SOME RESULTS ON THE GEOMETRY OF TRAJECTORY SURFACES

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Abstract-In this study, the relationships between the invariants of trajectory ruled surfaces generated by the oriented lines fixed, in a moving body, in E^3 are investigated. Some new results on the pitches and the angle of pitches of the trajectory surfaces generated by the Steiner and area vectors are obtained and new comments are given. Also, the area of projections of spherical closed images of these surfaces are studied.

1.Introduction

The geometry of path trajectory ruled surfaces, generated by the oriented lines, fixed in a moving rigid body is important in the study of rational design problem of spatial mechanism. An x-closed trajectory ruled surface (x-c.t.s.) is characterized by two real integral invariants, the pitch ℓ_x and the angle of pitch λ_x. Using the integral invariants, the closed trajectory surfaces have been studied in some papers [1], [2], [3].

In this study, based on [4], introducing a relationship between the dual integral invariant, Λ_x, and the dual area vector, V_x, of the spherical image of an x-c.t.s., new results on the feature of the trajectory surfaces are investigated. And also, since the dual angle of pitch, defined in [5], of an x-c.t.s. is a useful dual apparate in the study of line geometry, we use the dual representations of the trajectory surfaces with their dual angle of pitches.

Therefore, besides the results on the real angle of pitches, that some of them given [4] many other results on the pitches of closed trajectory ruled surfaces are obtained. And some relationships between the other invariants are given. Also, using the some other methods, the area of projections of spherical closed images of the trajectory surfaces are studied.

It is hoped that the findings will contribute to the geometry of trajectory surfaces, so the rational design of spatial mechanisms.

2. Basic Concepts

Let a moving orthonormal trihedron $\{v_1,\ v_2,\ v_3\}$ make a spatial motion along a closed space curve $r = r(t)$, $t \in \mathrm{IR}$, in E^3. In this motion, an oriented line fixed in the moving system generates a closed trajectory surface in E^3. A parametric equation of a closed trajectory surface generated by v_1-axis of the moving system is

$$x(t,\mu) = r(t)+\mu v_1(t), \;\; x(t+2\pi,\mu) = x(t,\mu), \; t,\mu \in IR \qquad (1)$$

and denoted by $v_1(t)$- c.t.s.

Consider the moving orthonormal system $\{v_1, v_2 = v_1'/\left\|v_1'\right\|, v_3 = v_1 \wedge v_2\}$, then the axes of the trihedron intersect at the striction point of v_1-generator of v_1-c.t.s. and v_2 and v_3 are the normal and tangent to the surface, at the striction point, respectively.

The structural equations of this motion are

$$dv_i = \sum_{j=1}^{3} \omega_i^j v_j, \omega_i^j(s) = -\omega_j^i(s), \quad s \in IR, \quad i,j = 1,2,3$$

(2)

$$\frac{db}{ds} = \cos\sigma v_1 + \sin\sigma v_3 \qquad (3)$$

$b = b(s)$ is the striction line of v_1-c.t.s. and the differential forms ω_1^2, ω_2^3 and σ are the natural curvature, the natural torsion and the striction of v_1-c.t.s., respectively. Here, the striction is restricted as $\frac{-\pi}{2} < \sigma < \frac{\pi}{2}$ for the orientation on v_1-c.t.s., and s is the arc-length of the striction line.

The pole vector and the Steiner vector are given by

$$p = \frac{\psi}{\|\psi\|}, \; d = \oint \psi \qquad (4)$$

spectively, where $\psi = \omega_2^3 v_1 + \omega_1^2 v_3$ is instantaneous Pfaffian vector of the motion.

The pitch (Öffnungsstrecke) of v_1-c.t.s. is defined by

$$\ell_{v_1} := \oint d\mu = -\oint \langle dr, v_1 \rangle \qquad (5)$$

The angle of pitch (Öffnungswinkel) of v_1-c.t.s. and is given by one of the forms

$$\lambda_{v_1} := \oint d\phi = -\oint \langle dv_2, v_3 \rangle = -\langle v_1, d \rangle = 2\pi - a_{v_1} = \oint g_{v_1}$$

(6)

a_{v_1} and g_{v_1} are the measures of the spherical surface area bounded by the spherical image of v_1-c.t. surface and the geodesic curvature of this image, respectively. The pitch and the angle of pitch are well-known integral invariants of a closed trajectory surface, [1], [3], [6].

The real area vector of an x-closed space curve in E^3 is given by

$$v_x = \oint x \wedge dx \qquad (7)$$

[7]. And in a spatial closed motion, the area of projection of x-closed spherical image along any y-closed trajectory surface is defined by

$$2f_{x,y} = \langle v_x, y\rangle \tag{8}$$

x and y are the unit vectors in the moving system, [4].

According to E. Study‘s transference principle, a unit dual vector $x = x + \varepsilon x^*$ corresponds to only one oriented line, in E^3, where the real part x shows the direction of this line and the dual part x^* shows the vectorial moment of the unit vector x with respect to the origin, in E^3, [8].

Let K be a moving dual unit sphere generated by a dual orthonormal system

$$\left\{V_1, V_2 = \frac{V_1'}{\|V_1'\|}, V_3 = V_1 \wedge V_2\right\},\ \ V_i = v_i + \varepsilon v_i^*,\ \ i = 1, 2, 3. \tag{9}$$

K' be a fixed dual unit sphere with the same center. Then, the derivative equations of the dual spherical closed motion of K with respect to K' are given as

$$dV_i = \sum \Omega_i^j V_j,\ \ \Omega_i^j(t) = \omega_i^j(t) + \varepsilon\omega_i^{*j}(t),\ \Omega_i^j = -\Omega_j^i,\ t \in \mathrm{IR},\quad i = 1, 2, 3. \tag{10}$$

The dual Steiner vector of the closed motion is defined by

$$D = \oint \Psi,\quad \Psi = \|\Psi\|\, P,\quad \Psi = \psi + \varepsilon\psi^* \tag{11}$$

$\Psi = \Omega_2^3 V_1 + \Omega_1^2 V_3$ and P are the instantaneous Pfaffian vector and the dual pole vector of the motion, respectively.

As known from the E.Study's transference principle, the dual equations (10) correspond to the real equations (2) and (3) of a closed spatial motion, in E^3. In this sense, the differentiable dual closed curve, $V_1 = V_1(t), t \in \mathrm{IR}$, is considered as a closed trajectory ruled surface in E^3 and denoted by $v_1(t)$-c.t.s..

A dual integral invariant which is called the dual angle of pitch of a v_1-c.t.s. is given by

$$\wedge_{v_1} = -\oint \langle dV_2, V_3\rangle = -\langle V_1, D\rangle = 2\pi - A_{v_1} = \oint G_{v_1} = \lambda_{v_1} - \varepsilon\ell_{v_1} \tag{12}$$

[5], [6], where $D = d + \varepsilon d^*, A_{v_1} = a_{v_1} + \varepsilon a_{v_1}^*$ and $G_{v_1} = g_{v_1} + \varepsilon g_{v_1}^*$ are the dual Steiner vector of the motion, the dual spherical surface area and the dual geodesic curvature of spherical image of v_1-c.t.s., respectively.

3. The Relationships and Results

Consider the differentiable unit dual spherical closed curve

$$X = X(t),\ \ X(t+2\pi) = X(t), \|X\| = 1, t \in \mathrm{IR}. \tag{13}$$

We know from E.Study's transference principle that the dual corresponds to an x-closed trajectory surface generated by an x-oriented line fixed in a moving rigid body, in E^3.

The dual area vector of an x-dual closed spherical curve can be defined by

$$V_x = \oint X \wedge dX \tag{14}$$

an analogy to the definition, in [7], where $dX = \Psi \wedge X$ is the differential velocity of a dual point,X, fixed in the moving sphere K.

From (4) and (14), the dual area vector may be developed as

$$\begin{aligned} V_x &= \oint X \wedge (\Psi \wedge X) \\ &= \oint (\langle X, X\rangle \Psi - \langle X, \Psi\rangle X) \\ &= \oint \Psi - \left\langle X, \oint \Psi \right\rangle X \\ &= D - \langle X, D\rangle X \end{aligned} \tag{15}$$

with the aid of (12)

$$V_x = D + \wedge_x X. \tag{16}$$

The statement shows that there is a relationship between the dual angle of pitch of an x-c.t.s. and the dual area vector of x-closed spherical image of this surface.

From (12) and (16), we may write

$$\langle V_x, D\rangle = \langle D, D\rangle + \wedge_x \langle X, D\rangle$$

$$\|V_x\| \left\langle \tfrac{V_x}{\|V_x\|}, D \right\rangle = \|D\|^2 - \wedge_x^2$$

$$\wedge_x^2 - \|V_x\| \wedge_{v_x} = \|D\|^2 \tag{17}$$

$\wedge_{v_x}$ is the dual angle of pitch of v_x-trajectory surface generated by the area vector of x-closed spherical image of x-c.t.s..

On the other hand, from (12), the dual angle of pitch of d-c.t.s. generated by the Steiner vector of the motion is

$$\begin{aligned} \wedge_d &= -\left\langle \tfrac{D}{\|D\|}, D \right\rangle \\ &= -\|D\|. \end{aligned} \tag{18}$$

Since $D = \oint \Psi$, the dual angle of pitch, $\wedge_d$, gives the total dual spherical rotation in the interval with one period.

Seperating (18) into real and dual parts, we may give the following theorem.

Theorem 1: The angle of pitch and the pitch of d-c.t.s. give the total rotation and the total translation of the closed spatial motion, i.e.;

$$\lambda_d = -\|d\|, \ \ell_d = \tfrac{\langle d, d^*\rangle}{\|d\|} \tag{19}$$

spectively.

Therefore, with the aid of (12) the following results may be given.

Result 1: There is the relationship

$$a_d = 2\pi + \|d\| \tag{20}$$

tween the spherical surface area bounded by the spherical image of d-c.t.s. and the total rotation of the motion.

Result 2: The total geodesic curvature of the spherical image of d-c.t.s. is equal to the total rotation of the motion, i.e.;

$$\oint g_d = -\|d\| . \tag{21}$$

On the other hand, from (19); we may write

$$\begin{aligned} \lambda_d \ell_d &= -\langle d^*, d \rangle \\ &= -\|d^*\| \left\langle \tfrac{d^*}{\|d^*\|}, d \right\rangle \\ &= \|d^*\| \lambda_{d^*} \end{aligned}$$

λ_{d^*} is the angle of pitch of d^*-moment trajectory surface.

Thus, the following result may be given.

Result 3: There is the relationship

$$\lambda_d \ell_d = \|d^*\| \lambda_{d^*} \tag{22}$$

tween the invariants of d- and d^*-c.t.surfaces.

From (12), (16) and (18), it follows that

$$\begin{aligned} \|V_x\| &= \sqrt{\langle D + \wedge_x X, D + \wedge_x \rangle} \\ &= \sqrt{\langle D, D \rangle + 2 \wedge_x \langle X, D \rangle + \wedge_x^2 \langle X, X \rangle} \\ &= \sqrt{\wedge_d^2 - \wedge_x^2}. \end{aligned} \tag{23}$$

Thus, with the aid of (17) and (23) the dual angle of pitch of v_x-trajectory surface is obtained as

$$\wedge_{v_x} = -\sqrt{\wedge_d^2 - \wedge_x^2} \tag{24}$$

there is the relationship

$$\wedge_{v_x}^2 = \wedge_d^2 - \wedge_x^2 \tag{25}$$

tween the dual angle of pitches of d-Steiner and an x-closed trajectory surfaces.

From (25), we may give the following theorem.

Theorem 2: Global invariants of the line surfaces generating by x, v_x, and d satisfy the following relations in E^3;

$$\lambda_{v_x}^2 = \lambda_d^2 - \lambda_x^2 \tag{26}$$

and

$$\ell_{v_x} = \frac{\lambda_x \ell_x - \lambda_d \ell_d}{\sqrt{\lambda_d^2 - \lambda_x^2}} \quad . \tag{27}$$

If $\ell_{v_x} = 0$ then v_x-closed trajectory surface is a cone. In this case, the relation (27) comes to

$$\lambda_x \ell_x = \lambda_d \ell_d . \tag{28}$$

Thus, the following result may be given.

Result 4: If v_x-closed trajectory surface is a cone, then there are the relationships

$$\frac{\ell_x}{\ell_d} = \frac{\lambda_d}{\lambda_x} = \frac{2\pi - a_d}{2\pi - a_x} = \frac{\oint g_d}{\oint g_x} \tag{29}$$

tween the global invariants of d-Steiner and x-closed trajectory surfaces and the spherical areas and the total geodesic curvatures of d and x-spherical images [6].

From (12) and (25), it follows that

$$(2\pi - A_{v_x})^2 = (2\pi - A_d)^2 - (2\pi - A_x)^2$$

$$A_{v_x}^2 + A_x^2 - A_d^2 = 4\pi(A_{v_x} + A_x - A_d - \pi) \tag{30}$$

$A_{v_x} = a_{v_x} + \varepsilon a_x^*$, $A_x = a_x + \varepsilon a_x^*$, $A_d = a_d + \varepsilon a_d^*$. And from (30)

$$a_{v_x}^2 + a_x^2 - a_d^2 = 4\pi(a_{v_x} + a_x - a_d - \pi) \tag{31}$$

$$a_{v_x} a_{v_x}^* + a_x a_x^* - a_d a_d^* = 2\pi(a_{v_x}^* + a_x^* - a_d^*) \tag{32}$$

obtained. Therefore, the following result may be given.

Result 5: There are the relationships (30)-(32) between the measures of the spherical surface areas bounded by the spherical images of v_x-, d-, and x-closed trajectory surfaces.

In case of the axes of V_x and D are perpendicular to each other, with the aid of (12), (17) and (18)

$$\left\langle \frac{V_x}{\|V_x\|}, \frac{D}{\|D\|} \right\rangle = 0 \Longleftrightarrow \frac{1}{\|D\|}\Lambda_{v_x} = 0$$
$$\Longleftrightarrow \Lambda_{v_x} = 0$$
$$\Longleftrightarrow \lambda_{v_x} = 0, \ell_{v_x} = 0$$
$$\Longleftrightarrow a_{v_x} = 2\pi, a_{v_x}^* = 0$$
$$\Longleftrightarrow \lambda_x^2 = \lambda_d^2, \lambda_x \ell_x = \lambda_d \ell_d$$

obtained. Thus, the following result may be given.

Result 6: In a closed spatial motion,

the axes of V_x-area vector and D-Steiner vector,$d \neq 0$, are perpendicular

$\Leftrightarrow \Lambda_{v_x} = 0$

$\Leftrightarrow$ v_x-closed trajectory surface is a cone, i.e., $\lambda_{v_x} = 0, \ell_{v_x} = 0$

$\Leftrightarrow$The spherical image of v_x-c.t.s. divides measure of the spherical surface area into two equal parts.

$\Longleftrightarrow$The relationships $\lambda_x^2 = \lambda_d^2, \lambda_x \ell_x = \lambda_d \ell_d$ are satisfied by the invariants of x- and d-c.t. surfaces.

As it's known the angle of pitch of the closed trajectory surface generated by the second axis of the moving trihedron is zero, i.e. $\Lambda_{v_2} = 0$. This means that the spherical image of the second trajectory surface of the moving trihedron divides the measure of the spherical surface area into two equal parts.

In a spatial motion, the dual area vectors of v_1, v_2 and v_3-dual closed spherical images, drawn by the axes of the moving system are given with the aid of (16) as

$$V_{v_1} = D + \Lambda_{v_1} V_1$$
$$V_{v_2} = D$$
$$V_{v_3} = D + \Lambda_{v_3} V_3$$

(33)

respectively. From (12), (18), (23) and (33)

$$\langle V_{v_1}, D\rangle = \langle D + \Lambda_{v_1} V_1, D\rangle \Leftrightarrow \|V_{v_1}\| \left\langle \frac{V_{v_1}}{\|V_{v_1}\|}, D \right\rangle = \langle D, D\rangle + \Lambda_{v_1} \langle V_1, D\rangle$$
$$\Leftrightarrow -\|V_{v_1}\| \Lambda_{v_{v_1}} = \|D\|^2 - \Lambda_{v_1}^2$$
$$\Leftrightarrow \Lambda_{v_{v_1}} = \frac{\Lambda_{v_1}^2 - \Lambda_d^2}{\sqrt{\Lambda_d^2 - \Lambda_{v_1}^2}}$$

$$\Lambda_{v_{v_1}}^2 = \Lambda_d^2 - \Lambda_{v_1}^2 \qquad (34)$$

obtained. Seperating (34) into real and dual parts

$$\left(\lambda_{v_{v_1}} - \varepsilon \ell_{v_{v_1}}\right)^2 = (\lambda_d - \varepsilon \ell_d)^2 - (\lambda_{v_1} - \varepsilon \ell_{v_1})^2$$

$$\lambda_{v_{v_1}}^2 - 2\varepsilon \lambda_{v_{v_1}} \ell_{v_{v_1}} = \lambda_d^2 - 2\varepsilon \lambda_d \ell_d - \lambda_{v_1}^2 + 2\varepsilon \lambda_{v_1} \ell_{v_1}$$

$$\lambda_{v_{v_1}}^2 = \lambda_d^2 - \lambda_{v_1}^2, \ \lambda_{v_{v_1}} \ell_{v_{v_1}} = \lambda_d \ell_d - \lambda_{v_1} \ell_{v_1} \qquad (35)$$

gained. These are the relationships between the integral invariants of d-, v_1 -, and v_{v_1} -trajectory surfaces.

Similar statements on the invariants of v_{v_2} - and v_{v_3} - trajectory surfaces may be given.

From (12) and (18)

$$\Lambda_{v_{v_2}} = \Lambda_d \qquad (36)$$

$$\lambda_{v_{v_2}} = \lambda_d, \quad \ell_{v_{v_2}} = \ell_d \qquad (37)$$

we find that

$$\Lambda_{v_{v_3}}^2 = \Lambda_d^2 - \Lambda_{v_3}^2 \quad . \qquad (38)$$

From (38)

$$\left(\lambda_{v_{v_3}} - \varepsilon \ell_{v_{v_3}}\right)^2 = (\lambda_d - \varepsilon \ell_d)^2 - (\lambda_{v_3} - \varepsilon \ell_{v_3})^2$$

$$\lambda_{v_{v_3}}^2 = \lambda_d^2 - \lambda_{v_3}^2, \ \lambda_{v_{v_3}} \ell_{v_{v_3}} = \lambda_d \ell_d - \lambda_{v_3} \ell_{v_3} \qquad (39)$$

obtained. Thus, the following theorem can be stated.

Theorem 3:

There are the relationships (35)-(39) between the real and dual integral invariants of area vector-closed trajectory surfaces and the corresponding axes-trajectory surfaces generated by the axes of the moving trihedron.

In a closed spatial motion, let X and Y be unit dual vectors fixed in the moving system, then the dual area of projection of x-dual closed spherical image of x-closed trajectory surface, in direction y-generator of y-closed trajectory surface is defined by

$$2F_{x,y} = \langle V_x, Y \rangle \tag{40}$$

is an analogy to the definition, given in IR^3, [4].

With the aid of (12), (16) and (40) we may write

$$\begin{aligned} 2F_{x,y} &= \langle V_x, Y \rangle \\ &= \langle D + \Lambda_x X, Y \rangle \\ &= \langle D, Y \rangle + \Lambda_x \langle X, Y \rangle \\ &= -\Lambda_y + \Lambda_x \cos \Phi \end{aligned} \tag{41}$$

$\Phi = \varphi + \varepsilon\varphi^*$, $0 \leq \varphi \leq \pi$, $\varphi^* \in \mathrm{IR}$, is a constant dual angle between X and Y-unit dual vectors and $F_{x,y} = f_{x,y} + \varepsilon f^*_{x,y}$. Here, φ and φ^* are the real angle and the real distance between x and y-generators of x and y-c.t. surfaces, respectively.

Considering $F_{x,y} = f_{x,y} + \varepsilon f^*_{x,y}$, we may give the following theorem.

Theorem 4: The relationships

$$2f_{x,y} = -\lambda_y + \lambda_x \cos\varphi \tag{42}$$

$$2f^*_{x,y} = \ell_y - \lambda_x \varphi^* \sin\varphi - \ell_x \cos\varphi \tag{43}$$

satisfied between the area of projection of x-closed spherical image and the invariants of x and y-closed trajectory surfaces.

Some special cases in (42) and (43) may be given as following;

(i) $\varphi = 0,\ \varphi^* = 0 \Leftrightarrow x \equiv y \Leftrightarrow \lambda_x = \lambda_y,\ \ell_x = \ell_y \Rightarrow f_{x,y} = 0,\ f^*_{x,y} = 0$ (44)

(ii) $\varphi = 0,\ \varphi^* \neq 0 \Leftrightarrow x//y \Leftrightarrow \lambda_x = \lambda_y,\ \ell_x = \ell_y \Rightarrow f_{x,y} = 0,\ f^*_{x,y} = 0$ (45)

(iii) $\varphi = \frac{\pi}{2}, \varphi^* = 0 \Leftrightarrow x \perp y \Rightarrow 2f_{x,y} = -\lambda_y, 2f^*_{x,y} = \ell_y$ (46)

(iv) $\varphi = \frac{\pi}{2}, \varphi^* \neq 0 \Leftrightarrow x \perp y \Rightarrow 2f_{x,y} = -\lambda_y, 2f^*_{x,y} = \ell_y - \lambda_x \varphi^*$ (47)

(v) $\varphi = \pi,\ \varphi^* = 0 \Leftrightarrow x \equiv -y \Leftrightarrow 2f_{x,y} = -\lambda_y + \lambda_{-y}(-1) = -\lambda_y + \lambda_y = 0,$
$2f^*_{x,y} = \ell_y - \ell_{-y}(-1) = \ell_y - \ell_y = 0$ (48)

(vi) $\varphi = \pi, \varphi^* \neq 0 \Leftrightarrow x//y, x = -y \Leftrightarrow f_{x,y} = 0, f^*_{x,y} = 0$ (49)

Thus, from (44)-(49) we may give the following results;

Result 7: The area of projection of an x-closed spherical image of x-closed trajectory surface in direction the same x or $(-x)$-generator is zero, i.e.

$$f_{x,x} = f^*_{x,x} = 0 \tag{50}$$

and

$$f_{x,-x} = f^*_{x,-x} = 0. \tag{51}$$

Result 8: If x and y intersect each other perpendicularly, then

$$2f_{x,y} = -\lambda_y, 2f^*_{x,y} = \ell_y \tag{52}$$

if x and y are perpendicular but they don't intersect each other, then

$$2f_{x,y} = -\lambda_y, 2f^*_{x,y} = \ell_y - \lambda_x \varphi^* . \tag{53}$$

Similarly, the dual area of projection of an x-dual closed spherical curve in direction of the dual unit Steiner vector of the motion is

$$\begin{aligned} 2F_{x,d} &= \left\langle V_x, \tfrac{D}{\|D\|} \right\rangle \\ &= \tfrac{1}{\|D\|} \langle V_x, D \rangle \\ &= \tfrac{1}{\|D\|} \langle D + \Lambda_x X, D \rangle \\ &= \tfrac{1}{\|D\|} \left(\langle D, D \rangle + \Lambda_x \langle X, D \rangle \right) \\ &= \tfrac{1}{\|D\|} \left(\|D\|^2 - \Lambda_x^2 \right) \end{aligned}$$

from (18)

$$2F_{x,d} = \tfrac{\Lambda_x^2 - \Lambda_d^2}{\Lambda^d} \quad . \tag{54}$$

Seperating the relation (54) into real and dual parts, the following theorem may be given.

Theorem 5: In a closed spatial motion, the relationships,

$$f_{x,d} = \tfrac{\lambda_x^2 - \lambda_d^2}{2\lambda_d}, \tag{55}$$

$$f_{x,d}^* = \tfrac{1}{\lambda_d} \left(\lambda_d \ell_d - \lambda_x \ell_x + \ell_d \tfrac{\lambda_x^2 - \lambda_d^2}{2\lambda_d} \right) \tag{56}$$

tween the area of projection of an x-closed spherical curve in direction d-generator of d-Steiner trajectory surface and the invariants of x and d-c.t. surfaces are satisfied.

From (54), the dual area of projections of v_i-dual closed spherical images (indicatrices) in direction d-generator are

$$2F_{v_i,d} = \frac{\Lambda_{v_i}^2 - \Lambda_d^2}{\Lambda_d}, \;\; i = 1, 2, 3.. \tag{57}$$

From (57), the relationships

$$f_{v_i,d} = \frac{\lambda_{v_i}^2 - \lambda_d^2}{2\lambda_d}, \tag{58}$$

$$f_{v_i,d}^* = \tfrac{1}{\lambda_d} \left(\lambda_d \ell_d - \lambda_{v_i} \ell_{v_i} + \ell_d \frac{\lambda_{v_i}^2 - \lambda_d^2}{2\lambda_d} \right), \;\; i = 1, 2, 3. \tag{59}$$

gained.

Since $\Lambda_{v_2} = 0$, it follows, from (36) and (57) that

$$2F_{v_2,d} = -\Lambda_{v_{v_2}} = -\Lambda_d \, . \tag{60}$$

Thus, the following result may be given.

Result 9: In a closed spatial motion, the angle of pitches and the pitches of v_{v_2}-, and d-trajectory surfaces can be stated in terms of the area of projections as

$$\lambda_{v_{v_2}} = \lambda_d = -2f_{v_2,d} \tag{61}$$

and

$$\ell_{v_{v_2}} = \ell_d = 2f_{v_2,d}^* \, . \tag{62}$$

From (12), (33) and (40), the area of projections of v_i -dual spherical closed images in directions V_j-generators are

$$\begin{aligned} 2F_{v_i,v_j} &= \langle V_{v_i}, V_j \rangle \\ &= \langle D + \Lambda_{v_i} V_i, V_j \rangle \\ &= -\Lambda_{v_j} + \Lambda_{v_i} \delta_{ij}, \quad i = 1, 2, 3. \end{aligned}$$

δ_{ij} is the Kronecker delta. Hence, we may write

$$2F_{v_i,v_j} = \{ \begin{array}{c} -\Lambda_{v_j};\ i \neq j \\ 0;\ i - j \end{array} \tag{63}$$

So, the following result may be given.

Result 10: The area of projections of v_i-closed spherical images of v_i-c.t. surfaces in direction v_j-generators of v_j-c.t. surfaces are given by

$$2f_{v_i,v_j} = \{ \begin{array}{c} -\lambda_{v_j};\ i \neq j \\ 0;\ i = j \end{array} \tag{64}$$

$$2f^*_{v_i,v_j} = \{ \begin{array}{c} \ell_{v_j};\ i \neq j \\ 0;\ i = j \end{array} \quad . \tag{65}$$

From (12) and (63), the following dual quantities may be obtained

$$2F_{v_2,v_1} = 2F_{v_3,v_1} = -\Lambda_{v_1} = A_{v_1} - 2\pi = -\oint G_{v_1} \tag{66}$$

$$2F_{v_1,v_2} = 2F_{v_3,v_2} = -\Lambda_{v_2} = A_{v_2} - 2\pi = -\oint G_{v_2} \tag{67}$$

$$2F_{v_1,v_3} = 2F_{v_2,v_3} = -\Lambda_{v_3} = A_{v_3} - 2\pi = -\oint G_{v_3} \quad . \tag{68}$$

Therefore, separating the formulas above the following relations between the invariants of closed trajectory surfaces can be given. It follows from (66), that

$$2f_{v_2,v_1} = 2f_{v_3,v_1} = -\lambda_{v_1} = a_{v_1} - 2\pi = -\oint g_{v_1} \tag{69}$$

$$2f^*_{v_2,v_1} = 2f_{v_3,v_1} = -\ell_{v_1} = a^*_{v_1} = -\oint g^*_{v_1} \tag{70}$$

from (68)

$$2f_{v_1,v_3} = 2f_{v_2,v_3} = -\lambda_{v_3} = a_{v_3} - 2\pi = -\oint g_{v_3} \tag{71}$$

$$2f^*_{v_1,v_3} = 2f^*_{v_2,v_3} = \ell_{v_3} = a^*_{v_3} = -\oint g^*_{v_3} \tag{72}$$

obtained. So, the following result may be given.

Result 11: The relations (69)-(72) are satisfied by the invariants of corresponding closed trajectory surfaces.

On the other hand, since $\Lambda_{v_2} = 0$, from (67) the relations

$$F_{v_1,v_2} = F_{v_3,v_2} = 0 \Leftrightarrow \Lambda_{v_2} = 0 \Leftrightarrow A_{v_2} = 2\pi \Leftrightarrow \oint G_{v_2} = 0$$

(73)

given. With the aid of (73) the following result on the invariants of trajectory surfaces may be given.

Result 12: $f_{v_1,v_2} = f_{v_3,v_2} = 0,\ f^*_{v_1,v_2} = f^*_{v_3,v_2} = 0$

$$\Leftrightarrow \lambda_{v_2} = 0, \;\; \ell_{v_2} = 0 \Leftrightarrow a_{v_2} = 2\pi, a^*_{v_2} = 0$$
$$\Leftrightarrow \oint g_{v_2} = 0, \;\; \oint g^*_{v_2} = 0$$
$\Leftrightarrow v_2$-c.t.s. is a cone and divides the measure of unit spherical surface area into two equal parts.

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