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Geometry of the Shilov boundary of bounded symmetric domains

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I.1 Riemannian symmetric space

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The group $Is(M)$ of isometries of M, with the compact-open topology is a Lie group (Myers-Steenrod). By composing symmetries, the group $Is(M)$ is esaily shown to be *transitive* on M. Let G be the neutral component of $Is(M)$. Then G is already transitive on M.

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Fix an origin o in M, and let K be the isotropy subgroup of o in G. Then K is a closed compact subgroup of G, and $M \simeq G/K$.

Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{k}$ the Lie algebra of K. The tangent space T_oM of M at o can be identified with $\mathfrak{g}/\mathfrak{k}$.

The map

$$\theta : G \longrightarrow G, \quad g \longmapsto s_0 \circ g \circ s_o$$

is an involutive isomorphism of G. Let $G^\theta = \{g \in G, \theta(g) = g\}$.
Then

$$(G^\theta)_o \subset K \subset G^\theta.$$

The differential of θ at the identity is a Lie algebra involution of $\mathfrak{g}$, still denoted by θ and yields a decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

$$\mathfrak{k} = \{X \in \mathfrak{g},\ \theta X = X\}, \mathfrak{p} = \{X \in \mathfrak{g},\ \theta X = -X\}.$$

Moreover

$$[\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k},\quad [\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p},\quad [\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}\ .$$

The projection from $\mathfrak{g}$ to $\mathfrak{k}$ along $\mathfrak{k}$ yields an isomorphism of $\mathfrak{g}/\mathfrak{k}$ with $\mathfrak{p}$, and hence there is a natural identification $T_oM\simeq\mathfrak{p}$.

Proposition 1. *Let X be in $\mathfrak{p}$. Let $g_t=\exp tX$ be the one-parameter group of G generated by X. Then $\gamma_X(t)=g_t(o)$ is the geodesic emanating from o with tangent vector X at o. Moreover*

$$g_t=s_{\gamma_X(\frac{t}{2})}\circ s_0\ .$$

The vector space $\mathfrak{p}$ is naturally equipped with a *Lie triple product* (LTS) , defined by

$$[X, Y, Z] = [[X, Y], Z].$$

Proposition 2. *The Lie triple product on $\mathfrak{p}$ satisfies the following identities*

$$[X, Y, Z] = -[Y, X, Z]$$

$$[X, Y, Z] + [Y, Z, X] + [Z, X, Y] = 0$$

$$[U, V, [X, Y, Z]] = [[U, V, X], Y, Z] + [X, [U, V, Y], Z] + [X, Y, [U, V, Z]]$$

This Lie triple product has a nice geometric interpretation, namely

$$R_o(X,Y)Z = -[[X,Y],Z] = -[X,Y,Z]$$

where R_o is the *curvature tensor* of M at o,

The *Ricci curvature* at o (also called the Ricci form) is the symmetric bilinear form on T_oM given by

$$r_o(X,Y) = -\,\mathbf{tr}(Z \longmapsto R_o(X,Z)Y)\ .$$

Proposition 3. *The Ricci cuvature at o satisfies*

$$r_o(X,Y) = -\frac{1}{2}B(X,Y)$$

where $B(X,Y) = \mathbf{tr}_{\mathfrak{g}}(\mathbf{ad}\,X\,\mathbf{ad}\,Y)$ *is the* Killing form *of the Lie algebra* $\mathfrak{g}$.

A Riemannian symmetric space $M \simeq G/K$ is said to be *irreducible* if the representation of K on the tangent space $T_oM \simeq \mathfrak{p}$ is irreducible (i.e. admits no invariant subspaces except $\{0\}$ and $\mathfrak{p}$).

If M is irreducible, then there exists a unique (up to a positive real constant) K-invariant inner product on $\mathfrak{p}$, and the Ricci form r_o has to be proportional to it.

An irreducible Riemannian symmetric space is said to be

of the Euclidean type if r_o is identically 0

of the compact type if r_o is positive definite

of the noncompact type if r_o is negative definite

Any simply connected Riemannian symmetric space M is a product of irreducible Riemannian symmetric spaces. If all factors are of the compact (resp. noncompact, Euclidean) type, then M is said to be of the compact (resp. noncompact, Euclidean) type.

If M is of compact type, then G is a compact semisimple Lie group, and if M is of the noncompact type, then G is a semisimple Lie group (with no compact factors) and θ is a Cartan involution of G.

For the Riemannian symmetric spaces of the noncompact type, the infinitesimal data characterize the space. More precisely, given a semisimple Lie algebra $\mathfrak{g}$ (with no compact factors), let G be any connected Lie group with Lie algebra $\mathfrak{g}$ and with finite center (there always exist such groups). Let θ be a Cartan involution of $\mathfrak{g}$ (there is hardly any choice, as two Cartan convolutions of $\mathfrak{g}$ are conjugate unde the adjoint action of G on $\mathfrak{g}$). Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g}$. The Killing form B of $\mathfrak{g}$ is negative definite on $\mathfrak{k}$ and positive-definite on $\mathfrak{p}$. The involution θ can be lifted to an involutive automorphism of G, still denoted by θ. Then $K = G^\theta$ is a compact connected subgroup of G. Let $X = G/K$, and set $o = eK$. Then the tangent space at o is naturally isomorphic to $\mathfrak{p}$ and $B_{|\mathfrak{p}\times\mathfrak{p}}$ is a K-invariant inner product on $\mathfrak{p}$.

Hence X can be equipped with a (unique) structure of Riemannian manifold, on which G acts by isometries. The space X does not depend on the choice of G (up to isomorphism), but only on $\mathfrak{g}$.

I.2 Hermitian symmetric spaces

Let M be a complex (connected) manifold with a Hermitian structure. M is said to be a *Hermitian symmetric space* is for each point m in M there exists an involutive *holomorphic* isometry s_m of M such that m is an isolated fixed point of s_m.

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There are special cases of Riemannian symmetric spaces, but we demand that the symmetries be holomorphic. As G (the neutral component of $Is(M)$ is generated by even products of symmetries, then G acts by holomorphic transformations on M. [One should however observe that G is *not* a complex Lie group].

Use same notation as before. In particular $\mathfrak{p}$ being isomorphic to the tangent space T_oM, admits a complex structure, i.e. a ($\mathbb{R}$-linear operator) $J = J_o$ which satisfies $J^2 = -\mathbf{id}$.

Proposition 4. *The complex structure operator J satisfies*

$$J([T,X]) = [T, JX], \textit{ for all } T \in \mathfrak{k}, X \in \mathfrak{p}$$

$$B(JX, JY) = B(X,Y), \ \textit{ for all } X, Y \in \mathfrak{p}$$

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Proposition 5. *There exists a unique element H in the center of $\mathfrak{k}$ such that $J = \mathbf{ad}_{\mathfrak{p}}\, H$.*

Proposition 6. *Let $\mathfrak{g}$ be a simple Lie algebra of the noncompact type, with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The associated Riemannian symmetric space $M \simeq G/K$ is a Hermitian symmetric space if and only if the center of $\mathfrak{k}$ is $\neq \{0\}$. Then there exists a unique (up to ± 1) element H in the center of $\mathfrak{k}$ such that $\mathbf{ad}\, H$ induces a complex structure operator on $\mathfrak{p}$ and G/K is, in a natural way a Hermitian symmetric space of the noncompact type.*

I.3 Jordan triple system

Let $M \simeq G/K$ be a Hermitian symmetric of the noncompact type. Then $\mathfrak{p} \simeq T_oM$ is equipped with its natural structure of Lie triple system, which coincides with the curvature tensor at o. The behaviour of the curvature tensor under the action of J the complex structure at o is rather intricate. It leads to the following definition.

Let, for X, Y, Z in $\mathfrak{p}$

$$\{X, Y, Z\} = \frac{1}{2}\big([[X, Y], Z] + J[[X, JY, Z]]\big)$$

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Theorem 7. *The triple product defined by the formula above satisfies the following identities, for X, Y, Z, U, V in $\mathfrak{p}$:*

$$(JT1) \quad J\{X, Y, Z\} = \{JX, Y, Z\} = -\{X, JY, Z\} = \{X, Y, JZ\}$$

$$(JT2) \qquad \{X, Y, Z\} = \{Z, Y, X\}$$

$(JT3)$
$\{U,V\,\{X,Y,Z\}\} = \{\{U,V,X\},Y,Z\} - \{X,\{V,U,Y\},Z\} + \{X,Y,\{U,V,$

Moreover, it satisfies

$$[[X,Y],Z] = \{X,Y,Z\} - \{Y,X,Z\}\ .$$

A complex vector space $\mathbb{V}$ together with a triple product $\{X, Y, Z\}$ which is $\mathbb{C}$-linear in X and Z, conjugate linear in Y, and satisfy $(JT2)$ and $(JT3)$ is called a (complex) *Jordan triple system*.

Let $L(X, Y)$ be the ($\mathbb{C}$-linear) operator on $\mathbb{V}$ defined by $L(X, Y)Z = \{X, Y, Z\}$, and consider the sesquilinear form

$$\tau(X, Y) = \mathbf{tr}\, L(X, Y)$$

If the form τ is nondegenerate, then τ is Hermitian $(\tau(X, Y) = \overline{\tau(Y, X)})$. The triple is said to be a *positive Hermitian Jordan triple system* (PHJTS) if the from τ is positive definite.

Theorem 8. *Let $M \simeq G/K$ be a Hermitian symmetric space of the noncompact type. Then $(\mathfrak{p}, J)$ (considered as a complex vector space) with its natural Jordan triple product is a PHJTS.*

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Summary

Riemannian symmetric spaces of the NC type $\equiv$ LTS with negative Ricci form

Hermitian symmetric spaces of the NC type $\equiv$ PHJTS

This correspondances can be extended to morphisms.