Noncommutative Calculus by Star Products

Akira Yoshioka.

Varna 200806

- Based on the Joint Work H. Omori Y. Maeda N. Miyazaki
- Interested in Noncommutativity
- Hope to construct Noncommutative elementary calculus.
- Primitive Elementary calculation by hands

Date . . No.

§0 Abstract of this talk

1 Star products

$\mathcal{P}(\mathbb{C}^{2m})$ complex polynomials of $2m$-variables

$\hbar > 0$

For $\forall f, g \in \mathcal{P}(\mathbb{C}^{2m})$, we consider a star product

$$f *_K g = f \, e^{\frac{i\hbar}{2} \overleftarrow{\partial} \overbrace{(J+K)}^{\overset{\Lambda}{\shortparallel}} \overrightarrow{\partial}} g$$

$$= f\left(1 + \frac{i\hbar}{2} \overleftarrow{\partial} \Lambda \overrightarrow{\partial} + \cdots + \frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n (\overleftarrow{\partial} \Lambda \overrightarrow{\partial})^n + \cdots \right) g$$

where $\Lambda = J + K$, $J = \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}$ fixed.

$K \in S = Sym(2m, \mathbb{C})$ symmetric $2m \times 2m$.

$$f \overleftarrow{\partial} \Lambda \overrightarrow{\partial} g = f\left(\sum_{\alpha\beta} \overleftarrow{\partial_\alpha} \Lambda^{\alpha\beta} \overrightarrow{\partial_\beta}\right) g = \sum_{\alpha,\beta} \Lambda^{\alpha\beta} \partial_\alpha f \, \partial_\beta g$$

$$f(\overleftarrow{\partial} \Lambda \overrightarrow{\partial})^n g = \sum_{\substack{\alpha_1 \cdots \alpha_n \\ \beta_1 \cdots \beta_n}} \Lambda^{\alpha_1\beta_1} \cdots \Lambda^{\alpha_n\beta_n} \partial_{\alpha_1} \cdots \partial_{\alpha_n} f \, \partial_{\beta_1} \cdots \partial_{\beta_n} g$$

$*_K$: associative product on $\mathcal{P}(\mathbb{C}^{2m})$

$(\mathcal{P}(\mathbb{C}^{2m}), *_K) \cong W_{2m}$ Weyl algebra of $2m$-generators

Date · · No.

2° Topology and completion

We want to consider star exponentials

$$e_*^{\frac{1}{i\hbar}H_*} = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\frac{1}{i\hbar}H_* * \cdots * \frac{1}{i\hbar}H_*}_{n}$$

for a noncommutative polynomial. H_*.

For this purpose, we introduce a certain topology T_p into $\mathcal{P}(\mathbb{C}^{2m})$ and take a completion $\mathcal{E}_p$.

3° Star exponentials

For a noncommutative polynomial H_* we define a star exponential function $e_*^{\frac{1}{i\hbar}H_*}$ by means of the evolution equation.

4° Elementary noncommutative f'ns

Using star exponential f'n $e_*^{\frac{t}{i\hbar}H_*}$ we hope to define elementary noncommutative f'ns such as

$$\begin{cases} H_*^{-1} = -\int_0^{\infty} e_*^{\frac{t}{i\hbar}H_*}\,dt \\ \delta_*(H_*) = \int_{-\infty}^{\infty} e_*^{\frac{t}{i\hbar}H_*}\,dt \\ \Gamma_*(H_*) = \int_{-\infty}^{\infty} e^{-e^{t}}\, e_*^{t\frac{1}{i\hbar}H_*}\,dt \quad \cdots \end{cases}$$

etc

Discuss examples only at present.

Date No.

§1 Star Products

$\mathcal{P} = \mathcal{P}(\mathbb{C}^{2m})$ complex, polynomials of $2m$ variables

$\hbar > 0$

1. $*_K$ - product

$J = \begin{pmatrix} 0 & 1_m \\ 1_m & 0 \end{pmatrix}$ fixed

For $\forall K \in S = Sym(2m.\mathbb{C})$ symmetric $2m \times 2m$ complex matrix,

we put $\Lambda = J + K$.

Def. 1.1 $\quad f *_K g = f\, e^{\frac{i\hbar}{2}\overleftarrow{\partial}\Lambda\overrightarrow{\partial}}\, g, \quad \forall f, g \in \mathcal{P}$

1) When $K = O$ zero $\quad *_K$ Moyal product

$K = \begin{pmatrix} 0 & 1_m \\ 1_m & 0 \end{pmatrix}$ normal product

$K = \begin{pmatrix} 0 & -1_m \\ -1_m & 0 \end{pmatrix}$ anti-normal product

$*_K$ is a generalization of Moyal, normal, anti-normal products.

2) $*_K : \mathcal{P} \times \mathcal{P} \longrightarrow \mathcal{P}$ well-defined associative product

$\mathcal{P}_{*_K} = (\mathcal{P}, *_K)$ associative algebra.

Date No.

2° Intertwiners

$\forall K, K' \in \mathcal{S} = Sym(2m, \mathbb{C})$

<u>Def 1.2</u> $I_K^{K'} : \mathcal{P} \longrightarrow \mathcal{P}$ linear by

$$I_K^{K'} f = e^{\frac{i\hbar}{4}(K'-K)\partial^2} f$$
$$= \left(1 + \frac{i\hbar}{4}(K'-K)\partial^2 + \cdots + \frac{1}{n!}\left(\frac{i\hbar}{4}\right)^n \left((K'-K)\partial^2\right)^n + \cdots\right) f$$

where $(K'-K)\partial^2 = \sum_{\alpha,\beta=1}^{2m} (K'-K)_{\alpha\beta}\, \partial_\alpha \partial_\beta$

<u>Th 1.3</u> $I_K^{K'}$ is a well-defined on $\mathcal{P}$, satisfies

1) $I_K^{K'}(f *_K g) = (I_K^{K'} f) *_{K'} (I_K^{K'} g)$

2) $I_{K'}^{K''} \cdot I_K^{K'} = I_K^{K''} \quad \forall K, K', K'' \in \mathcal{S}$

3) $(I_K^{K'})^{-1} = I_{K'}^{K} \quad \forall K, K' \in \mathcal{S}$

1) We have an intertwiner $I_K^{K'} : \mathcal{P}_{*K} \longrightarrow \mathcal{P}_{*K'}$
(an algebra isom.)

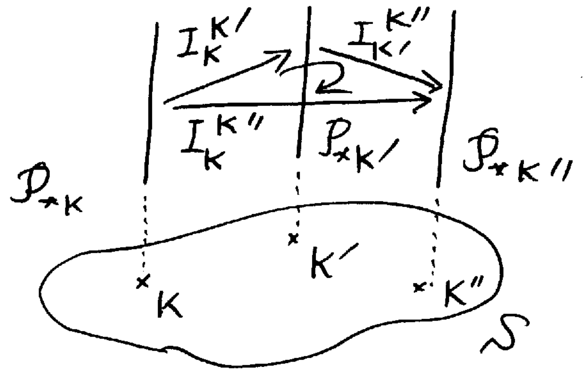
2) $(u^1 \cdots u^m, u^{m+1} \cdots u^{2m}) = (u^1 \cdots u^m, v^1 \cdots v^m) \in \mathbb{C}^{2m}$

$$\begin{cases} [u^k, u^l]_{*K} = u^k *_K u^l - u^l *_K u^l = 0 \\ [v^k, v^l]_{*K} = 0, \\ [u^k, v^l]_{*K} = \frac{\hbar}{i} \delta_{kl} \end{cases}$$

Hence $\mathcal{P}_{*K} \cong W_{2m}$ Weyl algebra

3° q-number polynomials = geometric picture of W_{2m}

We consider a trivial bundle $W_{2m} \longrightarrow \coprod_{K\in\mathcal{S}} \mathcal{P}_{\star K} \longrightarrow \mathcal{S}$



<u>Def 1.4</u> A section $f_\star \in \Gamma(\coprod_{K\in\mathcal{S}} \mathcal{P}_{\star K})$ is called a q-number polynomial $\Longleftrightarrow$ $I_K^{K'} f_\star(K) = f_\star(K')$, $\forall K, K' \in \mathcal{S}$

$\mathcal{P}_\star \equiv \{ f_\star ;\ q\text{-number polynomial} \}$

<u>Prop 1.5</u> $(f_\star * g_\star)(K) \equiv f_\star(K) *_K g_\star(K)$, $\forall K$ is a well-defined associative product on $\mathcal{P}_\star$, and $(\mathcal{P}_\star, *)$ is $\cong W_{2m}$.

Infinitesimal intertwiner $\left.\frac{d}{dt}\right|_{t=0} I_K^{c(t)} f_\star(K)$ where $c(t) \in \mathcal{S}$, $c(0)=K$, $\dot{c}(0) = X \in T_K\mathcal{S}$ gives a connection

$\nabla_X f_\star(K) = \left.\frac{d}{dt}\right|_{t=0} I_K^{c(t)} f_\star(K) \qquad \forall f \in \Gamma(\coprod_K \mathcal{P}_{\star K})$

<u>Prop 1.6</u> ∇ is a flat connection on $\coprod_K \mathcal{P}_{\star K}$ and $\mathcal{P}_\star = \{ f_\star \in \Gamma(\coprod_K \mathcal{P}_{\star K}) \mid \nabla f_\star = 0 \}$

§2. Topologies and Completions

1° Topology

We want to consider a complete topolgical vector space where $*_K$ gives a well-defined continuous product.

One space a well-know space of formal power series
$C^\infty(\mathbb{C}^{2m})[[\hbar]] \ni f_0 + f_1\hbar + f_2\hbar^2 + \cdots + f_n\hbar^n + \cdots$

This space is a good space for deformation quantization, however, $C^\infty(\mathbb{C}^{2m})[[\hbar]]$ drops certain important information.

We choose another one.

For $p>0$, we consider a system of semi-norm $\{\|\cdot\|_{p,s}\}_{s>0}$

$$\|f\|_{p,s} = \sup_{z\in\mathbb{C}^{2m}} |f(z)|\, e^{-s|z|^p} \qquad f\in \mathcal{P}(\mathbb{C}^{2m})$$

$|z| = \sqrt{|z_1|^2 + \cdots + |z_{2m}|^2}$. By this system, we put a topology T_p into $\mathcal{P}(\mathbb{C}^{2m})$. and we take the completion $\mathcal{E}_p$. Then we have

Theorem 2.1

1) $\mathcal{E}_p$ is a Fréchet space with $\{\|\cdot\|_{p,s}\}_{s>0}$.
2) $\mathcal{E}_p = \{ f \in \mathcal{H}ol(\mathbb{C}^{2m}) \mid \|f\|_{p,s} < \infty,\ \forall s>0\}$
3) $\mathcal{P}(\mathbb{C}^{2m}) \underset{\text{dense}}{\hookrightarrow} \mathcal{E}_p \hookrightarrow \mathcal{E}_{p'}$ continuous embedding for $0<\forall p<\forall p'$.
4) $\mathcal{E}_{p+} = \bigcap_{p'>p} \mathcal{E}_{p'}$ is also a Fréchet space.

For certain $\mathcal{E}_p$'s, $*_K$ and $I_K^{K'}$ are well-defined

<u>Th 2.2</u> For $\mathcal{E}_p$, $0 < p \leq 2$, we have the following:

1) For $\forall f, g \in \mathcal{E}_p$, $f *_K g = f e^{\frac{i\hbar}{2}\overleftarrow{\partial} \wedge \overrightarrow{\partial}} g$

converges to give an element of $\mathcal{E}_p$, for each $K \in \mathcal{S}$.

Moreover, $*_K : \mathcal{E}_p \times \mathcal{E}_p \to \mathcal{E}_p$ is a continuous associative product. Then $\mathcal{E}_{p *_K} = (\mathcal{E}_p, *_K)$ is a Fréchet algebra for every $K \in \mathcal{S}$.

2) For $\forall f \in \mathcal{E}_p$, $I_K^{K'} f = e^{\frac{i\hbar}{4}(K'-K)\partial^2} f$

converges in $\mathcal{E}_p$ to give an element of $\mathcal{E}_p$, for each $K \in \mathcal{S}$. Moreover,

$I_K^{K'} : \mathcal{E}_{p *_K} = (\mathcal{E}_p, *_K) \longrightarrow \mathcal{E}_{p *_{K'}} = (\mathcal{E}_p, *_{K'})$

is a continuous linear map s.t.

1) $I_K^{K'}(f *_K g) = (I_K^{K'} f) *_{K'} (I_K^{K'} g)$

2) $I_{K'}^{K''} I_K^{K'} = I_K^{K''}$

3) $(I_K^{K'})^{-1} = I_{K'}^{K}$. $\quad \forall K, K', K'' \in \mathcal{S}$

For $p > 2$, we have only the following.

<u>Th. 2.3</u> For $p > 0$, take $p' > 0$ s.t.

$0 < p' < 2 < p$ and $\frac{1}{p'} + \frac{1}{p} = 1$ then

$*_K : \mathcal{E}_p \times \mathcal{E}_{p'} \longrightarrow \mathcal{E}_p$ is a continuous bilinear product. Then $\mathcal{E}_p$ is a $\mathcal{E}_{p'}$-bimodule w.r.t $*_K$ for every $K \in \mathcal{S}$.

2°. <u>q-number functions</u>

Similarly to $\mathcal{P}_*$, we can consider flat bundle and parallel sections.

We consider a trivial bundle. $\coprod_{K \in \mathcal{S}} \mathcal{E}_p \xrightarrow{\pi} \mathcal{S}$

For the case $0 < p \leq 2$, due to Th. 2.2 we have the same concept of $\mathcal{P}_*$:

- A section $f_* \in \Gamma(\coprod_{K \in \mathcal{S}} \mathcal{E}_p)$ is called a q-number function $\Leftrightarrow$ $I_K^{K'} f_*(K) = f_*(K')$ $\forall K, K' \in \mathcal{S}$

$\mathcal{E}_{p*} = \{ f_* \text{ q-number functions of } p \}$.

Date No.

- For $\mathcal{E}_{p*}$ $(0 < p \leq 2)$, we have an associative product by

$$(f_* * g_*)(K) = f_*(K) *_K g_*(K) \quad \forall K \in \mathcal{S}.$$

- By the infinitesimal intertwiner $\nabla_X f_*(K)$

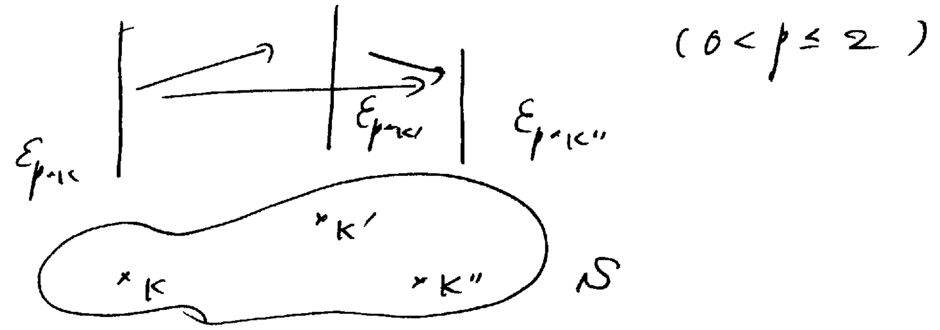
$$= \frac{d}{dt}\Big|_{t=0} I_K^{c(t)} f_*(K) \quad , \quad c(t) \in \mathcal{S}, \quad c(0) = K, \quad \dot{c}(0) = X$$

∇ is a flat connection of this bundle. and

$$\mathcal{E}_{p*} = \{ f_* \in \Gamma(\coprod_{K \in \mathcal{S}} \mathcal{E}_{p*_K}) \mid \nabla f_* = 0 \}$$

For $p > 2$, instead of $I_K^{K'}$ we use the infinitesimal intertwiner ∇ ; the connection of the bundle $\coprod_{K \in \mathcal{S}} \mathcal{E}_p \to \mathcal{S}$ to define a q-number function f_* by

$$\nabla f_* = 0.$$



Date . . No.

§3 Star Exponential functions

1° Method of definition

For a q-number polynomial $H_* \in \mathcal{P}_*$ we want to define a star exponential function $e_*^{\frac{1}{i\hbar}H_*}$ which is formally given as

$$e_*^{\frac{1}{i\hbar}H_*} = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\frac{1}{i\hbar}H_* * \cdots * \frac{1}{i\hbar}H_*}_{n}$$

In order to define, we follow the operator method, we use an evolution equation

$$\frac{d}{dt} f_t = \frac{1}{i\hbar} H_* * f_t, \qquad f_0 = 1.$$

and we set $e_*^{\frac{1}{i\hbar}H_*} = f_1$.

However, this is not so easy in general and at present, we investigate only the cases of linear q-number polynomials and quadratic q-number polynomials.

2° linear q-number polynomials.

$\mathfrak{a} = (a_1, -, a_{2m}) \in \mathbb{C}^{2m}$ constant

$u = (u^1, \dots, u^{2m}) \in \mathbb{C}^{2m}$ variables

We put $l_{\mathfrak{a}_*} = \langle \mathfrak{a}, u \rangle = \sum_{j=1}^{2m} a_j u^j \in \mathcal{P}_*$

We denote by $:l_{\mathfrak{a}*}:_K$ the $\mathcal{P}_{*K}$ representation of $l_{\mathfrak{a}} \in \mathcal{P}_*$ i.e, $:l_{\mathfrak{a}*}:_K = l_{\mathfrak{a}*}(K)$

It is easy to see $:l_{\mathfrak{a}*}:_K = \langle \mathfrak{a}, u \rangle$, hence the evolution equation is explicitly given as

$$\frac{d}{dt} f_t = \frac{1}{i\hbar} \langle \mathfrak{a}, u \rangle *_K f_t = \frac{1}{i\hbar} \langle \mathfrak{a}, u \rangle e^{\frac{i\hbar}{2} \overleftarrow{\partial} \wedge \overrightarrow{\partial}} f_t$$

$$= \frac{1}{i\hbar} \langle \mathfrak{a}, u \rangle f_t + \frac{1}{2} \mathfrak{a} \wedge \partial f_t , \quad f_0 = 1.$$

Then the solution is

$$f_t = e^{\frac{t^2}{4i\hbar} \mathfrak{a} K \mathfrak{a} + \frac{t}{i\hbar} \langle \mathfrak{a}, u \rangle}$$

and hence $:e_*^{\frac{t}{i\hbar} l_{\mathfrak{a}*}}:_K = e^{\frac{t^2}{4i\hbar} \mathfrak{a} K \mathfrak{a} + \frac{t}{i\hbar} \langle \mathfrak{a}, u \rangle}$

$\in \mathcal{E}_{1+}$

Date . . No.

Due to Theorem 2.2, the intertwiners are well-defined on $\mathcal{E}_{1+}$, we also obtain the same solution by means of $I_K^{K'}$ as follows.

When $K=0$, the product is the Moyal product $*_0$ and in this case the star exponential of a linear function $\langle a, u\rangle$ is obtained directly by the expansion $e_{*_0}^{\frac{t}{i\hbar}\langle a, u\rangle} = \sum \frac{1}{n!}\left(\frac{t}{i\hbar}\right)^n \langle a, u\rangle *_0 \cdots * \langle a, u\rangle$
$= e^{\frac{t}{i\hbar}\langle a, u\rangle}$

$$:e_*^{\frac{t}{i\hbar} l_{a*}}:_K = I_0^K\left(e^{\frac{t}{i\hbar}\langle a, u\rangle}\right)$$
$$= e^{\frac{i\hbar}{4}K\partial^2} e^{\frac{t}{i\hbar}\langle a, u\rangle}$$
$$= e^{\frac{t^2}{4i\hbar} aKa + \frac{t}{i\hbar}\langle a, u\rangle}$$

Date . . No.

3° quadratic g-number polynomials.

$$A = (A_{\alpha\beta}) \in S = Sym(2n, \mathbb{C})$$

$$H_* = u A u_* = \sum_{\alpha,\beta=1}^{2m} A_{\alpha\beta}\, u^\alpha * u^\beta$$

The representation at $\mathcal{P}_{*K}$ is given as

$$:H_*:_K = \sum_{\alpha,\beta} A_{\alpha\beta}\, u^\alpha *_K u^\beta$$

$$= \sum_{\alpha,\beta} A_{\alpha\beta}\, u^\alpha\, e^{\frac{i\hbar}{2}\overleftarrow{\partial}\Lambda\overrightarrow{\partial}}\, u^\beta$$

$$= \sum_{\alpha,\beta} A_{\alpha\beta}\, u^\alpha u^\beta + \frac{i\hbar}{2}\sum_{\alpha,\beta} A_{\alpha\beta}\Lambda^{\alpha\beta}$$

$$= u A u + \frac{i\hbar}{2}\operatorname{Tr}(AK).$$

$$\frac{d}{dt} f_t = :\frac{1}{i\hbar} H_*:_K *_K f_t, \qquad f_0 = 1.$$

The solution is given as

$$f_t = e^{\frac{i\hbar}{2}\operatorname{Tr}(AK)}\, g(t)\, e^{\frac{1}{i\hbar} u A(t) u}$$

where
$$\begin{cases} g(t) = \left(\det \dfrac{1+\tanh(tAJ)}{(1-\tanh(tAJ))KJ}\right)^{1/2} \\[2ex] A(t) = (1-\tanh(tAJ)KJ)^{-1}\tanh(tAJ) \end{cases}$$

$$\in \mathcal{E}_{2+}.$$

§4 Elementary noncommutative f'ns.

We hope to construct noncommutative f'ns by using star exponential f'ns of q-number polynomials. However, at present, we are just at the begining and far from the end.
We have only several examples.

1) linear q-number polynomial.

$$:e_*^{\frac{t}{i\hbar}\langle a, u\rangle}:_K = e^{\frac{t^2}{4i\hbar} aKa}\, e^{\frac{t}{i\hbar}\langle a, u\rangle}$$

If $Re\, \frac{t^2}{4i\hbar} aKa < 0$, then $:e_*^{\frac{t}{i\hbar}\langle a, u\rangle}:_K$ is rapidly decreasing w.r.t. t and we can define by

$$:\left(\frac{1}{i\hbar}\langle a, u\rangle\right)_*^{-1}:_K = -\int_0^\infty :e_*^{\frac{t}{i\hbar}\langle a, u\rangle}:_K\, dt$$

$$:\delta_*\left(\frac{1}{i\hbar}\langle a, u\rangle\right):_K = \int_{-\infty}^{\infty} :e_*^{\frac{t}{i\hbar}\langle a, u\rangle}:_K\, dt$$

$$:\Gamma_*\left(\frac{1}{i\hbar}\langle a, u\rangle\right):_K = \int_{-\infty}^{\infty} e^{-e^{\tau}} :e_*^{\frac{\tau}{i\hbar}\langle a, u\rangle}:_K\, dt$$

2) quadratic q-number polynomials.

$$(u^1, u^2) \in \mathbb{C}^2 \qquad [u^1, u^2]_* = \frac{\hbar}{i}$$
$$= (u, v)$$

We can define

$$\delta_*\left(\frac{1}{i\hbar} uv\right) = \int_{-\infty}^{\infty} e_*^{t \frac{1}{i\hbar} uv} dt$$

$$\left(\frac{1}{i\hbar} uv\right)_*^{-1} = -\int_0^{\infty} e_*^{t \frac{1}{i\hbar} uv} dt$$

$$\Gamma_*\left(\frac{1}{i\hbar} uv\right) = \int_{-\infty}^{\infty} e^{-e^{\tau}} e_*^{\tau \frac{1}{i\hbar} uv} d\tau$$

|
|
|