New results on the geometry of translation surfaces

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Outline

1. Translation surfaces in $\mathbb{E}^3$
2. On the geometry of the second fundamental form of translation surfaces in $\mathbb{E}^3$
   - $\{K_{II}, H\}$ - Generalized Weingarten translation surfaces
   - II-minimality
3. Translation surfaces in the hyperbolic space $\mathbb{H}^3$
4. Translation surfaces in the Heisenberg group Nil_3
5. Translation surfaces in $\mathbb{S}^3$
6. Final remarks

Darboux surfaces

Cartesian parametrization:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A(v) \begin{pmatrix} f(u) \\ g(u) \\ h(u) \end{pmatrix} + \begin{pmatrix} a(v) \\ b(v) \\ c(v) \end{pmatrix}$$

where $A(v) \in O(n)$

1

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where $A(v) \in O(n)$

A Darboux surface represents a union of "EQUAL" curves (i.e. the image of one curve[1], obtained by isometries of the space.

[1] generatrix

Darboux surfaces

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If the generatrix is
- a straight line : ruled surfaces
- a circle : circled surfaces including e.g. tubes

Tubes

$$r(s,t) = \gamma(t) + \cos s\, N(t) + \sin s\, B(t)$$

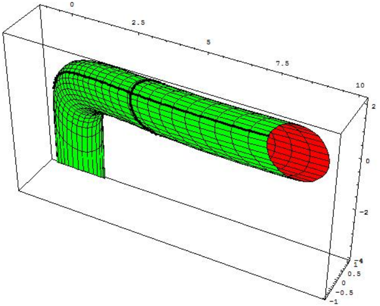

Figure: tube

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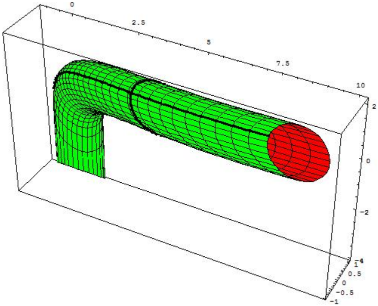

Figure: tube

$$r(s,t) = \gamma(t) + A(t)\, \mathbb{S}^1$$

Translation surfaces

Translation surface = ”sum” of two curves

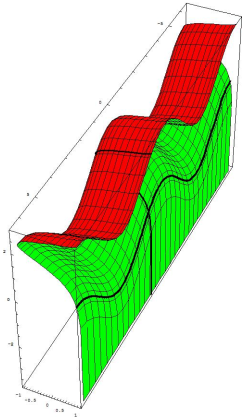

Figure: translation surface

Translation surfaces

If the two curves are situated in orthogonal planes

$$(x, y, z) \longmapsto (x, y, f(x) + g(y))$$

Examples:

1. planes

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2. cylinders
3. hyperbolic and elliptic paraboloids
4. the egg box surface
5. Scherk surface

Egg box surfaces

$$\left(x, y, a\left(\sin\frac{x}{b} + \sin\frac{y}{b}\right)\right)$$

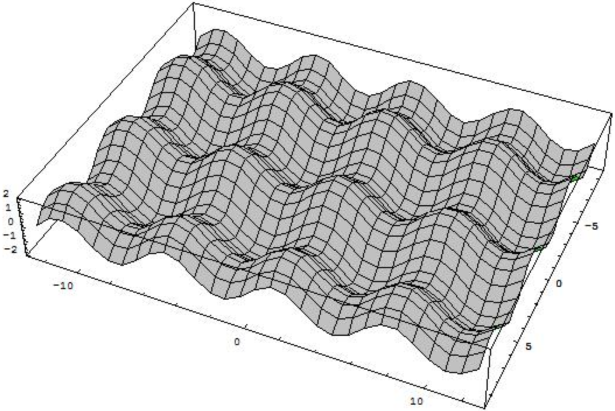

Figure: egg box surface

Scherk surfaces

$$\left(x, y, a \log \frac{\cos \frac{x}{a}}{\cos \frac{y}{a}}\right)$$

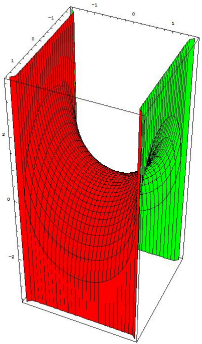

Figure: Scherk surface

Scherk surface - art

... much more beautiful

Figure: Scherk surface

Second fundamental form

ON THE GEOMETRY OF THE SECOND FUNDAMENTAL FORM OF TRANSLATION SURFACES IN $\mathbb{E}^3$
joint work with **A. I. Nistor**: arXiv:0812.3166v1 [math.DG]

M surface in $\mathbb{E}^3$
I = the first fundamental form – intrinsic object
II = the second fundamental form – extrinsic tool to characterize the twist of M in the ambient

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I = the first fundamental form – intrinsic object
II = the second fundamental form – extrinsic tool to characterize the twist of M in the ambient

II is a metric if and only if it is non-degenerate
curvature properties associated to II:
S. Verpoort, *The Geometry of the Second Fundamental Form: Curvature Properties and Variational Aspects,* PhD. Thesis, Katholieke Universiteit Leuven, Belgium, 2008

Second fundamental form

Lemma (Dillen, Sodsiri - 2005)

The second fundamental form II of M is non-degenerate if and only if M is non-developable.

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Remark (Verpoort - 2008)

Critical points of the area functional of the second fundamental form are those surfaces for which the mean curvature of the second fundamental form H_{II} vanishes.

Old results

- Koutroufiotis - 1974: a closed ovaloid with $K_{II} = cK\,,\ c \in \mathbb{R}$ or if $K_{II} = \sqrt{K}$ is a sphere

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Old and recent results

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Kim & Yoon - 2004, Sodsiri - 2005, Yoon - 2006 extends the study for 3-dimensional Lorentz-Minkowski spaces and for different relations between H, K, H_{II} and K_{II}

II-flat translation surfaces in $\mathbb{E}^3_1$

Theorem (Goemans, Van de Woestyne - 2007)

If a translation surface in $\mathbb{E}^3_1$ parametrized by $\bar{x}(s,t) = (s, t, f(s) + g(t))$ has $K_{II} = 0$, then

$$f(s) = \int F^{-1}(s+d)ds \quad \text{and} \quad g(t) = \int G^{-1}(t+m)dt$$

with F and G real functions determined by

$$F(x) = \int \frac{x^2}{ax^4+bx^2+c}dx \text{ and } G(x) = \int \frac{x^2}{-ax^4+(2a+b)x^2-a-b-c}dx,$$

and a, b, c, d şi m real numbers.

II-flat PT surfaces in $\mathbb{E}^3$

polynomial translation surfaces (in short, *PT surfaces*) : translation surfaces for which f and g are polynomials

Theorem (M., Nistor - 2009)

There are no II-flat polynomial translation surfaces in $\mathbb{E}^3$.

Proof.

$$K_{II} = \frac{1}{(|eg| - f^2)^2} \left(\begin{vmatrix} -\frac{1}{2}e_{vv} + f_{uv} - \frac{1}{2}g_{uu} & \frac{1}{2}e_u & f_u - \frac{1}{2}e_v \\ f_v - \frac{1}{2}g_u & e & f \\ \frac{1}{2}g_v & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_v & \frac{1}{2}g_u \\ \frac{1}{2}e_v & e & f \\ \frac{1}{2}g_u & f & g \end{vmatrix} \right)$$

II-flat PT surfaces in $\mathbb{E}^3$

(cont.)

$$K_{II} = \frac{num}{4\alpha'\beta'\Delta^{3/2}}$$

where

$$\begin{gathered} num = -2\alpha(u)^2\alpha'(u)^2\beta'(v) - 2\alpha'(u)\beta(v)^2\beta'(v)^2 + \\ 2\alpha(u)^2\alpha'(u)\beta'(v)^2 + 2\alpha'(u)^2\beta(v)^2\beta'(v) + \\ 2\alpha'(u)\beta'(v)^2 + 2\alpha'(u)^2\beta'(v) + \\ \alpha'(u)\beta(v)\beta''(v) + \alpha(u)\alpha''(u)\beta'(v) + \\ \alpha(u)^2\alpha'(u)\beta(v)\beta''(v) + \alpha(u)\alpha''(u)\beta(v)^2\beta'(v) + \\ \alpha'(u)\beta(v)^3\beta''(v) + \alpha(u)^3\alpha''(u)\beta'(v). \end{gathered}$$

$\square$

II-flat translation surfaces

example given by Blair & Koufogiorgos - 1992 : *II*-flat non-minimal translation surfaces, involving *power functions*, i.e.

$$\alpha = au^p \text{ and } \beta = bv^q \text{ with } a, b \in \mathbb{R},\ a, b \neq 0 \text{ and } p, q \in \mathbb{Q}.$$

Proposition (M., Nistor - 2009)

The only *II*-flat translation surfaces with *f* and *g* power functions can be parametrized by

$$r(u, v) = \left(u, v, c(u^{\frac{4}{3}} - v^{\frac{4}{3}})\right), c \in \mathbb{R}^*.$$

$K_{II} = H$

$\{A, B\}$ - generalized Weingarten surfaces : Dillen, Sodsiri - 2005

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Theorem (M., Nistor - 2009)

The only translation surfaces with non-degenerate second fundamental form having the property $K_{II} = H$ are given, up to a rigid motion of $\mathbb{R}^3$, by

$$r(u,v) = \left(u,\ v,\ \frac{2}{c}\ \log \left| \frac{\cos \frac{cu}{2}}{\cos \frac{cv}{2}} \right| \right),\ c \in \mathbb{R}^*.$$

More, we notice the parametrization of a Scherk type surface, so we have

$$K_{II} = H = 0.$$

$K_{II} = \lambda H,\ \lambda \neq 1, 2$

Theorem (M., Nistor - 2009)

The only $\{K_{II}, H\}$–generalized Weingarten translation surfaces with non-degenerate second fundamental form satisfying $K_{II} = \lambda H$ with $\lambda \in \mathbb{R} \setminus \{1, 2\}$, are given, up to a rigid motion of $\mathbb{R}^3$, by the parametrization

$$r(u,v) = \left(u, v, \frac{1}{p} \log \left| \frac{\cos(pv + r)}{\cos(pu + q)} \right| \right), \text{ where } p \neq 0 \text{ and } r, q \in \mathbb{R}$$

which represents a Scherk type surface. Moreover $K_{II} = H = 0$.

$K_{II} = 2H$

Theorem (M., Nistor - 2009)

The only translation surfaces with non-degenerate second fundamental form having the property $K_{II} = 2H$ are given, up to a rigid motion of $\mathbb{R}^3$, by the following parametrizations

i) Case 1.

$$r(u,v) = \left(u, v, -\frac{\nu}{2}\log\left(\sinh(pu)^{\frac{1}{p^2}}\cos(qv)^{\frac{1}{q^2}}\right)\right)$$

$$r(u,v) = \left(u, v, -\frac{\nu}{2}\log\left(\cosh(pu)^{\frac{1}{p^2}}\cos(qv)^{\frac{1}{q^2}}\right)\right)$$

Case 2.

$$r(u,v) = \left(u, v, \frac{\nu}{2}\log\frac{\cos(pu)^{\frac{1}{p^2}}}{\cos(qv)^{\frac{1}{q^2}}}\right)$$

$K_{II} = 2H$

i) Case 3.

$$r(u,v) = \left(u, v, -\frac{\nu}{2} \log \frac{\sinh(pu)^{\frac{1}{p^2}}}{\sinh(qv)^{\frac{1}{q^2}}}\right) \qquad r(u,v) = \left(u, v, -\frac{\nu}{2} \log \frac{\cosh(pu)^{\frac{1}{p^2}}}{\cosh(qv)^{\frac{1}{q^2}}}\right)$$

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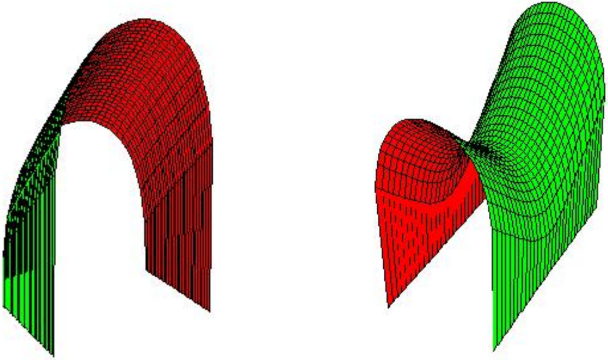
ii)

$$r(u,v) = (u, v, a(u-u_0)^2 - a(v-v_0)^2), \quad a, u_0, v_0 \in \mathbb{R}$$

hyperbolic paraboloid.

iii) combinations of the previous functions in (i) and a second order polynomial (as in (ii), for a certain *a*)

Figures



$$r(u,v) = (u, v, \log(\sinh u \cos v))$$
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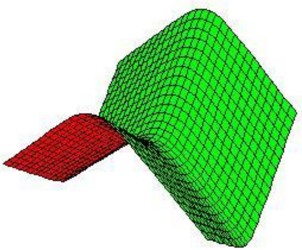
Figures



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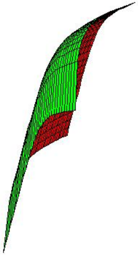

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II-minimal surfaces

Haesen, Verpoort, Verstraelen - 2008

$$H_{II} = -H - \frac{1}{4}\Delta^{II} \log(K)$$

where Δ^{II} is the Laplacian for functions computed with respect to the second fundamental form as metric. H_{II} can be equivalently expressed as

$$H_{II} = -H - \frac{1}{2\sqrt{\det II}} \sum_{i,j} \frac{\partial}{\partial u^i} \left(\sqrt{\det II}\, h^{ij}\, \frac{\partial}{\partial u^j} (\log \sqrt{K}) \right).$$

II-minimal translation surfaces

$(u, v) \mapsto (u, v, f(u) + g(v))$; $\alpha = f'$, $\beta = g'$

$H_{II} = 0$ is equivalent to

$$\frac{(1+\alpha^2)\beta' + (1+\beta^2)\alpha' - 4}{(1+\alpha^2+\beta^2)^2} + \frac{\alpha'''\alpha' - 2\alpha''^2}{2\alpha'^4} + \frac{\beta'''\beta' - 2\beta''^2}{2\beta'^4} = 0$$

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Theorem (M., Nistor - 2009)

There are NO II-minimal translation surfaces in Euclidean 3-space.

General things

R. López : arXiv:0902.4085v1 [math.DG]

$\mathbb{H}^3$ hyperbolic space : upper half-space $\mathbb{R}^3_+$
$ds^2 = \frac{1}{z^2}(dx^2 + dy^2 + dz^2)$

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x, y are interchangeable, but not with z
type 1 : $r(x,y) = \{x, y, f(x) + g(y)\}$
type 2 : $r(x,z) = \{x, f(x) + g(z), z\}$

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type 2 : $r(x,z) = \{x, f(x) + g(z), z\}$

Notice that there are NO isometries of $\mathbb{H}^3$ that carry surfaces of type 1 into surfaces of type 2 or vice-versa.

Minimal translation surface

Recall: in $\mathbb{E}^3 \Longrightarrow$ planes and Scherk surface

Known fact: Examples of minimal surfaces in $\mathbb{H}^3$: totally geodesic planes, minimal graphs (corresponding to Dirichlet problem)

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Known fact: Examples of minimal surfaces in $\mathbb{H}^3$: totally geodesic planes, minimal graphs (corresponding to Dirichlet problem)

Theorem (López - 2009)

There are NO minimal translation surfaces in $\mathbb{H}^3$ of type 1.
The only minimal translation surfaces in $\mathbb{H}^3$ of type 2 are totally geodesic planes.

Nil_3

Heisenberg group $Nil_3 \sim \mathbb{R}^3$

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) := \left(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1)\right)$$

$$g = dx^2 + dy^2 + \left[dz + \frac{1}{2}(ydx - xdy)\right]^2$$

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Lie algebra of $Iso(Nil_3)$ is generated by Killing v. f.

$$E_1 = \frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial z} \qquad E_2 = \frac{\partial}{\partial y} - \frac{x}{2}\frac{\partial}{\partial z}$$

$$E_3 = \frac{\partial}{\partial z} \qquad E_4 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

Nil_3

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Definition (Figueroa, Mercuri, Pedrosa - 1999)

A surface in Nil_3 is translation invariant if it is invariant under the action of 1-parameter subgroup generated by a Killing vector field of the form $a_1 E_1 + a_2 E_2 + a_3 E_3$, $a_1^2 + a_2^2 + a_3^2 \neq 0$.

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Proposition (Figueroa, Mercuri, Pedrosa - 1999)

Let M in Nil_3 be invariant under the 1-parameter group generated by

$$a_1E_1 + a_2E_2 + a_3E_3\ ,\ a_1^2 + a_2^2 \neq 0.$$

Then is it equivalent to a surface invariant under G_1.

Flat translation invariant surfaces

translation invariant surfaces : restrict to G_1 and G_3

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Proposition (Inoguchi - 2005)

Let M be a surface invariant under $G_3 = \{(0,0,t) \ : \ t \in \mathbb{R}\}$. Then M is locally expressed as

$$(0,0,v) \cdot (x(u), y(u), 0) \quad , \quad u \in I, v \in \mathbb{R}.$$

I - open interval, u - arclength parameter

Flat translation invariant surfaces

translation invariant surfaces : restrict to G_1 and G_3

Proposition (Inoguchi - 2005)

Let M be a surface invariant under $G_3 = \{(0,0,t) \ : \ t \in \mathbb{R}\}$. Then M is locally expressed as

$$(0,0,v)\cdot(x(u),y(u),0) \quad , \quad u \in I, v \in \mathbb{R}.$$

I - open interval, u - arclength parameter

Remark 1. $(x,y,0)$ and $(0,0,v)$ commute.
Remark 2. M is flat

Flat translation invariant surfaces

Proposition (Inoguchi - 2005)

Let M be a surface invariant under $G_1 = \{(t,0,0)\ ,\ t \in \mathbb{R}\}$. Then M is flat if and only if il is locally equivalent to the graph of

$$f(x,y) = \frac{xy}{2} + \frac{1}{2A}\left[y\sqrt{y^2 - A^2} - A^2 \log\left|y + \sqrt{y^2 - A^2}\right|\right]\ , \quad A \in \mathbb{R}^*.$$

Proof.

idea: the translation invariant surface (G_1) is locally parametrized as the graph

$$(x,0,0)\cdot(0,y,v(y)) = \left(x,y,v(y) + \frac{xy}{2}\right).$$

compute K + solve ODE □

Minimal G_1 - invariant surfaces

Proposition (Inoguchi - 2005)

Let M be a surfaces invariant under $G_1 = \{(t, 0, 0)\ ,\ t \in \mathbb{R}\}$. Then M is minimal if and only if il is locally equivalent to the graph of

$$f(x, y) = \frac{xy}{2} + a\left[y\sqrt{1 + y^2} + \log(y + \sqrt{1 + y^2})\right]\ , \quad a \in \mathbb{R}^*.$$

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Extensions: using translation to the right for curves in the xz-plane and yz-plane : no flatness results

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Why nothing about G_4?

G_4 invariant surfaces are nothing but rotational surfaces around z-axis ($G_4 = SO(2)$)

Classification results: Caddeo, Piu, Ratto - 1996

"Sum" of two curves

work in progress with Rafael López

$\mathbb{S}^3$ hypersurface in $\mathbb{R}^4 \equiv \mathbb{H}$ (noncommutative field of quaternions)
$\mathbb{S}^3$ group of unit quaternions

$\alpha(s)$, $\beta(t)$ curves on $\mathbb{S}^3$ (parametrized by arclength)

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Example (well known)

$$r(s,t) = (\cos s \cos t, \sin s \cos t, \cos s \sin t, \sin s \sin t).$$

- $\alpha = (\cos s, \sin s, 0, 0)$, $\beta(t) = (\cos t, 0, \sin t, 0)$: translation surface
- minimal and *II*-minimal

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From now on FIX $\alpha(s) = (\cos s, \sin s, 0, 0)$.

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The function x does not depend on s!!

First results

Proposition (López, M. - 2009)

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Example (the easiest: $q' = 0$)

$$\beta(t) = (\cos t, \sin t \sin\theta_0, \sin t \cos\theta_0 \cos\psi_0, \sin t \cos\theta_0 \sin\theta_0).$$

Proof.

$$\frac{\partial}{\partial t}\,\mathrm{ad}(r)(q) = \mathrm{ad}(r)(q') \qquad\qquad \beta'(t) = \xi_0\beta(t)$$

$$\xi_0 = \sin\theta_0\, i + j w_0, \quad w_0 \in \mathbb{C},\ |w_0| = \cos\theta_0,\ \theta_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

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□

Remark. All these surfaces are minimal.

Other results

Recall $N = j\zeta r$, $\zeta \in \mathbb{S}^1 \subset \mathbb{C}$

$$\zeta = \cos\varphi + \sin\varphi\, i \quad , \quad \varphi = \varphi(s,t)$$

$$\text{Weingarten operator } : A = \begin{pmatrix} -\dfrac{x}{\sqrt{1-x^2}} & \dfrac{1+x\varphi_t}{\sqrt{1-x^2}} \\ \dfrac{1}{\sqrt{1-x^2}} & -\dfrac{x+\varphi_t}{\sqrt{1-x^2}} \end{pmatrix}$$

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Proposition (López, M. - 2009)

The surface S cannot be totally geodesic in $\mathbb{S}^3$.

Minimality

Proposition (López, M. - 2009)

The surface S is minimal if and only if $\varphi(s,t) = -2\left(s + \int x(t)dt\right)$.
Moreover

$$\mathrm{ad}(r)(q) = x\,i - \sqrt{1-x^2}\left(-\sin\left(2\int x(t)dt + 2s\right)j + \cos\left(2\int x(t)dt + 2s\right)k\right)$$

where $x = x(t)$ is a smooth function.

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where $x = x(t)$ is a smooth function.

Difficulties: In order to give an explicit expression for β we have to solve the following QODE

$$\beta'(t) = \mu(t)\beta(t) \quad , \quad \mu(t) \text{ is known}$$

Problem

Find a 3-dimensional space and an embedding such that the following object becomes *II*-minimal or *II*-flat

Ceramic joke

Find a 3-dimensional space and an embedding such that the following object becomes II-minimal or II-flat

THANK YOU

FOR

ATTENTION !