Schur-Weyl Duality

and Natural Differential Operators

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Schur-Weyl duality – a connection between representations of the symmetric group S_n (the group of permutaitons of n elements) and representations of the linear group $GL(d) = \mathrm{Aut}\, L$, $\dim L = d$

Natural differential operators: $R(g)$ – the Riemann tensor, $W(g)$ – the Weyl conformal tensor, $\mathcal{N}(J)$ – the Nijenhuis tensor, $d(\omega)$ – the external differential, etc.

In the symbols of these operators, there are algebraic structures coming from the Schur-Weyl description of $GL(d)$

Basic example: $R(g)$

M – manifold, $T(M)$, $T^*(M)$ – tangent and cotangent bundles

$g \in C^\infty(S^2(T^*(M)))$ – metric with (some) fixed signature

$\phi \in \mathrm{Diff}\,(M)$ – diffeomorphism

$$\phi : g \mapsto \phi^*(g)_{\mu\nu}(x) = \frac{\partial \phi^\alpha}{\partial x^\mu} \frac{\partial \phi^\beta}{\partial x^\nu} g_{\alpha\beta}(\phi(x))$$

$$g_1 \sim g_2 \quad \Longleftrightarrow \quad \exists \phi : \phi^*(g_1) = g_2$$

If g_2 is flat, $R(g_1) \neq 0$ is an obstruction for the equivalence $g_1 \sim g_2$

Obstruction for local equvalence

Let $x_0 \in M$ be a given point, (x^μ) coordinates centered at x_0, The group $\text{Diff}_{x_0}(M)$ of all diffeos ϕ of M such that $\phi(x_0) = x_0$ plays a crucial role in the study of the local equivalence of metrics at x_0.

$\text{Diff}_{x_0}(M)$ has a natural action on the $j^k_{x_0}(g)$:

$$j^k_{x_0}(g) \mapsto j^k_{x_0}(\phi^*(g)) := j^k(\phi)(j^k_{x_0}(g)) \ .$$

If there exists $\phi \in \text{Diff}_{x_0}(M)$ such that $\phi^*(g_1) = g_2$, then

$$j^k(\phi)(j^k_{x_0}(g_1)) = j^k_{x_0}(g_2) \ ,$$

i.e., $j^k_{x_0}(g_1)$ and $j^k_{x_0}(g_2)$ lie in the same orbit of $\text{Diff}_{x_0}(M)$. If $j^k_{x_0}(g_1)$ and $j^k_{x_0}(g_2)$ belong to different orbits, then they are not equivalent in a neigborhood of x_0.

Therefore, we study the space of orbits of $\text{Diff}_{x_0}(M)$ on the space of k-th jets of metrics and look for the canonical projection.

We start with $k = 0, 1, 2, \ldots$ and look for the lowest k for which there is more than one orbit of $\mathrm{Diff}_{x_0}(M)$.

We work in the centered coordinates: $x^\mu(x_0) = 0$, and use the notation $j_0^k(g) := j_{x_0}^k(g)$.

Case $k = 0$: $$j_0^0(g)_{\mu\nu} = g_{\mu\nu}(0) =: \tilde{g}_{\mu\nu},$$

$$\tilde{g} \mapsto D(\phi)|_{x_0}\, \tilde{\eta}\, D(\phi)_{x_0}^T \ .$$

All metrics of the same signature in a vector space are equivalent, so there is no obstruction at this level.

Case $k = 1$: Fix $\tilde{g}_{\mu\nu}$, consider metrics with 1-jets starting with $\tilde{g}_{\mu\nu}$:

$$j_0(g)_{\mu\nu} = \tilde{g}_{\mu\nu} + \tilde{g}_{\mu\nu,\alpha}\, x^\alpha \ .$$

Without loss of generality,

$$j_0^1(\phi)^\rho(x) = x^\rho + \frac{1}{2} B^\rho_{\alpha\beta}\, x^\alpha\, x^\beta \ ,$$

$B^\rho_{\alpha\beta}$ symmetric in α and β. With this choice,

$$\begin{aligned} \tilde{g}_{\mu\nu} &\mapsto \tilde{g}_{\mu\nu} \\ \tilde{g}_{\mu\nu,\alpha} &\mapsto \tilde{g}_{\mu\nu,\alpha} + B_{\mu,\nu\alpha} + B_{\nu,\mu\alpha} \ , \end{aligned}$$

where $B_{\mu,\nu\alpha} := \tilde{g}_{\mu\rho}\, B^\rho_{\nu\alpha}$.

In a coordinate-free picture, we have a map

$$\mathcal{F} : L \otimes S^2(L) \mapsto S^2(L) \otimes L : \mathcal{F}(B)_{\mu\nu,\alpha} = B_{\mu,\nu\alpha} + B_{\nu,\mu\alpha} \ .$$

This action is

$$S^2(L) \otimes L \ni \tilde{g}_{\mu\nu,\alpha} \ \mapsto\ \tilde{g}_{\mu\nu,\alpha} + \mathcal{F}(B)_{\mu\nu,\alpha} \ , \quad B \in L \otimes S^2(L) \ .$$

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$$S^2(L) \otimes L \ni \tilde{g} \;\mapsto\; \tilde{g} + \mathcal{F}(B) \;, \quad B \in L \otimes S^2(L)$$

$\mathcal{F}$ is surjective, so $\left(S^2(L) \otimes L\right)/\mathcal{F}(L) = \{0\}$, hence at level $k = 1$ there is no obstruction. Therefore, there exist normal coordinates (“Riemann coordinates”) in which the derivatives of the metric tensor vanish at x_0:

$$\left.\frac{\partial g_{\mu\nu}}{\partial x^\rho}\right|_{x_0} = 0 \;.$$

Case $k = 2$: Fix the 1-jet $j_0^1(g)_{\mu\nu} = \tilde{g}_{\mu\nu} + 0$, i.e., we work in Riemann normal coordinates at x_0, where $\left.\frac{\partial g_{\mu\nu}}{\partial x^\rho}\right|_{x_0} = 0$, and consider the jets $j_0^2(g)$ over this 1-jet:

$$j_0^2(g)_{\mu\nu} = \tilde{g}_{\mu\nu} + \frac{1}{2}\tilde{g}_{\mu\nu,\alpha\beta}\, x^\alpha\, x^\beta \ ,$$

where $\tilde{g}_{\mu\nu,\alpha\beta} := \left.\frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha\, \partial x^\beta}\right|_{x_0}$. Without loss of generality,

$$j^2(\phi)^\rho(x) = x^\rho + 0 + \frac{1}{3!}\, B^\rho_{\alpha\beta\gamma}\, x^\alpha\, x^\beta\, x^\gamma \ ,$$

and the action is

$$\begin{array}{rcl} \tilde{g}_{\mu\nu} & \mapsto & \tilde{g}_{\mu\nu} \\ 0 & \mapsto & 0 \\ \tilde{g}_{\mu\nu,\alpha\beta} & \mapsto & \tilde{g}_{\mu\nu,\alpha\beta} + B_{\mu,\nu\alpha\beta} + B_{\nu,\mu\alpha\beta} \ , \end{array}$$

where $B_{\mu,\nu\alpha\beta} := \tilde{g}_{\mu\rho}\, B^\rho_{\nu\alpha\beta}$.

In a coordinate-free picture, we have a map

$$\mathcal{F} : L \otimes S^3(L) \mapsto S^2(L) \otimes S^2(L) : \mathcal{F}(B)_{\mu\nu,\alpha\beta} = B_{\mu,\nu\alpha\beta} + B_{\nu,\mu\alpha\beta} \ .$$

This action is

$$S^2(L) \otimes S^2(L) \ \ni \ \tilde{g}_{\mu\nu,\alpha\beta} \ \mapsto \ \tilde{g}_{\mu\nu,\alpha\beta} + \mathcal{F}(B)_{\mu\nu,\alpha\beta} \ , \qquad B \in L \otimes S^3(L) \ .$$

The map $\mathcal{F} : L \otimes S^3(L) \mapsto S^2(L) \otimes S^2(L)$ is not surjective: for $\dim(L) = 4$,

$$\dim \left(L \otimes S^3(L) \right) = 80 \ , \qquad \dim \left(S^2(L) \otimes S^2(L) \right) = 100 \ ,$$

and we must find the natural projection

$$\Pi : S^2(L) \otimes S^2(L) \to \left(S^2(L) \otimes S^2(L) \right) / \mathcal{F} \left(L \otimes S^3(L) \right) \ .$$

We will use the Schur-Weyl duality. Let S_n be the symmetric group, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$, $|\boldsymbol{\lambda}| = \lambda_1 + \cdots + \lambda_k = n$ be a partition of n. Graphically, this is a Young diagram. Each Young diagram is associated with an irreducible representation of S_n, denoted by $V(\boldsymbol{\lambda})$ ("Specht module"); $\dim V(\boldsymbol{\lambda}) =: \mathcal{N}(\boldsymbol{\lambda})$.

Let L be a linear space of dimension d. In $L^{\otimes n}$, there is a natural representation of $GL(d)$, which is generally reducible. A standard tableau on the diagram $\boldsymbol{\lambda}$ (with $|\boldsymbol{\lambda}| = n$) is the numbering of the boxes in the diagram with the entries from 1 to n, each ocurring once, and increasing across each row and each column.

With each standard tableau $T(\boldsymbol{\lambda})$ is associated a Young projection operator $P(\boldsymbol{\lambda}) : L^{\otimes n} \to L^{\otimes n}$. The image $P(\boldsymbol{\lambda})(L^{\otimes n}) =: L(\boldsymbol{\lambda})$ is an invariant subspace of $GL(d)$ and realizes an irreducible representation of $GL(d)$.

The representation of $GL(d)$ in $L^{\otimes n}$ is a direct sum of irreducible representations $V(\boldsymbol{\lambda})$ with multiplicities $\mathcal{N}(\boldsymbol{\lambda})$:

$$L^{\otimes n} = \bigoplus_{|\boldsymbol{\lambda}|=n} \mathcal{N}(\boldsymbol{\lambda})\, V(\boldsymbol{\lambda}) \ .$$

In the case $n = 2$, this is simply

$$L \otimes L = L_{(2)} \oplus L_{(1,1)} = S^2(L) \oplus \Lambda^2(L) \ .$$

In the case $n = 4$, $d = 4$,

$$L^{\otimes 4} = L_{(4)} \oplus 3\, L_{(3,1)} \oplus 2\, L_{(2,2)} \oplus 3\, L_{(2,1,1)} \oplus L_{(1,1,1,1)} \ ,$$

$$256 = 35 + 3(45) + 2(20) + 3(15) + 1 \ .$$

The tensor product of two irreps is a direct sum:

$$L_{(\boldsymbol{\lambda})} \otimes L_{(\boldsymbol{\mu})} = \bigoplus_{|\boldsymbol{\sigma}|=|\boldsymbol{\lambda}|+|\boldsymbol{\mu}|} \mathcal{C}_{\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\sigma}}\, L_{(\boldsymbol{\sigma})} \ ,$$

where $\mathcal{C}_{\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\sigma}}$ are the so-called Littlewood-Richardson numbers.
In our case,

$$\begin{aligned} L \otimes S^3(L) &= L_{(1)} \otimes L_{(3)} = L_{(4)} \oplus L_{(3,1)} \\ S^2(L) \otimes S^2(L) &= L_{(2)} \otimes L_{(2)} = L_{(4)} \oplus L_{(3,1)} \oplus L_{(2,2)} \ , \end{aligned}$$

so in these notations the map $\mathcal{F} : L \otimes S^3(L) \mapsto S^2(L) \otimes S^2(L)$ becomes

$$\mathcal{F} : L_{(4)} \oplus L_{(3,1)} \to L_{(4)} \oplus L_{(3,1)} \oplus L_{(2,2)} \ .$$

The map $\mathcal{F}$ is a splitting operator, therefore its kernel and image are invariant. In our case, $\ker \mathcal{F} = \{0\}$, and the image of $\mathcal{F}$ must be

$$L_{(4)} \oplus L_{(3,1)} \ \subset \ L_{(4)} \oplus L_{(3,1)} \oplus L_{(2,2)} \ .$$

Therefore, the Young projector

$$P_{(2,2)}: \ L_{(4)} \oplus L_{(3,1)} \oplus L_{(2,2)} \to L_{(2,2)}$$

is the canonical projection we needed. Therefore, the projector $P_{(2,2)}$ for the Young tableau

1	3
2	4

is the symbol of the Riemann tensor $R(g)$ considered as a differential operator.