$.
- Apply KDPW correspondence to get F, frame for harmonic map.
- CMC surface obtained from F by **Sym-Bobenko formula**.

"Weierstrass Representation" for CMC $H \neq 0$ Surfaces

- $a(z)$, $b(z)$ arbitrary holomorphic. Set

$$\alpha = \begin{pmatrix} 0 & a(z) \\ b(z) & 0 \end{pmatrix} \lambda^{-1} \mathrm{d}z.$$

- Automatically, $\mathrm{d}\alpha + \alpha \wedge \alpha = 0$. Integrate to get $F_- : \Sigma \to \Lambda G$, connection order $(-1, -1)$.
- Apply KDPW correspondence to get F, frame for harmonic map.
- CMC surface obtained from F by **Sym-Bobenko formula**.
- All CMC surfaces in $\mathbb{R}^3$ are obtained this way.

Iwasawa Decomposition

- In fact for harmonic maps, for the $\leftarrow$ direction of KDPW, need *Iwasawa splitting*

$$\Lambda G^{\mathbb{C}} \text{ ``}=\text{''} \; \Lambda G \cdot \Lambda^{+} G^{\mathbb{C}}.$$

- This holds globally if G is **compact.**
- F is obtained from F_- via:

$$F_- = F G_+.$$

- More generally, for the $\leftarrow$ direction, the holomorphic map F_- can be of order $(-1, b)$ where $b \geq -1$.

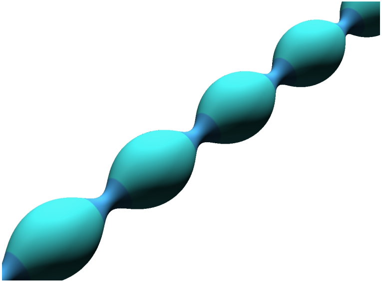


Figure: CMC Unduloid

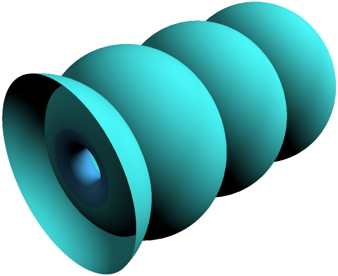


Figure: CMC Nodoid

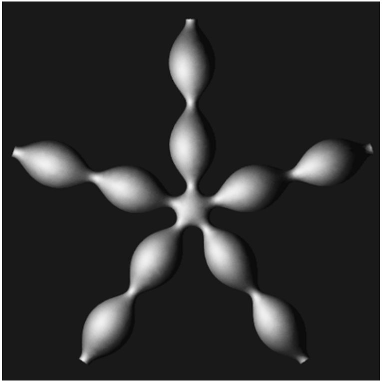


Figure: CMC 5-noid

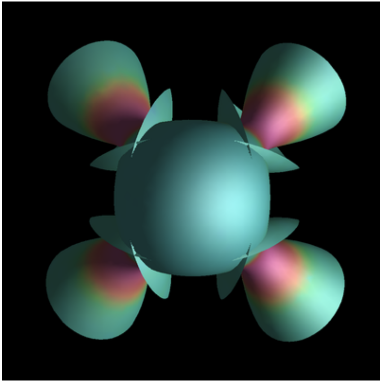


Figure: A Smyth Surface

CMC surfaces in Minkowski space, L^3

B.-, Rossman, Schmitt - *"Holomorphic representation of constant mean curvature surfaces in Minkowski space"* - Preprint

- Construction analogous to CMC in $\mathbb{R}^3$. Change $SU(2) \mapsto SU(1,1)$.

CMC surfaces in Minkowski space, L^3

B.-, Rossman, Schmitt - *"Holomorphic representation of constant mean curvature surfaces in Minkowski space"* - Preprint

- Construction analogous to CMC in $\mathbb{R}^3$. Change $SU(2) \mapsto SU(1,1)$.
- **Only difference:** $SU(1,1)$ *non-compact* $\Rightarrow$ Iwasawa decomposition not global.
- Iwasawa defined on an open dense set (the "big cell")

CMC surfaces in Minkowski space, L^3

B.-, Rossman, Schmitt - *"Holomorphic representation of constant mean curvature surfaces in Minkowski space"* - Preprint

- Construction analogous to CMC in $\mathbb{R}^3$. Change $SU(2) \mapsto SU(1,1)$.
- **Only difference:** $SU(1,1)$ *non-compact* $\Rightarrow$ Iwasawa decomposition not global.
- Iwasawa defined on an open dense set (the "big cell")
- Surface has singularities at boundary of this set.

Classification of surfaces with rotational symmetry

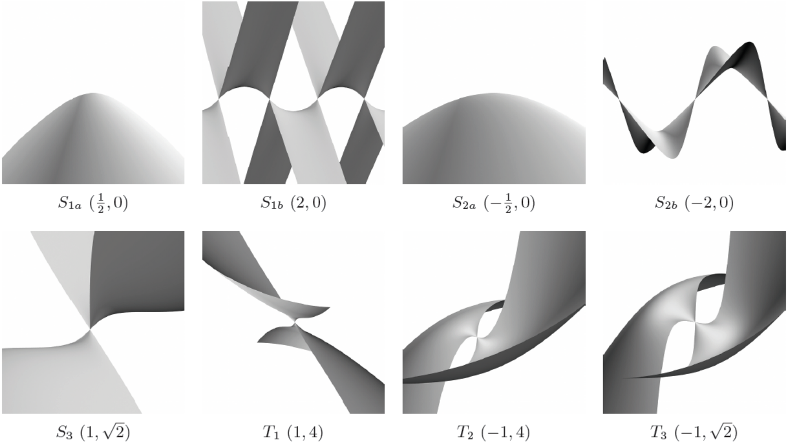


Figure: Examples from each of the eight families of surfaces with rotational symmetry in L^3. (Images made by Nick Schmitt's *XLab*.)

Smyth surfaces in L^3

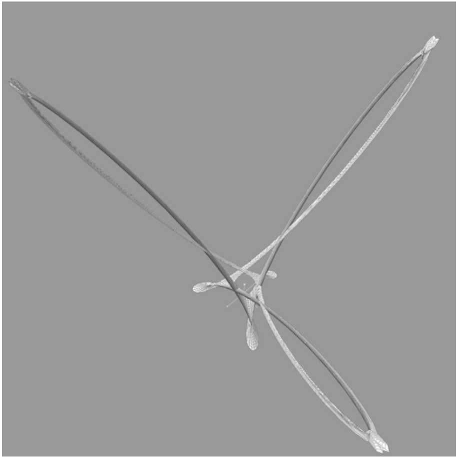


Figure: A Smyth surface in L^3

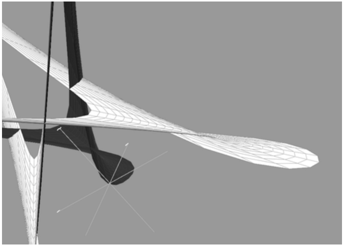


Figure: Swallowtail singularity

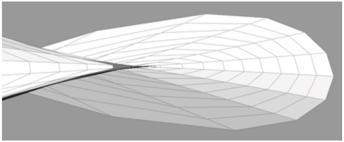


Figure: Swallowtail singularity

Outline

CMC Surfaces in Euclidean Space

CMC surfaces in Minkowski 3-Space

- The loop group construction

The loop group construction

- $\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The loop group construction

- $\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $G = SU(1,1) \cup i\sigma_1 \cdot SU(1,1)$
- $G^{\mathbb{C}} = SL(2,\mathbb{C})$

The loop group construction

- $\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $G = SU(1,1) \cup i\sigma_1 \cdot SU(1,1)$
- $G^{\mathbb{C}} = SL(2,\mathbb{C})$
- $\Lambda G^{\mathbb{C}} = \{\gamma : \mathbb{S}^1 \to G^{\mathbb{C}} \mid \gamma \text{ smooth}\}$

The loop group construction

- $\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $G = SU(1,1) \cup i\sigma_1 \cdot SU(1,1)$
- $G^{\mathbb{C}} = SL(2,\mathbb{C})$
- $\Lambda G^{\mathbb{C}} = \{\gamma : \mathbb{S}^1 \to G^{\mathbb{C}} \mid \gamma \text{ smooth}\}$
- $\Lambda G^{\mathbb{C}}_{\sigma} := \{x \in \Lambda G^{\mathbb{C}} | \, \sigma(x) = x\}$, where,
$$(\sigma(x))(\lambda) := \mathrm{Ad}_{\sigma_3} \, x(-\lambda).$$
- $\Lambda^{\pm} G^{\mathbb{C}}_{\sigma} := \Lambda G^{\mathbb{C}}_{\sigma} \cap \Lambda^{\pm} G^{\mathbb{C}}$

- $\Lambda G_\sigma := \Lambda G \cap \Lambda G_\sigma^{\mathbb{C}}$ - “real form".
- Note: $\Lambda G_\sigma = \Lambda SU(1,1)_\sigma \cup i\sigma_1 \cdot \Lambda SU(1,1)_\sigma$.

- $\Lambda G_\sigma := \Lambda G \cap \Lambda G_\sigma^{\mathbb{C}}$ - "real form".
- Note: $\Lambda G_\sigma = \Lambda SU(1,1)_\sigma \cup i\sigma_1 \cdot \Lambda SU(1,1)_\sigma$.
- Setting $x^*(\lambda) := \overline{x(\bar{\lambda}^{-1})}$, Then

$$\Lambda SU(1,1)_\sigma = \left\{ \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} \in \Lambda G_\sigma^{\mathbb{C}} \right\},$$

$$i\sigma_1 \cdot \Lambda SU(1,1)_\sigma = \left\{ \begin{pmatrix} a & b \\ -b^* & -a^* \end{pmatrix} \in \Lambda G_\sigma^{\mathbb{C}} \right\},$$

Loop group characterization for CMC surfaces in L^3

- Σ Riemmann surface
- $F : \Sigma \to \Lambda G_\sigma$ of connection order $(-1, 1)$

Loop group characterization for CMC surfaces in L^3

- Σ Riemmann surface
- $F : \Sigma \to \Lambda G_\sigma$ of connection order $(-1, 1)$
- $F_- : \Sigma \to \Lambda^- G_\sigma^{\mathbb{C}}$ of order $(-1, -1)$, associated to F via (normalized) Birkhoff splitting:

$$F = F_- G_+ .$$

Loop group characterization for CMC surfaces in L^3

- Σ Riemmann surface
- $F : \Sigma \to \Lambda G_\sigma$ of connection order $(-1, 1)$
- $F_- : \Sigma \to \Lambda^- G_\sigma^{\mathbb{C}}$ of order $(-1, -1)$, associated to F via (normalized) Birkhoff splitting:

$$F = F_- G_+ .$$

- For $\lambda_0 \in \mathbb{S}^1$, set

$$f^{\lambda_0} = -\frac{1}{2H} \mathcal{S}(F)\Big|_{\lambda=\lambda_0},$$
$$\mathcal{S}(F) := F i \sigma_3 F^{-1} + 2i\lambda \partial_\lambda F \cdot F^{-1} \ .$$

Loop group characterization for CMC surfaces in L^3

- Σ Riemmann surface
- $F : \Sigma \to \Lambda G_\sigma$ of connection order $(-1, 1)$
- $F_- : \Sigma \to \Lambda^- G_\sigma^{\mathbb{C}}$ of order $(-1, -1)$, associated to F via (normalized) Birkhoff splitting:

$$F = F_- G_+ .$$

- For $\lambda_0 \in \mathbb{S}^1$, set

$$f^{\lambda_0} = -\frac{1}{2H} \mathcal{S}(F)\Big|_{\lambda=\lambda_0},$$
$$\mathcal{S}(F) := F i \sigma_3 F^{-1} + 2 i \lambda \partial_\lambda F \cdot F^{-1} .$$

- *F_- holomorphic* if and only if $f^{\lambda_0} : \Sigma \to L^3$ has constant mean curvature H.

$SU(1,1)$ Iwasawa decomposition

Need:

$$\omega_m = \begin{pmatrix} 1 & 0 \\ \lambda^{-m} & 1 \end{pmatrix}, \quad m \text{ odd} \quad ; \quad \omega_m = \begin{pmatrix} 1 & \lambda^{1-m} \\ 0 & 1 \end{pmatrix}, \quad m \text{ even.}$$

Theorem

($SU(1,1)$ Iwasawa decomposition)

$$\Lambda G_\sigma^{\mathbb{C}} = \mathcal{B}_{1,1} \sqcup \bigsqcup_{n \in \mathbb{Z}^+} \mathcal{P}_n,$$

$$\textbf{big cell: } \mathcal{B}_{1,1} := \Lambda G_\sigma \cdot \Lambda^+ G_\sigma^{\mathbb{C}},$$

$$\textbf{n'th small cell: } \mathcal{P}_n := \Lambda SU(1,1)_\sigma \cdot \omega_n \cdot \Lambda^+ G_\sigma^{\mathbb{C}}.$$

- *$\mathcal{B}_{1,1}$, is an open dense subset of $\Lambda G_\sigma^{\mathbb{C}}$.*
- *Any $\phi \in \mathcal{B}_{1,1}$ can be expressed as*

$$\phi = FB, \qquad F \in \Lambda G_\sigma, \qquad B \in \Lambda^+ G_\sigma^{\mathbb{C}}, \tag{1}$$

  F unique up to right multiplication by $G_\sigma := \Lambda G_\sigma \cap G$.
- *The map $\pi : \mathcal{B}_{1,1} \to \Lambda G_\sigma / G_\sigma$ given by $\phi \mapsto [F]$, derived from (1), is a real analytic projection.*

Theorem

(Holomorphic rep. for spacelike CMC surfaces in L^3) Let

$$\xi = \sum_{i=-1}^{\infty} A_i \lambda^i \mathrm{d}z \quad \in \; Lie(\Lambda G_\sigma^{\mathbb{C}}) \otimes \Omega^1(\Sigma)$$

be a holomorphic 1-form over a simply-connected Riemann surface Σ, with

$$a_{-1} \neq 0,$$

on Σ, where $A_{-1} = \begin{pmatrix} 0 & a_{-1} \\ b_{-1} & 0 \end{pmatrix}$. Let $\phi : \Sigma \to \Lambda G_\sigma^{\mathbb{C}}$ be a solution of

$$\phi^{-1} d\phi = \xi.$$

On $\Sigma^\circ := \phi^{-1}(\mathcal{B}_{1,1})$, G-Iwasawa split:

$$\phi = FB, \qquad F \in \Lambda G_\sigma, \quad B \in \Lambda^+ G_\sigma^{\mathbb{C}}. \tag{2}$$

Then for any $\lambda_0 \in \mathbb{S}^1$, the map $f^{\lambda_0} := \hat{f}^{\lambda_0} : \Sigma^\circ \to L^3$, given by the Sym-Bobenko formula, is a conformal CMC H immersion, and is independent of the choice of F in (2).

Example 1: hyperboloid of two sheets.

$$\xi = \begin{pmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{pmatrix} dz, \qquad \Sigma = \mathbb{C}.$$

$$\phi = \begin{pmatrix} 1 & z\lambda^{-1} \\ 0 & 1 \end{pmatrix} : \Sigma \to \Lambda G_\sigma^{\mathbb{C}}.$$

Takes values in $\mathcal{B}_{1,1}$ for $|z| \neq 1$. G-Iwasawa:

$$\phi = F \cdot B, \quad F : \Sigma \setminus \mathbb{S}^1 \to \Lambda G, \quad B : \Sigma \setminus \mathbb{S}^1 \to \Lambda^+ G_\sigma^{\mathbb{C}},$$

$$F = \frac{1}{\sqrt{\varepsilon(1-|z|^2)}} \begin{pmatrix} \varepsilon & z\lambda^{-1} \\ \varepsilon\bar{z}\lambda & 1 \end{pmatrix},$$

$$B = \frac{1}{\sqrt{\varepsilon(1-|z|^2)}} \begin{pmatrix} 1 & 0 \\ -\varepsilon\bar{z}\lambda & \varepsilon(1-z\bar{z}) \end{pmatrix}, \qquad \varepsilon = \mathrm{sign}(1-|z|^2)\,.$$

Sym-Bobenko formula gives

$$\hat{f}^1(z) = \frac{1}{H(x^2+y^2-1)} \cdot [2y,\ -2x,\ (1+3x^2+3y^2)/2],$$

two-sheeted hyperboloid $\{x_1^2 + x_2^2 - (x_0 - \frac{1}{2H})^2 = -\frac{1}{H^2}\}$.

Hyperboloid: boundary of big cell behaviour

- In a small cell precisely when $|z| = 1$:
- There, have $\phi \in \Lambda SU(1,1)_\sigma \cdot \omega_2 \cdot \Lambda^+ G_\sigma^{\mathbb{C}}$:

$$\begin{pmatrix} 1 & z\lambda^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p\sqrt{z} & \lambda^{-1} q\sqrt{z} \\ \lambda q \sqrt{z}^{-1} & p\sqrt{z}^{-1} \end{pmatrix} \cdot \omega_2 \cdot \begin{pmatrix} (p+q)\sqrt{z}^{-1} & 0 \\ -\lambda q \sqrt{z}^{-1} & (p-q)\sqrt{z} \end{pmatrix}$$

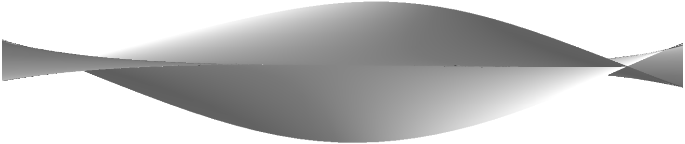
  where $p^2 - q^2 = 1$ and $p, q \in \mathbb{R}$.
- That is: $\phi \in \mathcal{P}_2$ for $|z| = 1$.
- Note: Surface blows up as $|z| \to 1$.

Example 2: numerical experiment

$$\xi = \lambda^{-1} \cdot \begin{pmatrix} 0 & 1 \\ 100\,z & 0 \end{pmatrix} dz,$$

Numerically:

1. Integrate with i.c. $\phi(0) = \omega_1$, to get $\phi : \Sigma \to \Lambda G_\sigma^{\mathbb{C}}$.
2. Iwasawa split to get $F : \Sigma \to \Lambda G_\sigma$.
3. Compute Sym-Bobenko formula to get $f^1 : \Sigma \to L^3$.
4. Use XLab to view the surface.



- Looks like *Shcherbak surface* singularity at $z = 0$.
- Since $\phi(0) = \omega_1$, the singularity occurs at $\mathcal{P}_1$.

Results on boundary of big cell behaviour

- **Proved:**
  1. The map $f^{\lambda_0} : \Sigma \to L^3$ always well defined (and real analytic) at $z_0 \in \phi^{-1}(\mathcal{P}_1)$, but *not immersed* at such a point.
  2. The map $f^{\lambda_0} : \Sigma \to L^3$ always blows up as $z \to z_0 \in \phi^{-1}(\mathcal{P}_2)$.

Results on boundary of big cell behaviour

- **Proved:**
  1. The map $f^{\lambda_0} : \Sigma \to L^3$ always well defined (and real analytic) at $z_0 \in \phi^{-1}(\mathcal{P}_1)$, but *not immersed* at such a point.
  2. The map $f^{\lambda_0} : \Sigma \to L^3$ always blows up as $z \to z_0 \in \phi^{-1}(\mathcal{P}_2)$.
- **Expect** (have not proved yet):
  - *generic* holomorphic data does not encounter $\mathcal{P}_n$ for $n > 2$.
  - Therefore: generic singularities of CMC surfaces occur only at points in $\mathcal{P}_1$.