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Integrable Dynamical Systems Associated with Plane Curves

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A. Suppose that the curvature $\kappa(s)$ of a plane curve Γ is given explicitly as a function of the arclength s, i.e., the intrinsic equation of the curve Γ is known.

Then, it is possible to recover the embedding (position vector)

$$\mathbf{x}(s) = (x(s), y(s)) \in \mathbb{R}^2$$

of the curve in the plane (up to a rigid motion) by quadratures in the standard manner.

First, recall that the unit tangent $\mathbf{t}\,(s)$ and normal $\mathbf{n}\,(s)$ vectors to the curve Γ:

$$\mathbf{t}\,(s) = \left(\frac{\mathrm{d}x(s)}{\mathrm{d}s}, \frac{\mathrm{d}y(s)}{\mathrm{d}s}\right), \qquad \mathbf{n}\,(s) = \left(-\frac{\mathrm{d}y(s)}{\mathrm{d}s}, \frac{\mathrm{d}x(s)}{\mathrm{d}s}\right)$$

are related to the curvature $\kappa(s)$ through Frenet-Serret formulas

$$\frac{\mathrm{d}\mathbf{t}\,(s)}{\mathrm{d}s} = \kappa\,(s)\,\mathbf{n}\,(s)\,, \qquad \frac{\mathrm{d}\mathbf{n}\,(s)}{\mathrm{d}s} = -\kappa\,(s)\,\mathbf{t}\,(s)\,.$$

The Frenet-Serret relations provide the following system of two second-order ODEs

$$\frac{\mathrm{d}^2 x(s)}{\mathrm{d}s^2} + \kappa(s)\frac{\mathrm{d}y(s)}{\mathrm{d}s} = 0, \qquad \frac{\mathrm{d}^2 y(s)}{\mathrm{d}s^2} - \kappa(s)\frac{\mathrm{d}x(s)}{\mathrm{d}s} = 0$$

which is readily integrable by quadratures to give the parametric equations of the curve Γ.

Indeed, in terms of the slope angle $\varphi(s)$ of the curve Γ one has

$$\kappa(s) = \frac{\mathrm{d}\varphi(s)}{\mathrm{d}s}, \qquad \frac{\mathrm{d}x(s)}{\mathrm{d}s} = \cos(\varphi(s)), \qquad \frac{\mathrm{d}y(s)}{\mathrm{d}s} = \sin(\varphi(s))$$

and hence, the parametric equations of the curve Γ can be expressed by quadratures

$$x(s) = \int \cos(\varphi(s))\,\mathrm{d}s, \qquad y(s) = \int \sin(\varphi(s))\,\mathrm{d}s$$

where

$$\varphi(s) = \int \kappa(s)\,\mathrm{d}s.$$

B. Suppose now that the curvature of the curve Γ is given as a function of the position of the points the curve is passing through, that is

$$\kappa = \mathcal{K}(x, y)$$

$\mathcal{K}(x, y)$ being a known function. In this case, the curve may be thought of as parametrized by a parameter t, the co-ordinates $x(t), y(t)$ of the position vector being determined by the system of equations

$$\ddot{x} + \mathcal{K}(x, y)\dot{y} = 0, \qquad \ddot{y} - \mathcal{K}(x, y)\dot{x} = 0 \tag{1}$$

where dots denote derivatives with respect to t.

The following two examples show that such a situation is not artificial:

Euler's elastica

$$\kappa = a_1 y + a_2, \qquad a_1, a_2 \in \mathbb{R}$$

Generalized (Lévy's) elastica

$$\kappa = b_1 \left(x^2 + y^2\right) + b_2, \qquad b_1, b_2 \in \mathbb{R}$$

Generalized elastica

MAURICE LEVY (1884)
JOURNAL DE MATHEMATIQUES PURES ET APPLIQUEES
Mémoire sur un nouveau cas intégrable du problème de l'élastique
et l'une de ses applications

Halphen (1888) Fonctions elliptiques (F. E.) II, Chap. V,
LA COURBE ELASTIQUE PLANE SOUS PRESSION NORMALE UNIFORME

Greenhil (1889) Mathematische Annalen LII,
THE ELASTIC CURVE UNDER UNIFOEM NORMAL PRESSURE

Tadjbakhsh I. and Odeh F. (1967) Equilibrium states of elastic rings
J. Math. Anal. Appl. 18, 59–74

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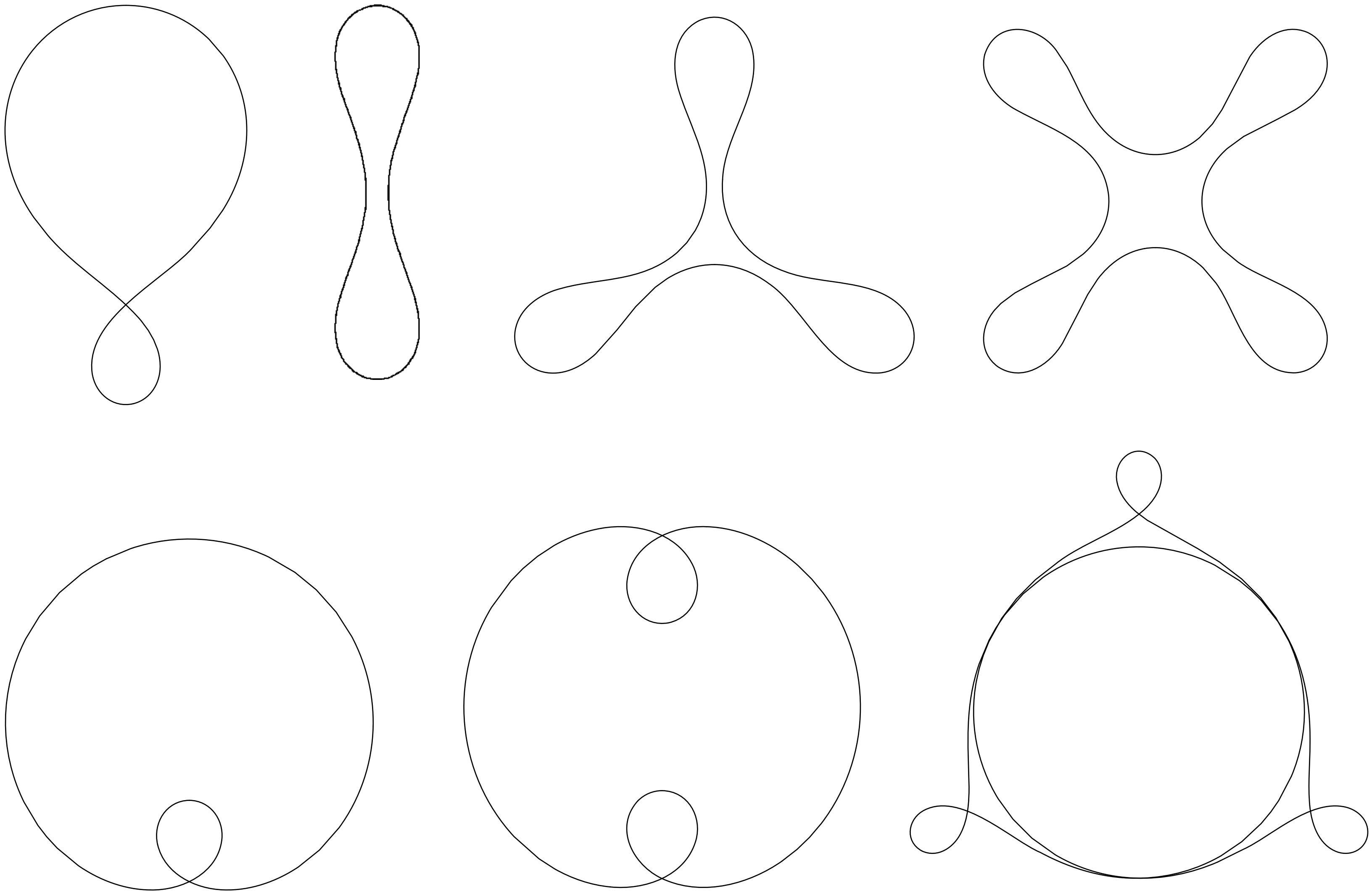
Vassilev V., Djondjorov P. and Mladenov I. (2008) Cylindrical equilibrium shapes of fluid membranes, arXiv:0803.0843v1 [math-ph], Submitted



C. The aim of the present note is to study the integrability of system (1) regarded as a dynamical system of two degrees of freedom describing the motion of a particle of unit mass, t playing the role of time.

- A sufficient condition for the integrability of a system of form (1) by quadratures is the respective function $\mathcal{K}(x, y)$ to be such that the system to possess two different constants of motion.
- Since the magnitude $\sqrt{\dot{x}^2 + \dot{y}^2}$ of the particle velocity is a constant of motion for any system of form (1), the problem is to find the conditions under which system (1) has at least one more constant of motion.
- For that purpose, first, it is shown that (1) is a Lagrangian system by constructing an appropriated Lagrangian L, determined explicitly through the function $\mathcal{K}(x, y)$, whose Euler-Lagrange equations are Eqs. (1).
- Then, a necessary and sufficient condition is found for the Lagrangian L to admit a symmetry group, which, by virtue of Noether's theorem, provides the existence of the sought constant of motion.
- Finally, a constructive procedure is suggested determining how, in such a case, to express the solutions of system (1) by quadratures.

The Lagrangian

- It is easy to check that equations (1), that is

$$\ddot{x} + \mathcal{K}(x, y)\dot{y} = 0, \qquad \ddot{y} - \mathcal{K}(x, y)\dot{x} = 0$$

are the Euler-Lagrange equations associated with the action functional

$$A = \int L(x, y, \dot{x}, \dot{y})\mathrm{d}t$$

whose Lagrangian L can be taken of the form

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + F\left(x, y\right)\dot{x} + G\left(x, y\right)\dot{y}$$

where the functions $F\left(x, y\right)$ and $G\left(x, y\right)$ are such that

$$\frac{\partial}{\partial y}F\left(x, y\right) - \frac{\partial}{\partial x}G\left(x, y\right) = \mathcal{K}(x, y). \tag{2}$$

Indeed, we have

$$L_{\mathsf{x}} = \dot{x}\frac{\partial}{\partial x}F(x,y) + \dot{y}\frac{\partial}{\partial x}G(x,y), \qquad L_{\underline{\mathsf{x}}} = \dot{x} + F(x,y)$$
$$L_{\mathsf{y}} = \dot{x}\frac{\partial}{\partial y}F(x,y) + \dot{y}\frac{\partial}{\partial y}G(x,y), \qquad L_{\underline{\mathsf{y}}} = \dot{y} + G(x,y)$$

and hence

$$L_{\mathsf{x}} - \frac{d}{dt}L_{\underline{\mathsf{x}}} = -\ddot{x} - \dot{y}\left(\frac{\partial}{\partial y}F(x,y) - \frac{\partial}{\partial x}G(x,y)\right)$$
$$L_{\mathsf{y}} - \frac{d}{dt}L_{\underline{\mathsf{y}}} = -\ddot{y} + \dot{x}\left(\frac{\partial}{\partial y}F(x,y) - \frac{\partial}{\partial x}G(x,y)\right).$$

Comparing the above formulas with system (1), we see that L can be regarded as the Lagrangian of this system provided that relation (2) hold.

Symmetries of the Lagrangian

- The invariance properties of the Lagrangian

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + F\left(x, y\right)\dot{x} + G\left(x, y\right)\dot{y}$$

with respect to local Lie groups of point transformations of the dependent variables x, y are studied. Using the standard procedure, the following conditions are obtained for a Lagrangian of the above form to admit a variational symmetry of the considered type:

$$\left(ay + b\right)\frac{\partial F\left(x, y\right)}{\partial x} - \left(ax + c\right)\frac{\partial F\left(x, y\right)}{\partial y} - aG\left(x, y\right) = 0$$
$$\left(ay + b\right)\frac{\partial G\left(x, y\right)}{\partial x} - \left(ax + c\right)\frac{\partial G\left(x, y\right)}{\partial y} + aF\left(x, y\right) = 0$$

where $a, b, c \in \mathbb{R}$ and consequently

$$\left(ay + b\right)\frac{\partial}{\partial x}\mathcal{K}(x, y) - \left(ax + c\right)\frac{\partial}{\partial y}\mathcal{K}(x, y) = 0$$

Group classification

Case I. $a \neq 0$ (w.l.g. one may set $a = 1$)

$$F = -V(\rho)\cos\vartheta + U(\rho)\sin\vartheta, \quad G = U(\rho)\cos\vartheta + V(\rho)\sin\vartheta$$

$$\mathcal{K}(x, y) = f(\rho)$$

$$\rho = (x + x_0)^2 + (y + y_0)^2 + \rho_0, \quad \vartheta = \arctan\left(-\frac{y + y_0}{x + x_0}\right)$$

Case II. $a = 0$

$$\mathcal{K}(x, y) = g(u), \qquad u = cx + by + u_0$$

Here $x_0, y_0, \rho_0, u_0 \in \mathbb{R}$, f, g, U and V are arbitrary functions.

Conservation laws and integrability

In Case I, Noether's theorem implies the following conservation law

$$p = (y + y_0)\,\dot{x} - (x + x_0)\,\dot{y} + (y + y_0)\,F(x, y) - (x + x_0)\,G(x, y)$$

and we have two constants of motion p and $v = \sqrt{\dot{x}^2 + \dot{y}^2}$ which under the transformation of the dependent variables of the form

$$x = \sqrt{\rho - \rho_0}\cos\vartheta - x_0, \qquad y = \sqrt{\rho - \rho_0}\sin\vartheta - y_0$$

give the relations

$$\dot{\vartheta} = \frac{1}{\rho - \rho_0}\left(p - U(\rho)\right), \qquad \dot{\rho}^2 = 4v(\rho - \rho_0) - 4\dot{\vartheta}^2(\rho - \rho_0)^2$$

and now, the integrability of the respective dynamical system is obvious.