NC Deformation of Vortexes and Instantons

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math-ph/0612041 [to appear in J.Geom.Phys.]
arXiv:0805.3373

10th Jun. 2008 Varna

Introduction

Instanton A is defined by

$$F^+ = \frac{1}{2}(1 + *)F = 0 \ ,$$

$F = dA + A \wedge A$: curv. 2-form , $*$: Hodge star.
Most studies for NC instanton are based on the ADHM method.

ADHM construction for $U(N)$

N.C. instanton by ADHM (Nekrasov-Schwarz)

2 complex vector spaces $V = \mathbf{C}^k$, $W = \mathbf{C}^N$.
ADHM data : B_1, $B_2 \in Hom(V,V)$,
$I \in Hom(W,V)$, $J \in Hom(V,W)$, s.t.

$$
\begin{aligned}
\mu_{\mathbf{R}} &:= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \hbar \ \mathsf{Id}_k \ , \\
\mu_{\mathbf{C}} &:= [B_1, B_2] + IJ = 0 \ .
\end{aligned}
$$

Using ADHM data we can construct instanton
(A.S. -Ishikawa -Kuroki, etc.)

[Known]:
ADHM instanton $\sharp = k$ (same as comm. instanton)
It does not depend on the NC parameter
(A.S. -Ishikawa -Kuroki, A.S.,Furuuchi,Tian)

Can we expect that ?
1. Instanton $\sharp$ are inv. under NC deform. in $\mathbb{R}^4$?
2.Top. charges in Y-M are preserved in $\mathbb{R}^n$??
(Vortex, Monopole and so on.)

[Unknown]:NC instantons $\Longleftrightarrow$ Comm. instanton
ADHM : $1/\hbar$ expansion
$\Downarrow$
Our method : $\hbar$ expansion

Contents

(1) Constructing a NC formal instanton deformed from a comm. instanton

(2) Proof : instanton $\sharp$ is independent of $\hbar$

(3) Constructing a NC vortex deformed from Taubes's commutative vortex

(4) Proof : vortex $\sharp$ is independent of the $\hbar$

(5) Conjectures and Open problems

Notations : Comm. relation, Moyal product

$$[x^\mu, x^\nu]_\star = x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu},\ \ \mu,\nu = 1,\ldots,2n\ ,$$

$(\theta^{\mu\nu})$: real,x-indep,skew-sym, NC parameters.

$$f(x)\star g(x) = f(x)g(x) + \sum_{n=1}^{\infty} \frac{1}{n!} f(x) \left(\frac{i}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu \right)^n g(x)\ .$$

Introduce $\hbar$ and a fixed constant $\theta_0^{\mu\nu} < \infty$ with

$$\theta^{\mu\nu} = \hbar \theta_0^{\mu\nu}$$

We define the commutative limit by letting $\hbar \to 0$.

The covariant derivative:

$$D_\mu := \partial_\mu + iA_\mu \ ,$$

The curvature two form F :

$$F := \frac{1}{2} F_{\mu\nu} dx^\mu \wedge\star dx^\nu = dA + A \wedge\star A$$

where $\wedge\star$ is defined by

$$A \wedge\star A := \frac{1}{2}(A_\mu \star A_\nu) dx^\mu \wedge dx^\nu.$$

(1) NC Instanton Construction

Consider the Yang-Mills theory on the NC $\mathbf{R}^4$

Formally we expand A as

$$A_\mu = \sum_{l=0}^{\infty} A_\mu^{(l)} \hbar^l$$

Using

$$P := \frac{1+*}{2}\ ;\ \ P_{\mu\nu,\rho\tau} = \frac{1}{2}(\delta_{\mu\rho}\delta_{\nu\tau} - \delta_{\nu\rho}\delta_{\mu\tau} + \epsilon_{\mu\nu\rho\tau}),$$

and covariant derivatives associated to $A_\mu^{(0)}$ by

$$D_\mu^{(0)} f := \partial_\mu f + i[A_\mu^{(0)}, f], \quad D_{A^{(0)}} f := d\ f + A^{(0)} \wedge f$$

l-th order Instanton Eq.

$$P^{\mu\nu,\rho\tau}\left(D_\rho^{(0)} A_\tau^{(l)} - D_\tau^{(0)} A_\rho^{(l)} + C_{\rho\tau}^{(l)}\right) = 0$$

$$P(D_{A^{(0)}} A^{(l)} + C^{(l)}) = 0.$$

where

$$C_{\rho\tau}^{(l)} := \sum_{(p;\ m,n)\in I(l)} \hbar^{p+m+n} \frac{1}{p\ !} \left(A_{[\rho}^{(m)} (\overleftrightarrow{\Delta})^p A_{\tau]}^{(n)} \right)$$

$$\overleftrightarrow{\Delta} \equiv \frac{i}{2} \overleftarrow{\partial}_{\mu} \theta_0^{\mu\nu} \overrightarrow{\partial}_{\nu}.$$

$$I(l) \equiv \{(p;\ m,n) \in \mathbb{Z}^3 | p+m+n = l, m \neq l, n \neq l\}.$$

Note that :

- $C_{\rho\tau}^{(l)}$ is consisited of $A^{(k)}$ $(k < l)$. i.e. given fun.
  We determine $A^{(l)}$ recursively.
- 0-th order is the comm. instanton Eq.

Asymptotic behavior of comm. instanton $A_\mu^{(0)}$

$$A_\mu^{(0)} = gdg^{-1} + O(|x|^{-2}), \;\; gdg^{-1} = O(|x|^{-1}),$$

where $g \in G$ and G is a gauge group.
Fix $A^{(0)}$ and impose a condition for $A^{(l)}(l \geq 1)$ as

$$A - A^{(0)} = D^*_{A^{(0)}} B \;, \;\; B \in \Omega^2_+,$$

where $D^*_{A^{(0)}}$ is defined by

$$(D^*_{A^{(0)}})_\rho^{\mu\nu} B_{\mu\nu} \;\; = \;\; \delta_\rho^\nu D^{(0)\mu} B_{\mu\nu} - \delta_\rho^\mu D^{(0)\nu} B_{\mu\nu}.$$

to deform the Eq. into elliptic DE.

We expand B in $\hbar$ as $B = \sum B^{(k)} \hbar^k$.
Using the fact that the $A^{(0)}$ is anti-selfdual,

$$\boxed{2D_{(0)}^2 B^{(l)\mu\nu} + P^{\mu\nu,\rho\tau} C_{\rho\tau}^{(l)} = 0,} \quad : \quad \text{Main Eq.}$$

where

$$D_{(0)}^2 \equiv D^{\rho}_{A^{(0)}} D_{A^{(0)}\rho} \ .$$

Let's solve the Main Eq. by the Green's fun. of $D_{(0)}^2$.

$$D_{(0)}^2 G_0(x, y) = \delta(x - y),$$

$G_0(x,y)$ was constructed (Corrigan et.al)

$$G_0(x,y) = \frac{[v_1(x) \otimes v_2(x)]^\dagger (1-\mathfrak{M})[v_1(y) \otimes v_2(y)]}{4\pi^2(x-y)^2}.$$

Here $\mathfrak{M}$ and v_1, v_2 are determined by the ADHM data and v_i is a bounded function. (Comm. Instanton has 1 to 1 corresp. with ADHM) Then,

$$B^{(l)\mu\nu} = -\frac{1}{2}\int_{\mathbb{R}^4} G_0(x,y) P^{\mu\nu,\rho\tau} C^{(l)}_{\rho\tau}(y) d^4 y$$

and the NC instanton $A = \sum A^{(l)}$ is given by

$$A^{(l)} = D^*_{A^{(0)}} B^{(l)}.$$

The key fact to get the main result is

$$G_0(x, y) = O(|x - y|^{-2}) \ , |x - y| >> 1 \ .$$

Using this, we can prove

$$\boxed{|A^{(l)}| < O(|x|^{-3+\epsilon}), \quad \forall \epsilon > 0}$$

(2) Proof: Instanton $\sharp$ is indep. of $\hbar$

The first Pontrjagin number is defined by

$$I_\hbar := \frac{1}{8\pi^2} \int tr\ F \wedge \star F.$$

We rewrite this as Cycl. Sym. Break.$\downarrow$

$$\frac{1}{8\pi^2} \int tr\ d(A \wedge \star dA + \frac{2}{3} A \wedge \star A \wedge \star A +) + \frac{1}{8\pi^2} \int tr P_\star$$

$\int trP_\star$ is typically written as

$$\int_{\mathbb{R}^d} tr(P \wedge \star Q - (-1)^{n(4-n)} Q \wedge \star P).$$

P and Q are an n-form and a $(4-n)$-form
The term of order $\hbar$ is given by

$$\int_{\mathbb{R}^4} tr\{\hbar \theta_0^{\mu\nu} (\partial_\mu P \wedge \partial_\nu Q)\}$$
$$\sim \int_{\mathbb{R}^4} \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} tr \ d\{(*\theta) \wedge (P_{\mu_1 \ldots \mu_n} dQ_{\mu_{n+1} \ldots \mu_4})\}$$

$$\sim \int_{\mathbb{R}^4} \epsilon^{\mu_1\mu_2\mu_3\mu_4} tr\ d\{(*\theta) \wedge (P_{\mu_1\ldots\mu_n} dQ_{\mu_{n+1}\ldots\mu_4})\}$$

where $*\theta = \epsilon_{\mu\nu\rho\tau}\theta^{\rho\tau} dx^\mu \wedge dx^\nu / 4$.
These integrals are zero if
$P_{\mu_1\ldots\mu_n} dQ_{\mu_{n+1}\ldots\mu_4}$ is $O(|x|^{-(4-1+\epsilon)})$ $(\epsilon > 0)$.

Similarly, higher order terms are written as total div.
Hence vanish under the decay hypothesis.

$\Longrightarrow$ $$\int trP_\star = 0.$$

From these and $|A^{(l)}| < O(|x|^{-3+\epsilon})$,

$$\frac{1}{8\pi^2}\int trF\wedge\star F = \frac{1}{8\pi^2}\int trF^{(0)}\wedge F^{(0)},$$

Summarizing the above discussions,

Theorem 1. *Let $A_\mu^{(0)}$ be a comm. instanton in $\mathbb{R}^4$. There exists a formal NC instanton $A_\mu = \sum_{l=0}^{\infty} A_\mu^{(l)}\hbar^l$ such that the instanton number is independent of the NC parameter $\hbar$.*

(3) Deformation of Vortex

Review the Abelian-Higgs model in com. $\mathbb{R}^2$

ϕ: a complex scalar field
G: $U(1)$ gauge group
Complex coordinates : $z = \frac{1}{\sqrt{2}}(x^1 + ix^2)$,
Complex gauge fields by $A = \frac{1}{\sqrt{2}}(A_1 - iA_2)$,
Curvature : $F_{zz} = F_{\bar{z}\bar{z}} = 0$, $F_{z\bar{z}} = iF_{12} = \partial\bar{A} - \bar{\partial}A$

We define the magnetic field

$$B := -iF_{z\bar{z}} \ .$$

The Vortex Eqs.:

$$\bar{D}\phi = (\bar{\partial} - i\bar{A})\phi = 0 \quad , \quad B + \phi\bar{\phi} - 1 = 0 \quad .$$

The vortex number,

$$N_0 := \frac{1}{2\pi}\int d^2x B_0 \ \in \mathbb{Z}$$

Theorem[Taubes]
For a smooth, finite vortex.

$$
\begin{aligned}
|\phi_0| &\sim 1 - Ce^{-r(1-\epsilon)} \\
|\partial\phi_0| &\sim |\bar{\partial}\phi_0| \sim C'\frac{1}{r} \\
|A_0| &\sim C''\frac{1}{r}\ .
\end{aligned}
$$

where $r = |x|$.
Let's investigate the NC deformations of this theory.

The NC Abelian Higgs Model

$$[x^\mu, x^\nu] = i\hbar\epsilon^{\mu\nu}, \quad \mu, \nu = 1, 2 \quad ,$$

$$F_{z\bar{z}} \;=\; iF_{12} = \partial\bar{A} - \bar{\partial}A - i[A, \bar{A}]_\star \; ,$$

The NC vortex Eqs.

$$\bar{D} \star \phi = (\bar{\partial} - i\bar{A}) \star \phi = 0 \quad , \quad B + \phi \star \bar{\phi} - 1 = 0 \quad .$$

The formal expansions of the fields:

$$\phi = \sum_{n=0}^{\infty} \hbar^n \phi_n(z, \bar{z}) \; , \quad A = \sum_{n=0}^{\infty} \hbar^n A_n(z, \bar{z}) \; .$$

The k-th order equations:

$$\begin{aligned} -i(\partial\bar{A}_k + \bar{\partial}A_k) + \phi_k\bar{\phi}_0 + \phi_0\bar{\phi}_k - \delta_{k0} + C_k(z,\bar{z}) &= 0 \\ \bar{\partial}\phi_k - i\bar{A}_k\phi_0 - i\bar{A}_0\phi_k + D_k(z,\bar{z}) &= 0. \end{aligned}$$

Here $C_k(z,\bar{z})$ and $D_k(z,\bar{z})$ are composite functions of lower order A_n, ϕ_n

In particular in the case of $k=0$, these are comm. Vortex Eqs.

Setting

$$\varphi_k := \frac{\phi_k}{\phi_0} + \frac{\bar{\phi}_k}{\bar{\phi}_0} = 2Re(\frac{\phi_k}{\phi_0}) \quad \text{and} \quad d_k = \frac{D_k}{\phi_0} \; ,$$

Vortex Eqs. are simplified as

$$\boxed{(-\Delta + |\phi_0|^2)\varphi_k = E_k}$$

where

$$E_k := -C_k + \partial d_k - \bar{\partial}\bar{d}_k.$$

NC Vortex Number

Let's see conditions which preserve the vortex number under a NC deformation.

Theorem 2. *If* $\frac{1}{2\pi}\int d^2xB_0 = N_0$ *and* $|\phi_k| < Cr^{-\epsilon}$, $|\partial_r\phi_k| < Cr^{-\epsilon+1}$, *then*

$$\frac{1}{2\pi}\int d^2xB = N_0 \ .$$

We can prove this by using asymptotic behavior of commutative vortex.

(4) Proof that Vortex $\sharp$ is preserved

The Schrödinger equation and Vortex Solutions

To show that there exists a unique NC vortex solution deformed from the Taubes' vortex, consider the Schrödinger equation

$$(-\Delta + V(x))u(x) = f(x)$$

Assumptions for $V(x)$

$(a1)$ $V(x) \geq 0$, $^{\forall}x \subset \mathbb{R}^2$

$(a2)$ $^{\exists}K \subset \mathbb{R}^2$ and $^{\exists}c > 0$ s.t. K is a compact and for $x \in \mathbb{R}^2 \backslash K$, $V(x) \geq c$

$(a3)$ $^{\exists}x_1, \ldots, x_N \in \mathbb{R}^2$ s.t. $V(x_i) = 0, V(x) > 0$ for $x \not\in \{x_1, \ldots, x_N\}$

$(a4)$ $\forall\ \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2,\ \exists\ C_\alpha$ such that $|\partial_x^\alpha (V - c)| \leq C_\alpha$ for any $x \in \mathbb{R}^2$

Note that our system satisfies $(a1) - (a4)$.

We set

$$H_l(n) := \{f| \ \ ||f||: \ = \ \sup_{x\in\mathbb{R}^2}(1+|x|^n)|\partial_x^\alpha f(x)| < \infty \quad \text{for any } |\alpha| \le l\}$$

From standard way of Green's function, we can prove the following

Theorem 3. *Under the assumptions* $(a1) - (a4)$*, there exists a unique solution* $u \in H_l(n)$ *of* $(-\Delta + V(x))u(x) = f(x)$ *for any* $f \in H_l(n)$*.*

Vortex Eq. $(-\Delta + |\phi_0|^2)\varphi_k = E_k$ is a particular example.

These 2 theorems imply the following theorem.

Theorem 4. (A_0, ϕ_0) *satisfy the Vortex Eqs. Then there exists a unique solution* (A, ϕ) *of the NC vortex equations with* $A|_{\hbar=0} = A_0$, $\phi|_{\hbar=0} = \phi_0$, *and its vortex number is preserved:*

$$N = N_0 \ , \ \ i.e. \ \ \frac{1}{2\pi}\int d^2x\, B = \frac{1}{2\pi}\int d^2x\, B_0 \ .$$

[Outline of the Proof]

(1) We found $V(x) = |\phi_0|^2$ satisfies $(a1) - (a4)$. From asympt. behavior, $E_1 \in H_\infty(4)$.

(2) If $E_i \in H_\infty(2i+2)$, as a result of asym. behavior estimation, there exist unique solutions $\varphi_1, \ldots, \varphi_{k-1}$.

(3) Then we find $E_k \in H_\infty(2k+2)$. Therefore $E_k \in H_\infty(2k+2)$ is proved for arbitrary k.

(4) Theorem 3 is applicable to $(-\Delta + |\phi_0|^2)\varphi_k = E_k$ for arbitrary k, then it is shown that each φ_k is determined uniquely and $\varphi_k \in H_\infty(2k+2)$

(5) Finally, Theorem 2 imply that $N = N_0$.

$\square$

(5) Conjectures and Open problems

Conjecture: The instanton numbers in Euclidean 4-space are invariant under NC deformation. Furthermore Top charges might be preserved under the NC deformation for any other solitons in gauge theories in Euclidean spaces.

Open problem:
"Which instantons (solitons) preserve their instanton number (Top charge) under NC deformation?"

Hint

1 Hint has already appeared ?

The key point is the vol. of the space is ∞, in the previous proofs to show the Top. charges are not deformed.

$$\Downarrow$$

Therefore it is natural to expect that
instanton $\sharp$ depends on the NC parameter in a finite vol. NC space.

Example: Instanton ♯ on NC Torus

Instanton on T^4 for $U(N^2)$ gauge theory

$$D_1 = \partial_1, \qquad D_2 = \partial_2 + \frac{1}{2}\frac{k}{N}(x_1 \mathbf{1}_N) \otimes \mathbf{1}_N,$$
$$D_3 = \partial_3, \qquad D_4 = \partial_4 - \frac{1}{2}\frac{k}{N}(x_3 \mathbf{1}_N) \otimes \mathbf{1}_N,$$

The instanton number is given by k^2.

After NC deformation, the instanton number is also deformed to

$$\frac{1}{8\pi^2}\int_{T^4} tr\ F \wedge \star F = \frac{k^2 N^2}{(N - k\hbar)^2}.$$