Torsion and Einstein-Gauss-Bonnet gravity in D=5

F. C. **arXiv:0706.3538**.

F. C., A. Giacomini, S. Willison, **PHYSICAL REVIEW D 76, 044021 (2007)**

F. C., A. Giacomini, R. Troncoso, PHYSICAL REVIEW D 77,024002 (2008)

Fabrizio Canfora

CECS, Valdivia

Motivations

- In 4 dimensions the EH action tells that torsion vanishes (in vacuum). But in D>4 the Lovelock actions do not imply that torsion vanishes in vacuum.
- The geometrical structure of torsion shares many features of BPS dynamics and allows a stability analysis which without torsion would be impossible.
- One can obtain exact solutions which have very nice structure such as the purely gravitational analogue of the Bertotti-Robinson space-time where the torsion can be seen as the dual of the covariantly constant electromagnetic field.

But…

- Usually in a generic Lovelock Lagrangian it is very difficult to find exact solutions with torsion: to the best of my knowledge exact solutions with torsion only exist in the Chern-Simons case.
- Even in the Chern-Simons case, in which the equations simplify due to the enhanced gauge invariance, the known solutions with non-vanishing torsion appear to be such that one can make torsion vanishing with a gauge transformation.
- The analogy with BPS states will help us to overcome these difficulties…

Plan of the talk

- The analogy of gravity with torsion and BPS states and how one can get an ansatz for the torsion is shortly discussed.
- Then, how this ansatz works in the non-Chern-Simons case providing one with a vacuum solution very similar to Bertotti-Robinson is showed. Apparently, this is the first exact solution with torsion in the non-Chern-Simons case.
- Eventually, how this ansatz works in the Chern-Simons case providing the spherically symmetric BH with a half BPS ground state is discussed.

BPS analogy

In the Yang-Mills-Higgs theory, the BPS equations involve typically linear relations (such as higher dimensional self-duality conditions) among $D^a\phi$ (the covariant derivative of the Higgs field) and F^{ab} (the Yang-Mills field strength) in which ϕ can enter quadratically (as, for instance, in the vortex case). Inspired by this, a natural ansatz for gravity with torsion is the linear relation

$$T^c = f^c_{ab}(\alpha R^{ab} + \beta e^a e^b),$$

where f^c_{ab} is an appropriately chosen three index tensor and α and β are two constants.

A useful trick which helps in dealing with torsion in four dimensions is available in the presence of a non vanishing cosmological constant
one can define the following "higher order" connection

$$W^{AB} = \begin{bmatrix} \omega^{ab} & -\frac{e^a}{l} \\ -\frac{e^b}{l} & 0 \end{bmatrix}$$

in such a way that the "higher order" curvature is

$$F^{AB} = \begin{bmatrix} R^{ab} - \frac{e^a \wedge e^b}{l^2} & \frac{T^a}{l} \\ -\frac{T^b}{l} & 0 \end{bmatrix}$$

and the "higher dimensional" topological term reads

$$F^{AB} \wedge F_{AB} = R^{ab} \wedge R_{ab} + \frac{2}{l^2} N.$$

The energy of a static vortex configuration (in which the vortex is along the third axis) is

$$E_{AH} = \int d^2x \left[\frac{1}{2}B^2 + \left|\overrightarrow{D}\phi\right|^2 + \frac{e^2}{2}\left(|\phi|^2 - v^2\right)^2\right] \geq v^2 \left|\Phi_B\right|$$

The Bogomol'nyi equations can be deduced assuming that the bound is saturated:

$$B \mp e\left(|\phi|^2 - v^2\right) = 0, \quad \left(D_1 \pm iD_2\right)\phi = 0.$$

also in the case of monopoles in non Abelian Yang-Mills-Higgs theory the Bogomol'nyi equations are linear relations among the magnetic components of the curvature and the covariant derivative of the Higgs field of the type

$$B^a = \pm\left(D\phi\right)^a$$

where B^a is the non abelian magnetic field.

In gauge theories…

$$(DA)_i^{\ a} = B_i^{\ a}$$

$$B_i^{\ a} \approx \pm (D_i \phi)^a \Rightarrow$$

$$\overline{A} = (A, \phi)$$

$$F(\overline{A}) = \overline{F} = (F(A), D\phi) \Rightarrow$$

$$BPS - eq : \overline{F} = \pm * \overline{F}$$

In gravitational theories…

$$(De)_i{}^a = T_i{}^a$$

$$R_{ij}{}^{ab} = (d\omega + \omega \wedge \omega)^{ab}{}_{ij}$$

$$\overline{\omega} = (\omega, e)$$

$$R(\overline{A}) = \overline{R} = (R(\omega), T) \Rightarrow$$

$$BPS - like - eq : \overline{R} = \pm * \overline{R} \Leftrightarrow$$

$$T^c = f^c_{ab}(\alpha R^{ab} + \beta e^a e^b),$$

- One of the features of topological defects is that they partially break Lorentz invariance (the surviving generators are the ones leaving invariant the defects). For instance, when in quantum field theory one expands around the trivial vacuum (all the fields equal to zero) the Lorentz generators annihilate the vacuum. When expanding around nontrivial saddle points (that is, topological defects) this is not so since the position and the structure of the topological defects make the action of the Lorentz generators on the vacuum nontrivial.

In the case of our solutions, there is a three-dimensional submanifold of constant curvature. Thus it is natural to choose the tensor $f_{ijk} = \epsilon_{ijk}$ consistent with the unbroken Lorentz generators.

An interesting feature is related to the rigidity of the solutions due to the presence of torsion. In the non–Chern-Simons case, the 5D equations of motion have to fulfil strong compatibility conditions which arise taking the covariant derivative of the equation of motion:

$$e^b R^{cd} T^e \epsilon_{abcde} = 0, \qquad e^b e^c e^d T^e \epsilon_{abcde} = 0.$$

The "BPS" ansatz solves the above strong consistency conditions.

On the notation…

$$\int d^5x\sqrt{g}\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right),$$

$$\int R^{ab} \wedge R^{cd} \wedge e^e \epsilon_{abcde}$$

Lagrangians and Bianchi identities

$$I_D = \kappa \int \sum_{p=0}^{[D/2]} \alpha_p L^{(D,p)},$$

$$e^a = e^a_\mu dx^\mu$$

$$\omega^{ab} = \omega^{ab}{}_\mu dx^\mu$$

$$L^{(D,p)} \equiv \varepsilon_{a_1 \dots a_D} R^{a_1 a_2} \wedge \cdots \wedge R^{a_{2p-1} a_{2p}} \wedge e^{a_{2p+1}} \wedge \cdots \wedge e^{a_D}.$$

$$R^a{}_b \equiv (d\omega + \omega \wedge \omega)^a{}_b,$$

$$T^a \equiv De^a = de^a + \omega^a{}_b \wedge e^b.$$

$$DR^a{}_b = dR^a{}_b + \omega^a{}_c \wedge R^c{}_b + \omega^c{}_b \wedge R^a{}_c = 0,$$

$$DT^a = R^a{}_b \wedge e^b.$$

Extremizing the Lovelock Lagrangian of order *p* one obtains the following types of equations:

$$\sum_{p=0}^{[D/2]} (D - 2p)\alpha_p \Xi_a^{(p)} = 0,$$

$$\sum_{p=0}^{[D/2]} p(D - 2p)\alpha_p \Xi_{ab}^{(p)} = 0,$$

where

$$\Xi_a^{(p)} \equiv \varepsilon_{ab_2 \ldots b_D} R^{b_2 b_3} \ldots R^{b_{2p} b_{2p+1}} e^{b_{2p+2}} \ldots e^{b_D},$$

$$\Xi_{ab}^{(p)} \equiv \varepsilon_{aba_3 \ldots a_D} R^{a_3 a_4} \ldots R^{a_{2p-1} a_{2p}} T^{a_{2p+1}} e^{a_{2p+2}} \ldots e^{a_D}.$$

The action in D=5:

$$I = \kappa \int d^5x \sqrt{g}(R - 2\Lambda + \alpha(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})),$$

$$I = \int \left(\frac{c_0}{5} e^a e^b e^c e^d e^e + \frac{c_1}{3} R^{ab} e^c e^d e^e + c_2 R^{ab} R^{cd} e^e \right) \epsilon_{abcde},$$

The D=5 equations of motions

$$\mathcal{E}_{ab} \equiv T^c (c_1 e^d e^e + 2 c_2 R^{de}) \epsilon_{abcde} = 0.$$

$$\mathcal{E}_e \equiv (c_0 e^a e^b e^c e^d + c_1 R^{ab} e^c e^d + c_2 R^{ab} R^{cd}) \epsilon_{abcde} = 0.$$

$$\alpha = \frac{c_2}{2c_1}, \Lambda = -6\frac{c_0}{c_1}, \quad \kappa = 2c_1$$

Compatibility conditions in the generic case:

$$e^b R^{cd} T^e \epsilon_{abcde} = 0, \qquad e^b e^c e^d T^e \epsilon_{abcde} = 0.$$

First Part:Non-Chern-Simons

$c_1^2 \neq 4c_0c_2$ $\Longleftrightarrow$ $\frac{4\alpha\Lambda}{3} \neq -1$

$\mathrm{AdS}_2 \times S_3$ solution

$$ds^2 = \frac{l^2}{x^2}(-dt^2 + dx^2) + \frac{r_0^2}{4}(d\phi^2 + d\phi^2 + d\psi^2 + 2\cos\theta d\phi d\psi).$$

$$e^0 = \frac{l}{x}dt, \qquad e^1 = \frac{l}{x}dx, \qquad e^i = r_0\tilde{e}^i,$$

$\mathbf{AdS_2 \times S_3}$ solution **continuation**

$$T^1 = 0, \qquad T^0 = 0, \qquad T_i = \frac{H}{r_0} e^j e^k \epsilon_{ijk}, \quad \text{where } H \text{ is a constant.}$$

This ansatz for the torsion is suggested by the analogy with BPS states.

$$\omega^{01} = -\frac{1}{x} dt, \qquad \omega^{ij} = (H+1)\tilde{\omega}^{ij} = -(H+1)\epsilon^{ijk}\tilde{e}_k.$$

$$R^{01} = -\frac{1}{l^2} e^0 e^1, \qquad R^{ij} = \frac{1-H^2}{r_0^2} e^i e^j.$$

Now the equation of motions…

From the equations of the torsion: $$c_1 - \frac{2c_2}{l^2} = 0 \quad \text{or} \quad H = 0.$$

From the equations of the curvature: $$4c_0 + 2c_1 \frac{(1 - H^2)}{r_0^2} = 0.$$

$$12c_0 + 2c_1\left(-\frac{1}{l^2} + \frac{1 - H^2}{r_0^2}\right) - \frac{4c_2}{l^2}\frac{(1 - H^2)}{r_0^2} = 0.$$

Key identities of the BPS-ansatz:

$$\epsilon_{ijk} T^i e^j e^k = 0$$

$$\epsilon_{ijk} T^i e^j e^0 \simeq \epsilon_{ijk} (\epsilon^{imn} e_m e_n) e^j e^0$$
$$= (\delta^m_j \delta^n_k - \delta^m_k \delta^n_j) e_m e_n e^j e^0 = 0$$

These key identities fulfilled by our BPS ansatz make the equations of motions (which in general are quite non-trivial) very easy to solve.

The solution

$$c_1^2 = 12c_0c_2.$$

The coupling constants are fixed to be of a non-Chern-Simons theory.

$$\frac{1}{l^2} = \frac{c_1}{2c_2}. \quad 1 - H^2 = -\frac{2c_0}{c_1}r_0^2 \Rightarrow H^2 = 1 + \frac{r_0^2}{3l^2}.$$

torsion *cannot be small.*

$$R^{01} = -\frac{1}{l^2}e^0e^1, \qquad R^{ij} = -\frac{1}{3l^2}e^ie^j,$$

In particular, in the vacuum analogue of the Bertotti-Robinson metric it appears that the torsion is bounded from below for a finite sized sphere and AdS length scale.

Maxwell interpretation

$$T \equiv T_{ijk} e^i \wedge e^j \wedge e^k = 3! \frac{H}{r_0} e^2 \wedge e^3 \wedge e^4$$

$$DT = 0, \qquad D * T = 0.$$

This follows from Bianchi identities

$$F \equiv *T.$$

F is seen to obey the source free Maxwell equation.

Second Part:Chern-Simons

- In three dimensions, the BTZ black hole has a half-BPS stable ground state but in five dimensions the analogue of the BTZ black hole can decay into naked singularity…
- The torsion can save the situation.

Chern-Simons coupling

$$c_2 = \frac{c_1^2}{4c_0}$$

The spherically symmetric BH ansatz

$$ds^2 = -f^2(r)\,dt^2 + \frac{dr^2}{f^2(r)} + r^2 d\Sigma_3^2$$

where $d\Sigma_3^2$ stands for the metric of a three-dimensional base manifold Σ_3, The vielbein can then be chosen as

$$e^0 = f(r)dt; \quad e^1 = f^{-1}(r)dr; \quad e^m = r\tilde{e}^m$$

$$\omega^{ab} = \mathring{\omega}^{ab} + K^{ab}$$

$$T^m = K(r)\epsilon^{mnp}e_m e_p$$

$$R^{ab} = \mathring{R}^{ab} + \mathring{D}K^{ab} + K^a{}_c K^{cb}$$

$$\mathring{R}^{ab} = d\mathring{\omega}^{ab} + \mathring{\omega}^a{}_c\mathring{\omega}^{cb} \qquad K^{mn} = -K(r)\epsilon^{mnp}e_p$$

The curvature…

$$\mathring{R}^{01} = -\frac{\left(f^2\right)''}{2} e^0 e^1 \quad ; \quad \mathring{R}^{0n} = -\frac{\left(f^2\right)'}{2r} e^0 e^n$$

$$\mathring{R}^{1n} = -\frac{\left(f^2\right)'}{2r} e^1 e^n \quad ; \quad \mathring{R}^{mn} = \tilde{R}^{mn} - \frac{f^2}{r^2} e^m e^n$$

$$R^{01} = \mathring{R}^{01} \quad ; \quad R^{0m} = \mathring{R}^{0m} \quad ; \quad R^{1n} = \mathring{R}^{1n} - \frac{f}{r} T^n ,$$

$$R^{mn} = \mathring{R}^{mn} - \frac{d\left(rK\right)}{r} \epsilon^{mnp} e_p - K^2 e^m e^n .$$

Equation of motions

$\mathcal{E}_{mn} = 0$ → $f^2 = \frac{c_1}{2c_2} r^2 + \alpha r - \mu$

the components $\mathcal{E}_{1m} = 0$, and $\mathcal{E}_{0m} = 0$ are identically fulfilled.

$\mathcal{E}_{01} = 0$, is solved provided $d(rK) = 0$,

$$K = -\frac{\delta}{r}$$

$\mathcal{E}_1 = 0$, reduces to

$$6A(r) + \tilde{R}B(r) = 0$$

where $\tilde{R}$ is the Ricci scalar of the base manifold

$$A = 4c_0 - \frac{c_1^2}{c_2} + 2c_2\alpha\left(\frac{\delta^2 - \mu}{r^3} + \frac{\alpha}{r^2}\right)$$

$$B = -2\frac{c_2\alpha}{r}.$$

Therefore

$$A + \gamma B = 0$$

where γ is a constant

Thus, the solution reads

$$ds^2 = -\left(\frac{r^2}{l^2} - \mu\right) dt^2 + \frac{dr^2}{\frac{r^2}{l^2} - \mu} + r^2 d\Sigma_3^2$$

$$T^m = -\frac{\delta}{r}\epsilon^{mnp} e_m e_p$$ where $l = \sqrt{\frac{2c_2}{c_1}}$ is the AdS radius

For this solution, the torsion and the Riemannian curvature are singular at the origin, but nevertheless these singularities are surrounded by an event horizon located at $r^2 = \mu l^2$, so that this geometry describes a black hole whose horizon geometry is endowed with a nontrivial torsion.

It is worth to remark that the field equations are solved for any fixed base manifold Σ_3.

- It is worth pointing out the similarity of the black hole solution found here with the BTZ black hole. In the case of the 5D black hole without torsion, since the solution for $\mu=-1$ corresponds to AdS spacetime, while for the range $-1< \mu <0$ it describes timelike naked singularities.
- However, it has been shown that in the absence of torsion and matter fields, the only solution possessing Killing spinors within this family is the maximally supersymmetric AdS spacetime, so that the zero mass black hole breaks all the supersymmetries.
- Hence, it is natural to wonder whether the presence of torsion helps to improve the situation in 5D, in the sense that if the black hole vacuum with torsion had Killing spinors, its stability would be guaranteed preventing the black hole from decaying into naked singularities.

Killing spinors

The Killing spinor equation is obtained requiring a purely bosonic configuration to possess unbroken global supersymmetries. In the absence of matter fields, i.e., for the purely gravitational sector, the Killing spinor equation is given by

$$\nabla\epsilon := \left(d + \frac{1}{4}\omega^{ab}\Gamma_{ab} + \frac{1}{2l}e^{a}\Gamma_{a}\right)\epsilon = 0\ .$$

$$\nabla\nabla\epsilon = \left(\frac{1}{4}\left(R^{ab} + \frac{1}{l^2}e^{a}e^{b}\right)\Gamma_{ab} + \frac{1}{2l}T^{a}\Gamma_{a}\right)\epsilon = 0$$

$$\left[\frac{1}{2}\left(\tilde{R}^{mn}+(\mu-\delta^2)\tilde{e}^m\tilde{e}^n\right)\Gamma_{mn}+r\delta\epsilon^{mnp}\tilde{e}_n\tilde{e}_p\left(\frac{1}{l}-\sqrt{\frac{1}{l^2}-\frac{\mu}{r^2}}\Gamma_1\right)\Gamma_m\right]\epsilon=0$$

Note that there is no way to collect terms having the same functional dependence on r *unless* $\mu=0$. In this case the consistency condition reduces to

$$\left[\frac{1}{2}\left(\tilde{R}^{mn}-\delta^2\tilde{e}^m\tilde{e}^n\right)\Gamma_{mn}+\frac{r}{l}\delta\epsilon^{mnp}\tilde{e}_n\tilde{e}_p\Gamma_m\left(1+\Gamma_1\right)\right]\epsilon=0\ ,$$

$$\Gamma_1\epsilon=-\epsilon$$

$$\left(\tilde{R}^{mn} - \delta^2 \tilde{e}^m \tilde{e}^n\right) \Gamma_{mn} \epsilon = 0$$

$$\tilde{R}^{mn} = \delta^2 \tilde{e}^m \tilde{e}^n$$

Note that for the generic black hole solution the constant δ is arbitrary. However, the above condition implies that δ cannot longer be a truly integration constant, since it can be brought into the form $\delta = 1$ under suitable rescalings of time and radial coordinates.

$$\left(d - \frac{1}{2r}dr\right)\epsilon = 0$$

Therefore, the Killing spinor does not depend neither on time nor on the coordinates of the base manifold (since $\partial_t\epsilon = \partial_m\epsilon = 0$). The Killing spinors are given by the solution radial equation

$$\left(\partial_r - \frac{1}{2r}\right)\epsilon = 0\,,$$

which can be integrated as

$$\epsilon = \sqrt{\frac{r}{l}}\eta_0$$

where η_0 is a constant spinor satisfying the chirality condition

$$\Gamma_1\eta_0 = -\eta_0\ .$$

- Therefore, unlike for the torsion-free case, one concludes that the black holes described above which generically have an arbitrary but fixed base manifold, possess a half-BPS groundstate only for base manifolds which are locally a parallelized (combed) three-sphere with a torsion given by

$$T^m = -\frac{1}{r}\epsilon^{mnp} e_m e_p$$

Conclusion

- Torsion may play the role of the Higgs field in the theory of BPS states in YM theory. It helps in stabilizing solutions, it provides with solutions looking like topological excitation…

- It is worth to explore these interesting features in more general context and to see if observable effects are possible.