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Theoretical models for astrophysical objects in General Relativity

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Outline

Outline

- Introduction

Outline

- Introduction
- Theoretical background

Outline

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- Some examples

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- Some examples
- Recent works

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- Recent works
- Toward new insight

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- Theoretical background
- Some examples
- Recent works
- Toward new insight
- Conclusions

Introduction

Astrophysical objects

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- Evolution processes among the most complex phenomena known in nature (disruption, supernova events etc.)
- End-products: Withe Dwarfs, Neutron Stars, Black Holes

Neutron Stars

<u>Outer crust:</u>

lattice of ionized nuclei,
+ degenerate relativistic e^- gas

<u>Inner crust:</u>

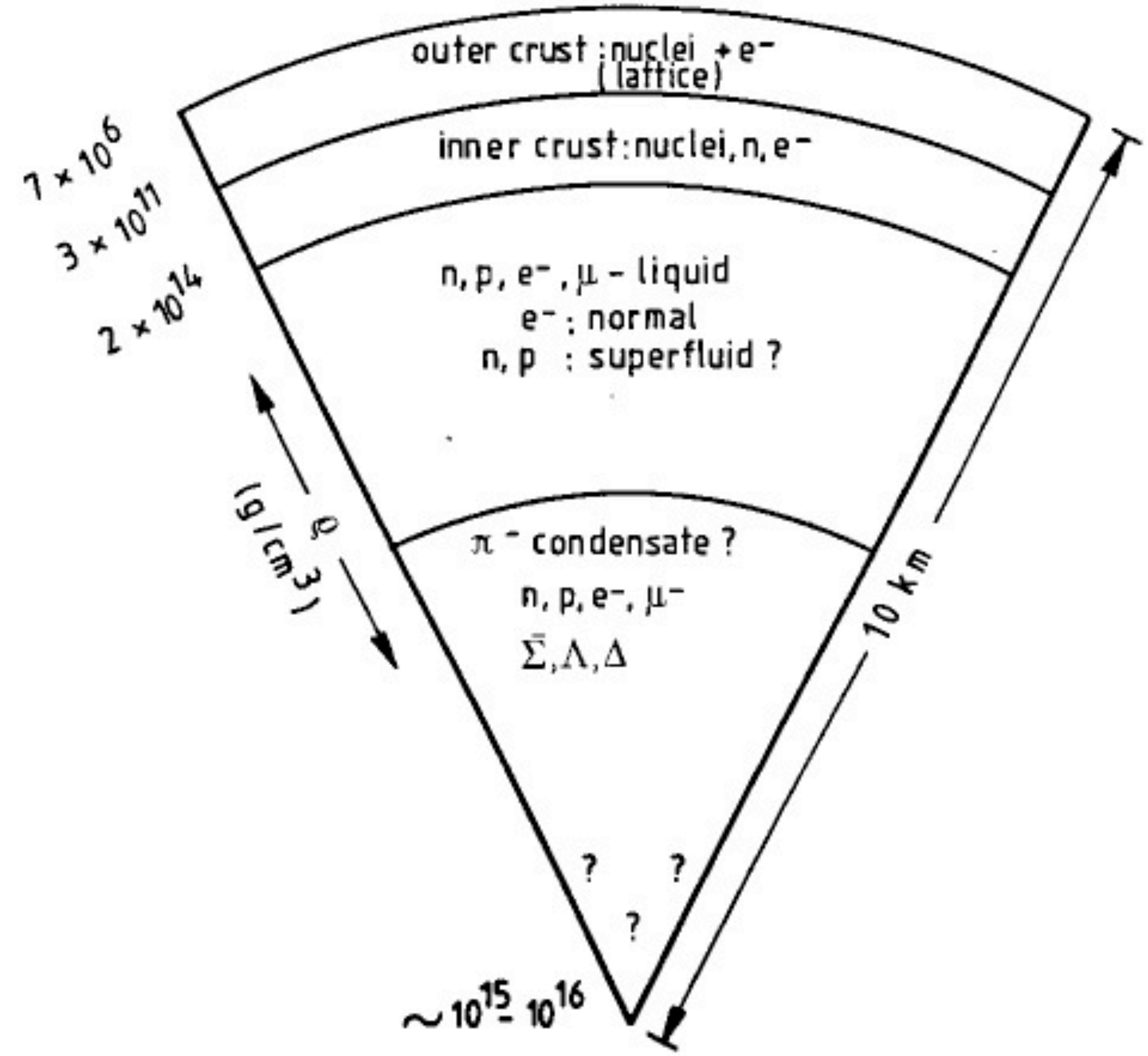
n rich nuclei in β-equilibrium,
+ degenerate relativistic e^- gas
+ degenerate n gas (<u>superfluid</u>)

<u>Outer core:</u>

n (superfluid)
+ p, e^-, μ^-
+ p (superconducting)

<u>Inner core:</u>

π^- condensate
+ quarks



N. Straumann, *General Relativity*, Springer (2004)
Padmanabhan, *Theoretical astrophysics Vol.2*, CUP

Relativity & QM

It is evident that **a description of these objects requires the employment of both Quantum Mechanics and Relativity.**

For instance, the quantum statistic of identical particles, namely the Fermi-Dirac distribution (Dirac, 1926), was applied for the first time just to the description of an astrophysical body, the White Dwarf Sirius B (Fowler, 1926).

This model was non-relativistic as noticed by Chandrasekhar who, trough relativistic kinematic corrections, provided a better description leading to the discovery of a limiting mass (1934).

.... Eddington... Landau... etc.

Theoretical background

General Relativity

In GR the spacetime in presence of a gravitational field is described by the pair (M, g) where M is a four-dimensional manifold and g a Lorentzian metric.

The matter content is described by a suitable stress-energy tensor T, e.g. a perfect fluid: $T^{\mu\nu} = pg^{\mu\nu} + (\rho + p)u^{\mu}u^{\nu}$

The interplay between gravity and matter is ruled by the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Vacuum solutions

Schwarzschild: spherically symmetric static asymptotically flat spacetime

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2\, d\Omega^2$$

Kerr: axisymmetric stationary asymptotically flat spacetime

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right) \mathrm{d}t^2 - \frac{2GMar\sin^2\theta}{\rho^2}(\mathrm{d}t\,\mathrm{d}\phi + \mathrm{d}\phi\,\mathrm{d}t)$$

$$+ \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2 \mathrm{d}\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2 + a^2)^2 - a^2\Delta\sin^2\theta\right]\mathrm{d}\phi^2$$

with $\Delta(r) = r^2 - 2GMr + a^2 \quad \rho^2(r,\theta) = r^2 + a^2\cos^2\theta \quad a = J/M$

where M and J are the Komar mass and angular momentum respectively.

General relativistic stellar structure equations

The metric describing non-rotating, static, spherically symmetric (compact) stars can be described by a metric of the form:

$$ds^2 = -\exp(2\nu)dt^2 + \exp(2\lambda)dr^2 + r^2 d\Omega^2$$

$$\nu = \nu(r), \quad \lambda = \lambda(r)$$

the matter content being described as a perfect fluid parametrized by the stress-energy tensor:

$$T^{\mu\nu} = diag(\rho, p, p, p)$$

TOV equations

Einsten equations + energy conservation imply:

$$\begin{aligned} e^{-2\lambda} &= 1 - \frac{2GM(r)}{r} \\ M(r) &= 4\pi \int_0^r \rho(r') r'^2 dr' \\ \frac{dp}{dr} &= -\frac{G(\rho + p)(M(r) + 4\pi r^3 p)}{r^2(1 - 2GM(r)/r)} \end{aligned}$$

Tolmann-Oppenheimer-Volkoff
equation of hydrostatic equilibrium

Remarks

- To construct a model of star an equation of state $P=P(\rho)$ is required
- We are provided with bounds from GR
- Example - The simplest model: Incompressible Perfect Fluid Schwarzschild Solution (Bondi limit - $R>9m/4$)
- First model of neutron star (Oppenheimer-Volkoff, 1939): ideal mixure of nuclear particles

Remarks

- The strong and weak energy conditions should be obeyed, i.e. the density ρ is always positive and the density is always greater than the pressure P (i.e. $\rho \geq 0$; $\rho \geq p$)
- P and ρ are monotonically decreasing as we move out from the center
- The interior should be matched smoothly to the exterior
- The generalization to the (slowly or fast) rotating case is quite complicate, it is usually approached via numerical techniques

Inside vs outside

We are left with the problem of joining the interior solutions and the exterior solutions discussed above. This problem can also be stated as follows.

A hypersurface Σ (either spacelike or timelike) divides a spacetime in two regions: $\mathcal{M}^{+}, \quad \mathcal{M}^{\square}$.

In each region we have a different coordinate systems and a metrics.

What conditions must be imposed in order for the two regions to be joined smothly on the hypersurface and for the resulting metric to be a solution of Einstein field equations?

Junction Conditions

W. Israel, Nuovo Cimento 44, 1 (1966).
W. Israel, Phys. Rev. D 2, 641 (1970).

Let us introduce the notation to indicate the jump discontinuity in the value of a quantity X as calculated by the two metrics and evaluated at the surface: $[X] = X^{+}|_{\Sigma} - X^{-}|_{\Sigma}$

Then, in order to match two spacetimes the Dormois-Israel matching conditions must be fulfilled:

- continuity if the first fundamental form $$[h_{ab}] = 0$$
- continuity if the second fundamental form $$S_{ab} = -\frac{\epsilon}{8\pi}([K_{ab}] - [K]h_{ab})$$

S_{ab} being the stress-energy three tensor of the hypersurface.

Some examples

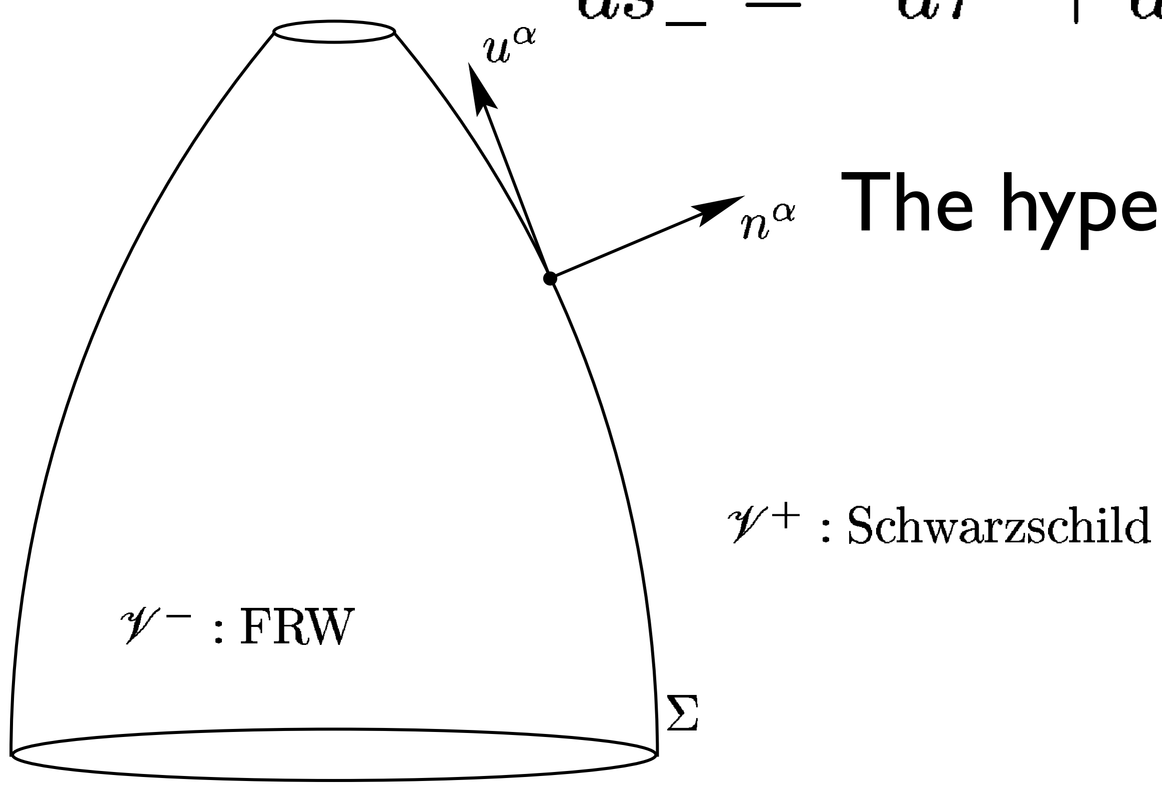
Example I

Spherical dust distribution collapse

(Oppenheimer-Snyder, 1939)

A simplified model of collapse to a black hole. The star is modeled as a spherical ball of pressureless matter with uniform density. The metric inside the dust is FRW while the metric outside the matter distribution is Schwarzschild.

$$ds_+^2 = -f\,dt^2 + f^{-1}\,dr^2 + r^2\,d\Omega^2, \qquad f = 1 - 2M/r$$
$$ds_-^2 = -d\tau^2 + a^2(\tau)\big(d\chi^2 + \sin^2\chi\,d\Omega^2\big)$$



The hypersurface is parametrized by t=T(τ), r=R(τ)

$$M = \frac{4\pi}{3}\,\rho R^3$$

Example II

Slowly rotating thin shell

Consider slowly rotation sphericaly shaped sphere. Assume exterior metric to be the slow rotation limit of Kerr solution while the metric inside the shell to be Minkowski. Perform the junction on a hypersurface of fixed radius R.

$$ds^2_+ = -f\,dt^2 + f^{-1}\,dr^2 + r^2\,d\Omega^2 - \frac{4Ma}{r}\sin^2\theta\,dtd\phi.$$
$$ds^2_- = -(1-2M/R)\,dt^2 + d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\,d\psi^2)$$

The discontinuity in the second fundamental form can be interpreted as the stress-energy due to a perfect fluid: $S^{ab} = \sigma u^a u^b + p(h^{ab} + u^a u^b)$

The density and pressure are found to be:

$$\sigma = \frac{1}{4\pi R}\left(1 - \sqrt{1-2M/R}\right) \qquad p = \frac{1 - M/R - \sqrt{1-2M/R}}{8\pi R\sqrt{1-2M/R}}.$$

in the milit R>>2M one gets:

$\omega \sim 3a/(2R^2)$ $\quad$ $p \sim M^2/(16\pi R^3)$ $\quad$ $\sigma \sim M/(4\pi R^2)$

Recent advances

Conformal degrees of freedom

F. Canfora, A. Giacomini, S. Willison, arxiv: gr-qc/0710.3193v2

An inner core undergoing a phase transition characterized by conformal degrees of freedom on the phase boundary (e.g. quantum Hall effect, superconductivity, superfluidity), is considered.

By solving the ID junction conditions for the conformal matter on a spherical hypersurface, one can determine a range for the parameters in which a stable equilibrium configuration for the phase boundary is found (e.g. a physically reasonable model for a neutron star).

What about rotation?

The problem of finding possible Kerr sources is that, in order to obtain a physically sensible mass distribution, many restrictions must be imposed.

The metric must be joined smoothly to the Kerr one on a reasonable surface for a rotating body and the hydrostatics pressure
must be zero on such a surface.

The energy conditions must hold.

The star must be a non-radiating source and in the static limit a reasonable Schwarzschild interior metric must be obtained.

Interior Kerr solutions

S. P. Drake and R. Turolla, Class. Quantum Grav. 14, 1883 (1997)
S. P. Drake and Szekeres, Gen. Rel. Grav. 32, 445 (2000)
S. Viaggiu, arxiv.org:gr-qc/0603036

A possible approach to obtain interior solutions of the Kerr metric is to apply the **Newman-Janis Algorithm*** to a static physically reasonable seed Space-Time (going to SAS metrics from SSS ones).

$$ds^2 = e^{2\mu(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - e^{2\nu(r)}dt^2$$

This metric can be written in the terms of null tetrad vectors as:

$$g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu \bar{m}^\nu - m^\nu \bar{m}^\mu$$

with

$$l^\mu = \delta_1^\mu$$

$$n^\mu = -\frac{1}{2}e^{-2\lambda(r)}\delta_1^\mu + e^{-\lambda(r)-\phi(r)}\delta_0^\mu$$

$$m^\mu = \frac{1}{\sqrt{2}\bar{r}}\left(\delta_2^\mu + \frac{i}{\sin\theta}\delta_3^\mu\right).$$

NJ-Algorithm

* E.T. Newman and A. Janis, J. Math. Phys. 6, 915 (1965)

Perform the complex transformation:

$$u' = u - \imath a \cos\theta \;,\; r' = r + \imath a \cos\theta$$

The new null tetrad vectors basis is:

$$l^\mu = \delta_1^\mu$$

$$n^\mu = -\frac{1}{2} e^{-2\lambda(r,\theta)} \delta_1^\mu + e^{-\lambda(r,\theta) - \phi(r,\theta)} \delta_0^\mu$$

$$m^\mu = \frac{1}{\sqrt{2}(r + ia\cos\theta)} \left(ia \sin\theta (\delta_0^\mu - \delta_1^\mu) + \delta_2^\mu + \frac{i}{\sin\theta} \delta_3^\mu \right)$$

The new metric

$$g_{tt} = -e^{2\nu(r,\theta)} \ , \ g_{rr} = \frac{\Sigma}{\Sigma e^{-2\mu(r,\theta)} + a^2 \sin^2\theta} \ , \ g_{\theta\theta} = \Sigma \ ,$$

$$g_{\phi\phi} = \sin^2\theta[\Sigma + a^2 \sin^2\theta e^{\nu(r,\theta)}(2e^{\mu(r,\theta)} - e^{\nu(r,\theta)})] \ ,$$

$$g_{t\phi} = a e^{\nu(r,\theta)} \sin^2\theta (e^{\mu(r,\theta)} - e^{\nu(r,\theta)}),$$

with $\Sigma = r^2 + a^2 \cos^2\theta$

This metric reduces the Kerr (-Neuman) solution for a suitable choice of the functions

Junction hypersurface

The separating surface is both static and axially symmetric with vanishing surface stress-energy (no thin shells).

The hypersurface is left unspecified leading to a complete set of boundary conditions for the joining of any two stationary axially symmetric metrics generated by the NJA when applied to any SSS seed metric.

Then, consider "physically reasonable" source for the interior spacetime and simplify the approach restricting to surfaces described by:
$\partial R(\theta)/\partial \theta = 0$.

It turns out that $R(\theta) = R =$ constant, is a sensible choice of boundary surface. The surface defined by this condition is and **oblate spheroid**.

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The task is to enrich the models adding more physically motivated features:

1. Conformal degrees of freedom
2. Dynamical behaviour of junction hyperurfaces
3. Rotation (exact, approximated...)

Points 1 & 2

- Consider (if possible) oblate spheroid or determine the geometry of other suitable hypersurfaces
- Consider the Israel - Darmois matching conditions
- Extend to the case of a dymamical boudary

Point 3

- If the employment of the external Kerr exact solutions will prove to be too restrictive we will consider approximated solutions (slowly rotating stars as perturbed Schwarzschild)
- We also consider to non stationary rotation (conformal Killing vectors, warped geometries, *bi-conformal vector fields...*)

Conclusions?

At the moment we are considering both to generate new solutions and, more in general, to get a better understanding of the NJA itself.

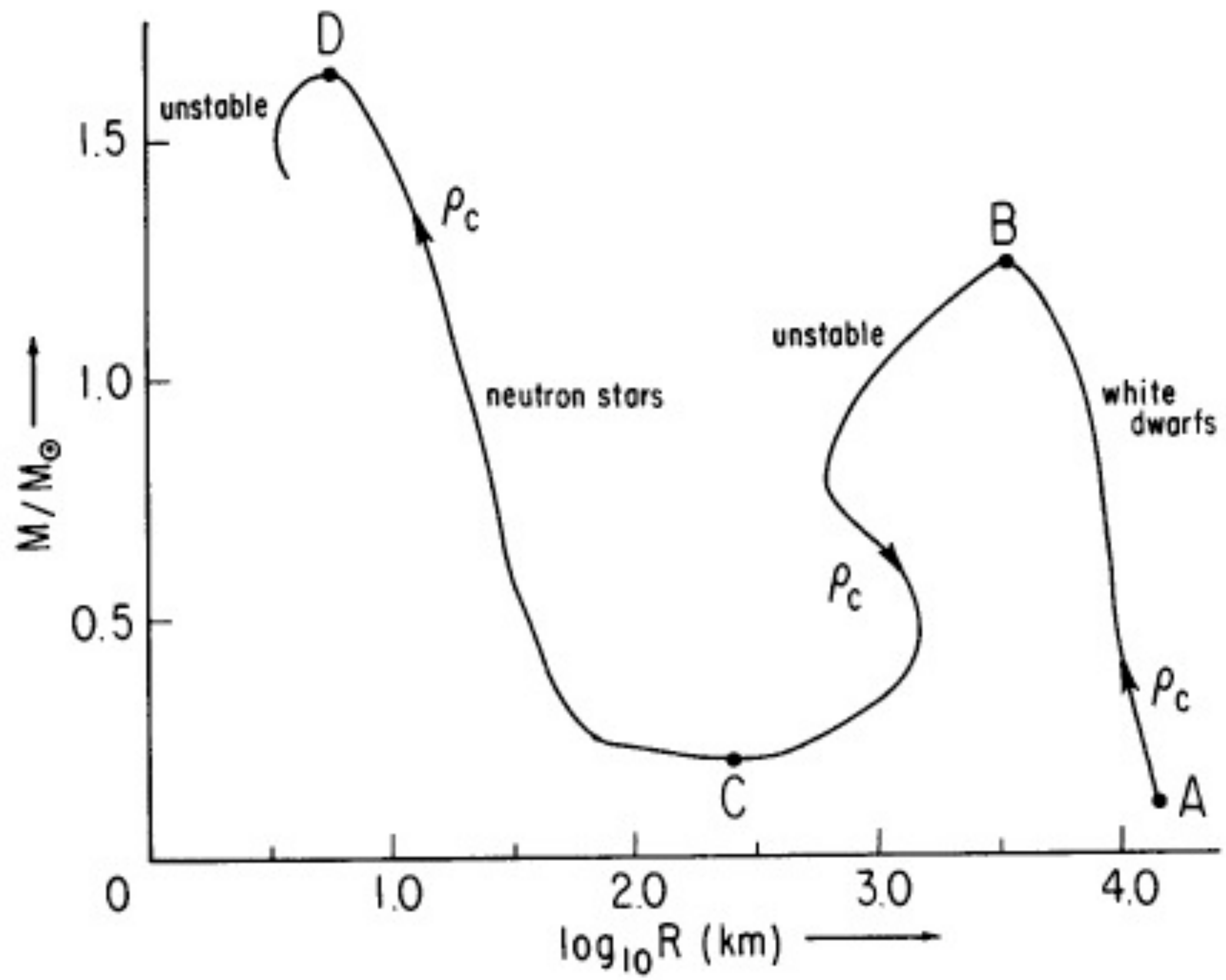
A possibility is to modified the NJA to get wider classes of metrics (i.e. containing also the K.-N.-d.S.)

Open problems

- Rigidity of exact external solutions
- Difficulty with internal solutions (i.e. Geroch conjecture: *Kerr metric might have no sources other than a black hole)*
- Suitable description of matter (via a stress-energy tensor) and the joining hypersurfaces
- .....

Thank you!

Stars in GR



R.M. Wald, *General Relativity*, University of Chicago (1984)

Example III

Dynamics of false-vacuum bubbles

(S. K. Blau, E. I. Guendelman, A. H. Guth, Phys. Rev. **D**35, 1987)

Dynamics of false-vacuum bubbles

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(Received 3 November 1986)

The possibility of localized inflation is investigated by calculating the dynamics of a spherically symmetric region of false vacuum which is separated by a domain wall from an infinite region of true vacuum. For a range of initial conditions, the false-vacuum region will undergo inflation. An observer in the exterior true-vacuum region will describe the system as a black hole, while an observer in the interior will describe a closed universe which completely disconnects from the original spacetime. We suggest that this mechanism is likely to lead to an instability of Minkowski space: a region of space might undergo a quantum fluctuation into the false-vacuum state, evolving into an isolated closed universe; the black hole which remains in the original space would disappear by quantum evaporation. The formation of these isolated closed universes may also be relevant to the question of information loss in black-hole formation.