A fiber structure of Teichmüller space and conformal field theory

David Radnell[1] Eric Schippers[2]

[1]Department of Mathematics and Statistics
American University of Sharjah
Sharjah, UAE

[2]Department of Mathematics
University of Manitoba
Winnipeg, Canada

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Conformal Field Theory (CFT):

- Special class of 2D quantum field theories.
- Mathematical definition (G. Segal, Kontevich $\approx$ 1986)
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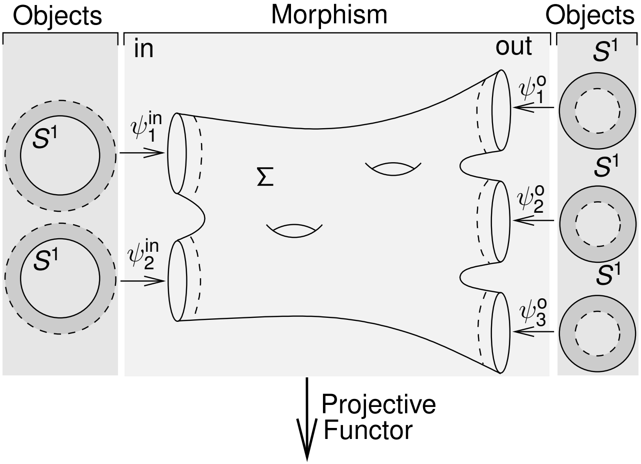
Our General Aim:

- Provide a natural analytic setting for the rigorous definition of CFT in higher genus. Definitions and Theorems.
- Use CFT ideas (especially sewing) to prove new results in Teichmüller theory and geometric function theory.

Motivation/Application: Conformal Field Theory



$$H \otimes H \xrightarrow{A([\Sigma,\psi])} H \otimes H \otimes H$$

Quasiconformal Maps I

$f : \Omega \subset \mathbb{C} \to \mathbb{C}$. Homeomorphism. Orientation Preserving.
Jacobian$(f) = \cdots = |f_z|^2 - |f_{\bar{z}}|^2 > 0$. So, $|f_{\bar{z}}/f_z| < 1$.

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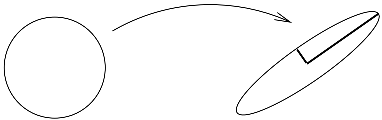

Circular Dilatation = $\frac{\text{major axis}}{\text{minor axis}} = \frac{1+|\mu|}{1-|\mu|}$

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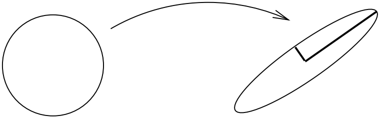

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Note: $f(z)$ conformal $\iff f_{\bar{z}} = 0 \iff \mu(z) = 0 \iff$ *Circ.Dil.* $= 1$.

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Geometric Definition:

f is K**-quasiconformal** if its circular dilatation is globally bounded by K. (i.e. Infinitesimally, circles map to ellipses of bounded eccentricity).

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Analytic Definition:

f is K-quasiconformal if it satisfies the Beltrami Equation

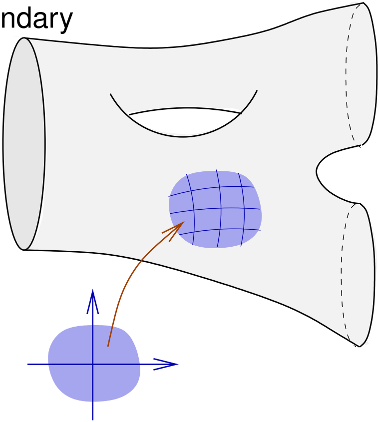
$$\frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z}$$

for some $\mu(z)$ with $||\mu||_\infty = k < 1$. $K = (1+k)/(1-k)$.

Note: Technical conditions skipped. QC maps are only differentiable almost everywhere etc.

Basic Objects

- Riemann Surfaces with boundary



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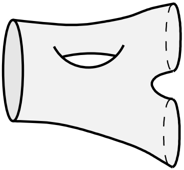

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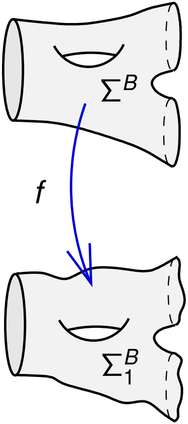


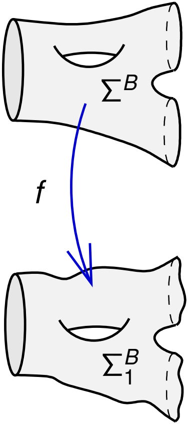
Basic Objects

- Riemann Surfaces with boundary
- Fix: g = genus, n = # of boundary components. The **moduli space** is the space of conformal equivalence classes of surfaces.
- Quasiconformal map
- Quasisymmetric map

Definition:

$h : S^1 \to S^1$

h has quasiconformal extensions to $\mathbb{C}$.



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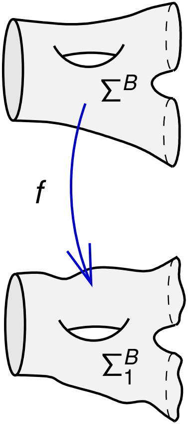
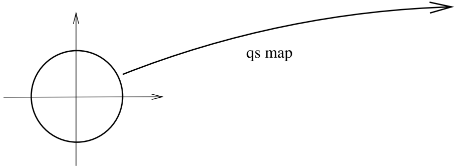
- Quasisymmetric map
- Quasisymmetric boundary parametrization





Teichmüller Space $=$ space of Riemann surfaces

Fix a base Riemann surface Σ.
Given Σ_1 and quasiconformal $f : \Sigma \to \Sigma_1$, write (Σ, f, Σ_1).

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$T(\Sigma) = \{(\Sigma, f, \Sigma_1)\}/\sim$.

$$(\Sigma, f, \Sigma_1) \sim (\Sigma, g, \Sigma_2) \iff \exists \text{ conformal } \sigma : \Sigma_1 \to \Sigma_2 \text{ such that } g^{-1} \circ \sigma \circ f \approx \text{id (rel. boundary)}$$

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Teichmüller metric:

$$\mathit{distance}([\Sigma, f, \Sigma_1], [\Sigma, g, \Sigma_2]) = \inf_{f,g} \log(\text{circular dilatation of } g \circ f^{-1})$$

This measures how close (in the quasiconformal sense) to a conformal map there is from Σ_1 to Σ_2.

Teichmüller space facts

Fix Σ. $f : \Sigma \to \Sigma_1$. $T(\Sigma) =$ Teichmüller space.

Why?

- $\mu(f) = f_{\bar{z}}/f_z$ is a differential form on the base surface.
- Study the Teichmüller space by studying certain spaces of forms.
- This is classical work from the 50's and 60's of Ahlfors and Bers et al. Well developed theory.

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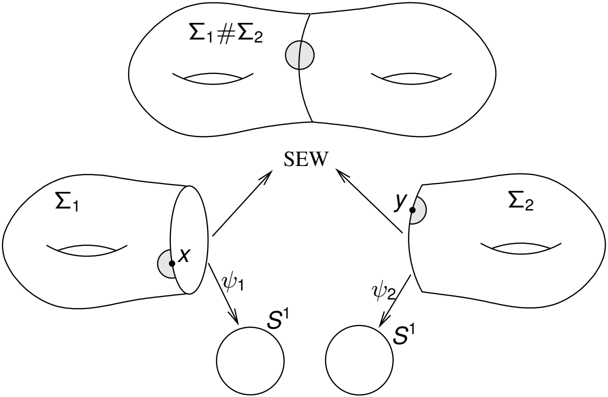
Classical Facts:

1. T(torus) = upper half-plane.
2. If Σ is closed (with punctures) then $T^P(\Sigma)$ is a finite-dimensional complex manifold.
3. If Σ is a surface with boundary then $T^B(\Sigma)$ is an ∞-dimensional complex manifold.
4. Moduli space $= T(\Sigma)/$ (Mapping Class Group).
5. The moduli space is not a manifold.

Sewing

$$\Sigma_1 \# \Sigma_2 = (\Sigma_1 \sqcup \Sigma_2)/(\psi_1(x) = \psi_2(y))$$



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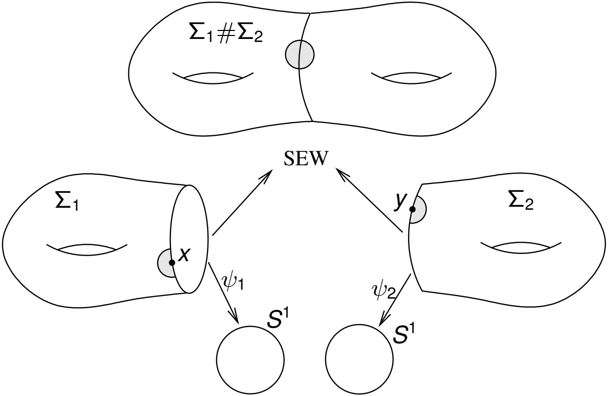


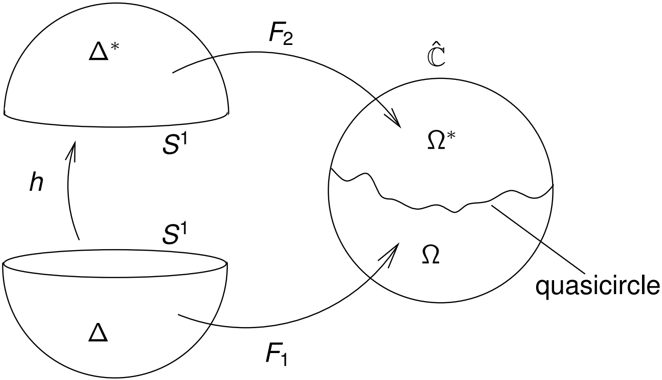
Note: If ψ_i are conformal then $\Sigma_1 \# \Sigma_2$ immediately becomes a Riemann surface. This is what was previously used in CFT.

Conformal Welding

Δ – unit disk, $\Delta^* = \hat{\mathbb{C}} \setminus \bar{\Delta}$, $h : S^1 \to S^1$ (quasisymmetry)

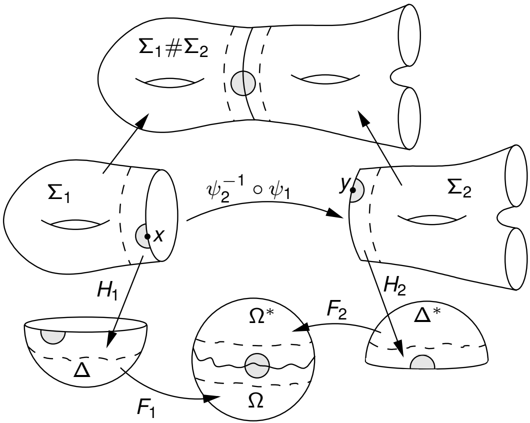
Theorem (conformal welding:)

There exists conformal maps F_1 and F_2 such that $F_2^{-1} \circ F_1 = h$ on S^1.



Quasisymmetric Sewing

ψ_1 and ψ_2 – quasisymmetric boundary parametrizations.
Define charts on $\Sigma_1 \# \Sigma_2$ by:



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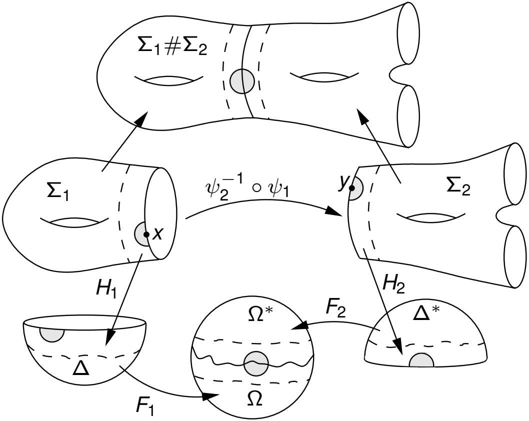


Proposition (R-S 06)

This gives the unique complex structure on $\Sigma_1 \# \Sigma_2$ which is compatible with Σ_1 and Σ_2.

Holomorphicity of sewing

Key idea:

Fix τ to be a quasisymmetric boundary parametrization of Σ.
$[\Sigma, f, \Sigma_1] \in T^B(\Sigma)$ contains boundary parametrization information for Σ_1 via $\psi = \tau \circ f^{-1}$.

Theorem (R-S 2006)

The sewing operations are holomorphic. That is,

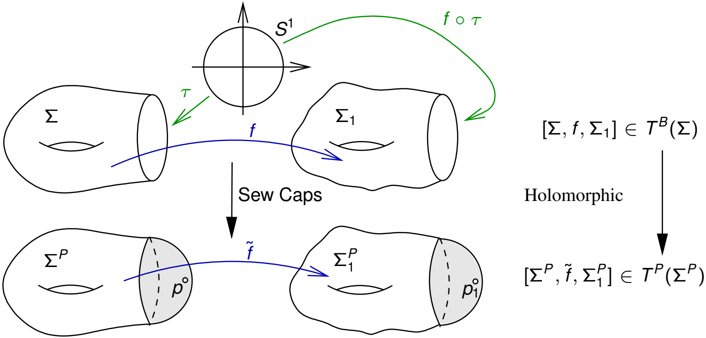
$$T^B(\Sigma_1) \times T^B(\Sigma_2) \xrightarrow{sew} T^B(\Sigma_1 \# \Sigma_2)$$

is holomorphic.

Cap Sewing: $T^B \to T^P$

Theorem (RS 08)

1. *T^B is a holomorphic fiber space over T^P.*
2. *The fibers are complex Banach manifolds modeled on* $\mathcal{O}_{qc} = \{f : \mathbb{D} \to \mathbb{C} \mid f \text{ is univalent, has qc extension, and } f(0) = 0.\}$



HELP!!

- In one (and several) complex variables an injective holomorphic map automatically has a holomorphic inverse
- This is not true in infinite dimensions in general.
- In the Banach space setting do there exist nice conditions to guarantee holomorphicity of the inverse?