Natural principal connections on the principal gauge prolongation of a principal bundle

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Contents

Let Γ be a principal connection on a principal bundle $\pi : P \to M$ and let Λ be a linear connection on M. We describe all possible natural prolongations of Γ, with respect to Λ, to principal connections on the principal gauge prolongation $W^r P$ of P. For $r = 1, 2$ we give the full classification, for $r \geq 3$ we give a base of natural operators which generates all possible natural connections on $W^r P$.

1. Gauge natural-bundles and natural operators

The theory of gauge-natural bundles started in 1981 by the paper by D. Eck as a generalization of natural bundles by A. Nijenhuis (1972) and it is a geometrical background of physical gauge-invariant field theories (see, for instance Fatibene and Francaviglia).

- D. J. Eck: *Gauge-natural bundles and generalized gauge theories*, Mem. Amer. Math. Soc. **33** No. 247 (1981).
- P. W. Michor: *Gauge theory for fiber bundles,* Napoli, Bibliopolis 1988.
- I. Kolář, P. W. Michor, J. Slovák: *Natural Operations in Differential Geometry,* Springer–Verlag 1993.
- L. Fatibene, M. Francaviglia: *Natural and Gauge Natural Formalism for Classical Field Theories*, Kluwer Academic Publishers, Dordrecht/Boston/London 2003.

1. Gauge natural-bundles and natural operators

Eck, 1981: A **gauge-natural bundle functor** (g.-n.b.f.) over m-dimensional manifolds is a covariant functor $F : \mathcal{PB}_m(G) \to \mathcal{FM}$ transforming

$$\begin{array}{ccc} P & \xrightarrow{f} & \bar{P} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\bar{\pi}} \\ M & \xrightarrow{\underline{f}} & \bar{M} \end{array} \quad \text{in } \mathcal{PB}_m(G) \text{ to} \quad \begin{array}{ccc} FP & \xrightarrow{Ff} & F\bar{P} \\ {\scriptstyle\pi_P}\downarrow & & \downarrow{\scriptstyle\pi_{\bar{P}}} \\ M & \xrightarrow{\underline{f}} & \bar{M} \end{array}$$

in the category $\mathcal{FM}$. Moreover, we have the **locality** condition and the **regularity** condition which allows to transform right G-invariant vector fileds Ξ on P into v.f. $F(\Xi)$ on FP over the same v.f. on M. ♣

A **gauge-natural bundle** (g.-n.b.) is a fibered manifold $\pi_P : FP \to M$.

1. Gauge natural-bundles and natural operators

In the theory of gauge-natural bundles the key role is played by the g.-n.b.f. W^r, which transforms a principal bundle (p.b.) $P = (P, M, \pi; G)$ into the p.b. $W^r P = P^r M \times_M J^r P \equiv$ $\equiv (W^r P, M, p; W^r_m G)$, where $P^r M$ is the r-th order **frame bundle** and $W^r_m G$ is the semidirect product $W^r_m G = G^r_m \rtimes T^r_m G$, and each $(\varphi, \underline{\varphi}) \in \mathrm{Mor}(\mathcal{PB}_m(G))$ into $W^r \varphi = (P^r \underline{\varphi}, J^r \varphi)$. The p.b. $W^r P$ is called the **principal r-th order gauge prolongation** of P. Let us note that $W^r P = \{j^r_{(0,e)} \varphi | \varphi : \mathbb{R}^m \times G \to P \in \mathrm{Mor}(\mathcal{PB}_m(G))\}$.

Theorem: (Eck, 1981) Any g.-n.b.f. is of the form

$$FP = [W^r P, S_0], \quad Ff = [W^r f, \mathrm{id}].$$

r is the **order** of g.-n.b.f. F and S_0 is the **standard fibre** of F. ♣

Example: The **adjoint bundle** $\mathrm{Ad}(P) \to M$ is the vector g.-n.b. of order 0 with the standard fibre $\mathfrak{g}$. If $VM \to M$ is a tensor bundle then the tensor product $\mathrm{Ad}(P) \otimes_M VM$ is the vector g.-n.b. of order $(1, 0)$. ♣

Let F be a g.-n.b.f., $(\varphi : P \to \bar{P}) \in \mathrm{Mor}(\mathcal{PB}_m(G))$ over $\underline{\varphi} : M \to \bar{M}$. Let $\sigma \in C^\infty(FP)$, then $\varphi^*\sigma \in C^\infty(F\bar{P})$ given by $\varphi^*\sigma = F\varphi \circ \sigma \circ \underline{\varphi}^{-1}$.

Eck, 1981: A **natural differential operator** (n.d.o.) D from a g.-n.b.f. F_1 to a g.-n.b.f. F_2 is a family of differential operators

$$\{D(P) : C^\infty(F_1 P) \to C^\infty(F_2 P)\}_{P \in \mathrm{Ob}(\mathcal{P}B_m(G))}$$

such that

i) (**naturality**) $D(\bar{P})(\varphi^*\sigma) = \varphi^* D(P)(\sigma)$ for every section $\sigma \in C^\infty(F_1 P)$ and every $\varphi : P \to \bar{P}$ in $\mathrm{Mor}(\mathcal{P}B_m(G))$,

ii) (**locality**) $D_{\pi^{-1}(U)}(\sigma|U) = (D_P\sigma)|U$ for every section $\sigma \in C^\infty(F_1 P)$ and every open submanifold $U \subset M$,

iii) (**regularity**) every smoothly parameterized family of sections of $F_1 P$ is transformed into a smoothly parametrized family of sections of $F_2 P$. ♣

If a n.d.o. $D : F_1 P \to F_2 P$ is of order k (for any $P \in \mathrm{Ob}\mathcal{PB}_m(G)$) then we have the one-to-one correspondence with a **natural transformation** $\mathcal{D} : J^k F_1 \to F_2$. We have

Theorem: (Eck, 1981) Let F_1 and F_2 be g.-n.b.f. of order $\leq r$. Then we have a one-to-one correspondence between n.d.o. of order k from F_1 to F_2 and $W_m^{r+k} G$-equivariant mappings from $(J^k F_1)_0$ to $(F_2)_0$. ♣

We consider a principal bundle $P = (P, M, \pi; G)$ with a structure group G. We denote by (x^λ, z^a) fibered coordinates on P, $\lambda = 1, \dots, \dim M$, $a = 1, \dots, \dim G$.
A **principal connection** on P is defined as a lifting linear mapping $\Gamma\colon TM \to TP/G$. In coordinates

$$\Gamma = d^\lambda \otimes \left(\partial_\lambda + \Gamma^a{}_\lambda(x)\widetilde{\mathfrak{b}}_a\right),$$

where $\Gamma^a{}_\lambda(x)$ are functions on M and $(\widetilde{\mathfrak{b}}_a)$ is the base of vertical right invariant vector fields on P which are induced by the base $(\mathfrak{b}_a)$ of $\mathfrak{g}$.
If we identify Γ with the functions $\Gamma^a{}_\lambda(x)$, then Γ can be considered as a section of the bundle $QP \to M$ of principal connections on P.
Moreover, QP is a 1-order G-gauge-natural affine bundle associated with the vector bundle $\mathrm{Ad}(P) \otimes T^*M \to M$.

We denote by $R[\Gamma]$ the **curvature tensor field** of Γ considered as the 1st order natural operator $R[\Gamma]\colon J^1QP \to \mathrm{Ad}(P) \otimes \bigwedge^2 T^*M$.
Now, let Λ be a linear connection on M. Then we can define the **covariant derivative** of the curvature tensor $R[\Gamma]$ of Γ with respect to the pair (Λ, Γ), as a natural operator

$$\nabla R[\Gamma]\colon QP^1M \times_M J^2QP \to \mathrm{Ad}(P) \otimes \bigwedge^2 T^*M \otimes T^*M\,.$$

Then, by iteration, we can define the r-th order covariant derivative and obtain the natural operator

$$\nabla^r R[\Gamma]\colon J^{r-1}QP^1M \times_M J^{r+1}QP \to \mathrm{Ad}(P) \otimes \bigwedge^2 T^*M \otimes \otimes^r T^*M\,.$$

Let Γ_r be a principal connection on $W^r P$ given in coordinates by

$$\Gamma_r = d^\lambda \otimes \Big(\partial_\lambda + \sum_{i=1}^{r} \Lambda^\nu_{\mu_1\dots\mu_i\lambda}(x)\, \widetilde{\mathfrak{b}}_\nu^{\mu_1\dots\mu_i} + \sum_{j=0}^{r} \Gamma^a{}_{\kappa_1\dots\kappa_j\lambda}(x)\, \widetilde{\mathfrak{b}}_a^{\kappa_1\dots\kappa_j} \Big) \,.$$

We have the projections

$$\pi^r_{r-1} \colon QW^r P \to QW^{r-1} P \,, \qquad p_1 \colon QW^r P \to QP^r M \,,$$

so any principal connection Γ_r on $W^r P$ projects on a principal connection Λ_r on $P^r M$ and on a principal connection Γ_{r-1} on $W^{r-1} P$.

Theorem: Let Γ_r and $\bar{\Gamma}_r$ be two principal connections on $W^r P$ such that they are over the same principal connections Λ_r on $P^r M$ and Γ_{r-1} on $W^{r-1}P$, then the difference $\Gamma_r - \bar{\Gamma}_r$ is identified with a section

$$\Psi_r = \Gamma_r - \bar{\Gamma}_r \colon M \to \mathrm{Ad}(P) \otimes S^r T^* M \otimes T^* M \,. \qquad \clubsuit$$

This Theorem is a consequence of

Lemma: The intersection of kernels of the projections $\pi^r_{r-1} \colon \mathfrak{w}^r_m \mathfrak{g} \to \mathfrak{w}^{r-1}_m \mathfrak{g}$ and $p_1 \colon \mathfrak{w}^r_m \mathfrak{g} \to \mathfrak{g}^r_m$ is $\mathfrak{g} \otimes S^r \mathbb{R}^m$ with the action of the group $G^1_m \times G$ given as the tensor product of the adjoint action of G on $\mathfrak{g}$ and the tensor action of G^1_m on $S^r \mathbb{R}^m$. $\clubsuit$

Let ξ be a vector field on M, Γ be a principal connection on P and Λ be a principal connection on P^1M such that the horizontal lift $h^{\Lambda}(\xi) = \mathcal{P}^1(\xi)$. Let $h^{\Gamma}(\xi)$ denote the horizontal lift of ξ with respect to Γ. Let us denote by $Fl_t(h^{\Gamma}(\xi))$ the flow of $h^{\Gamma}(\xi)$. Then the expression

$$W^r(Fl_t(h^{\Gamma}(\xi))) = (P^r(Fl_t(\xi)), J^r(Fl_t(h^{\Gamma}(\xi))) = Fl_t(h^{\mathcal{W}^r\Gamma}(\xi))$$

gives a principal connection $\mathcal{W}^r\Gamma$ on W^rP which depends on Γ in order r and on Λ in order $(r-1)$. So $\mathcal{W}^r\Gamma$ is a natural operator

$$\mathcal{W}^r\Gamma\colon J^{r-1}QP^1M \times_M J^rQP \to QW^rP$$

called the **flow prolongation** of Γ with respect to Λ and we will denote it by $\mathcal{W}^r\Gamma(\Lambda, \Gamma)$.

Remark: Let us remark that the flow prolongation $\mathcal{W}^r\Gamma(\Lambda, \Gamma)$ projects on the natural principal connection on P^rM which depends on Λ only. We denote it by $\mathcal{P}^r\Lambda(\Lambda)$ and we call it the **flow prolongation** of Λ to P^rM. ♣

In the second order we have the coefficients of $\mathcal{W}^2\Gamma(\Lambda, \Gamma)$

$$\Lambda^\nu_{\mu_1\mu_2\lambda} = \frac{1}{2}(\partial_{\mu_1}\Lambda^\nu_{\mu_2\lambda} + \partial_{\mu_2}\Lambda^\nu_{\mu_1\lambda} + \Lambda^\nu_{\mu_1\alpha}\Lambda^\alpha_{\mu_2\lambda} + \Lambda^\nu_{\mu_2\alpha}\Lambda^\alpha_{\mu_1\lambda})\,, \tag{1}$$

$$\Gamma^a{}_{\mu\lambda} = \partial_\mu\Gamma^a{}_\lambda + \Gamma^a{}_\varrho\,\Lambda^\varrho_{\mu\lambda}\,, \tag{2}$$

$$\begin{aligned}\Gamma^a{}_{\mu_1\mu_2\lambda} = {} & \partial_{\mu_1\mu_2}\Gamma^a{}_\lambda + \partial_{\mu_1}\Gamma^a{}_\varrho\,\Lambda^\varrho_{\mu_2\lambda} + \partial_{\mu_2}\Gamma^a{}_\varrho\,\Lambda^\varrho_{\mu_1\lambda} + \\ & + \frac{1}{2}\Gamma^a{}_\varrho\,(\partial_{\mu_1}\Lambda^\varrho_{\mu_2\lambda} + \partial_{\mu_2}\Lambda^\varrho_{\mu_1\lambda} + \Lambda^\varrho_{\mu_1\sigma}\Lambda^\sigma_{\mu_2\lambda} + \Lambda^\varrho_{\mu_2\sigma}\Lambda^\sigma_{\mu_1\lambda})\,.\end{aligned} \tag{3}$$

In what follows we will need the gauge version **exponential reduction** given by

I. Kolář: *On the gauge version of exponential map*, preprint 2009.

Consider a torsion free principal connection Λ on P^1M and a principal connection Γ on P. Then we have a local map

$$\exp^{\Lambda,\Gamma}_{(u,p)} : \mathbb{R}^m \times G \to P$$

$u \in P^1_x M$, $p \in P_x$. This map is $(G^1_m \times G)$-invariant and it is the inverse of the (Λ, Γ) **adapted trivialization** $P \to \mathbb{R}^m \times G$ by

M. Doupovec, W. M. Mikulski: *Reduction theorems for principal and classical connections*, to appear in Acta Mathematica Sinica.

4. Exponential reduction

The rule

$$E_r(\Lambda,\Gamma)(u,p) = j^{r+1}_{(0,e)} \exp^{\Lambda,\Gamma}_{(u,p)} \in W^{r+1}P$$

defines the **exponential reduction**

$$E_r(\Lambda,\Gamma)\colon P^1M \times_M P \to W^{r+1}P$$

corresponding to the canonical injection

$$i = (i^{r+1}_m \times j^{r+1}_m)\colon G^1_m \times G \to W^{r+1}_m G\,,$$

where the injection $j^{r+1}_m\colon G \to T^{r+1}_m G$ is given by $g \mapsto j^{r+1}_0 \hat{g}$ and $\hat{g}$ is the constant mapping on $g \in G$. Moreover, this reduction corresponds to a (torsion free) natural principal connection $E_r(\Lambda,\Gamma)$ on W^rP called the **exponential prolongation** of (Λ,Γ).

With respect to the above exponential reduction we have the $W^r_m G$-natural isomorphism

$$\Phi^{\Lambda,\Gamma} \colon \oplus_{i=1}^r (TM \otimes S^i T^* M) \oplus \oplus_{j=0}^r (\mathrm{Ad}(P) \otimes S^j T^* M) \to \mathrm{Ad}(W^r P)\,.$$

Remark: The exponential prolongation is defined for torsion-free connections Λ, but a non-symmetric connection Λ can be decomposed in a unique way as the sum of the classical symmetric connection $\widetilde{\Lambda}$ (obtained by symmetrization of Λ) and the torsion tensor T of Λ, i.e. $\Lambda = \widetilde{\Lambda} + T$. Then $E_r(\widetilde{\Lambda}, \Gamma)$ is a principal connection on $W^r P$ naturally given by the pair (Λ, Γ). ♣

Main theorem: Let Γ be a principal connection on P and Λ be a classical connection on M. Then any natural principal connection Γ_r on $W^r P$ given by Λ and Γ is of the form

$$\begin{aligned} \Gamma_r &= \mathcal{W}^r\Gamma + \Sigma_r = \\ &= \mathcal{W}^r\Gamma + (\Phi^{\widetilde{\Lambda},\Gamma} \otimes \mathrm{id}_{T^*M})(\Phi_1, \dots, \Phi_r, \Psi_0, \dots, \Psi_r)\,, \end{aligned}$$

where $\Phi_k \colon M \to TM \otimes S^k T^*M \otimes T^*M$, $k = 1, \dots, r$, and $\Psi_l \colon M \to \mathrm{Ad}(P) \otimes S^l T^*M \otimes T^*M$, $l = 0, 1, \dots, r$, are natural tensor fields given by the pair (Λ, Γ). ♣

Hence $\Gamma_r \approx (\Phi_1, \dots, \Phi_r, \Psi_0, \dots, \Psi_r)$ and $\Lambda_r \approx (\Phi_1, \dots, \Phi_r)$.

To classify natural tensor fields we can use the higher order Utiyama's reduction method by

J. Janyška: *Higher order Utiyama invariant interaction*, Rep. Math. Phys. **59** (2007) 63–81.

and we get

Theorem: 1. Any natural tensor field $\Phi_k(\Lambda, \Gamma)$ has the maximal order $(k-1)$ and is of the form

$$\Phi_k(j^{k-1}\Lambda, j^{k-1}\Gamma) = \bar{\Phi}_k(c, \widetilde{\nabla}^{(k-2)} R[\widetilde{\Lambda}], \widetilde{\nabla}^{(k-2)} R[\Gamma], \widetilde{\nabla}^{(k-1)} T)\,,$$

where $\widetilde{\nabla}^{(k-2)} R[\Gamma]$ are covariant derivatives of the curvature tensor of Γ with respect to the pair $(\widetilde{\Lambda}, \Gamma)$ and $c = (c^a_{bd})$ are the structure constants of G. $\bar{\Phi}_k$ is a zero order operator.

2. Any natural tensor field $\Psi_l \colon M \to \mathrm{Ad}(P) \otimes S^l T^* M \otimes T^* M$ has the maximal order l and is of the form

$$\Psi_l(j^l\Lambda, j^l\Gamma) = \bar{\Psi}_l(c, \widetilde{\nabla}^{(l-1)} R[\widetilde{\Lambda}], \widetilde{\nabla}^{(l-1)} R[K], \widetilde{\nabla}^l T)\,,$$

where $\bar{\Psi}_l$ is a zero order operator.

Lemma:

1. All natural tensor fields $\Phi_1(\Lambda, \Gamma)$ form a 3-parameter family
$$\Phi_1(\Lambda) = a_1\, T + a_2\, \mathrm{id}_{TM} \otimes \widehat{T} + a_3\, \widehat{T} \otimes \mathrm{id}_{TM}\,, \qquad a_i \in \mathbb{R}\,,$$
where $\widehat{T}$ denote the contraction.
2. All natural tensor fields $\Phi_2(\Lambda, \Gamma)$ form a 17-parameter family constructed by tensorial operations from c, $R[\widetilde{\Lambda}]$, $R[\Gamma]$, T and $\widetilde{\nabla} T$. ♣

Collorary:

1. All natural principal connections $\Lambda_1(\Lambda, \Gamma)$ on $P^1 M$ form a 3-parameter family $\Lambda_1(\Lambda, \Gamma) = \Lambda + \Phi_1(\Lambda)$.
2. All natural principal connections $\Lambda_2(\Lambda, \Gamma)$ on $P^2 M$ form a 20-parameter family. ♣

Collorary: All natural connections on $P^2 M$ given by symmetric connection Λ on M and by a principal connection Γ on P form a 5-parameter family. ♣

Remark: Let $A \in \mathfrak{g}$ be an Ad-invariant element, i.e. $\mathrm{Ad}_g(A) = A$ for all $g \in G$. Then A determines the invariant section $\widetilde{A}\colon M \to \mathrm{Ad}(P)$ which is an "absolute" natural tensor field (independent of Λ and Γ). Then $\widetilde{A} \otimes \omega$, where ω is a natural $(0, r)$-tensor field on M given by Λ and Γ, is a natural tensor field $M \to \mathrm{Ad}(P) \otimes \otimes^r T^*M$. ♣

Remark: Let $\varphi\colon \mathfrak{g} \to \mathfrak{g}$ be an Ad-invariant linear map, i.e. $\varphi(\mathrm{Ad}_g(X)) = \mathrm{Ad}_g(\varphi(X))$ for all $g \in G$ and $X \in \mathfrak{g}$. Then φ determines the invariant homomorphism $\widetilde{\varphi}\colon \mathrm{Ad}(P) \to \mathrm{Ad}(P)$ which is "absolute" natural, i.e. independent of Λ and Γ. Then the section $\widetilde{\varphi} \otimes \mathrm{id}_{\otimes^2 T^*M} \circ R[\Gamma]\colon M \to \mathrm{Ad}(P) \otimes \bigwedge^2 T^*M$ is a natural tensor field given by Γ. In KMS this operator is called **modified curvature operator**. ♣

Lemma:

1. All natural tensor fields $\Psi_0 \colon M \to \mathrm{Ad}(P) \otimes T^*M$ are of the form $\widetilde{A} \otimes \widehat{T}$, where $A \in \mathfrak{g}$ is an Ad-invariant element.
2. All natural tensor fields $\Psi_1 \colon M \to \mathrm{Ad}(P) \otimes T^*M \otimes T^*M$ are of the form

$$\Psi_1 = \widetilde{\varphi} \otimes \mathrm{id}_{\otimes^2 T^*M} \circ R[\Gamma] + \sum_{i=1}^{8} \widetilde{B}_i \otimes \omega_i \,,$$

   where $\varphi \colon \mathfrak{g} \to \mathfrak{g}$ is an Ad-invariant linear mapping, $B_i \in \mathfrak{g}$ are Ad-invariant elements and ω_i are natural $(0,2)$-tensor fields given by Λ.
3. All natural tensor fields $\Psi_2 \colon M \to \mathrm{Ad}(P) \otimes S^2 T^*M \otimes T^*M$ are of the maximal order two and depend on Ad-invariant linear mappings $\psi_i \colon \mathfrak{g} \to \mathfrak{g}$, $i = 1, 2, 3$, and Ad-invariant elements $C_k \in \mathfrak{g}$, $k = 1, \ldots, 28$. ♣

Collorary:

1. Natural principal connections Γ_0 on P are of zero order and are of the form
$$\Gamma_0 = \Gamma + \widetilde{A} \otimes \widehat{T}\,,$$
where $A \in \mathfrak{g}$ is an Ad-invariant element.
2. Natural principal connections Γ_1 on W^1P form a family of connections depending on 3 real parameters, an Ad-invariant linear mapping $\varphi\colon \mathfrak{g} \to \mathfrak{g}$ and 9 Ad-invariant elements A, $B_j \in \mathfrak{g}$, $j = 1, \dots, 8$.
3. Natural principal connections Γ_2 on W^2P form a family of connections depending on 20 real parameters, 4 Ad-invariant linear mappings $\varphi, \psi_i\colon \mathfrak{g} \to \mathfrak{g}$, $i = 1, 2, 3$, and 37 Ad-invariant elements A, $B_j, C_k \in \mathfrak{g}$, $j = 1, \dots, 8, k = 1, \dots, 28$. ♣

Let us note that to give the coordinate expression of $\Gamma_r(\Lambda, \Gamma)$ for $r \geq 1$ is not a trivial problem, because we have not the coordinate expression of the exponential identification $\Phi^{\widetilde{\Lambda},\Gamma} \otimes \mathrm{id}_{T^*M}$. But we can give it in the case of $r = 1$. If we consider Λ_1 and Γ_0, then $\mathcal{W}^1\Gamma_0(\Lambda_1, \Gamma_0)$ is a principal natural connection on $W^1 P$ over Λ_1 and Γ_0. Then any other natural connection $\Gamma_1(\Lambda, \Gamma)$ on $W^1 P$ over Λ_1 and Γ_0 is of the form

$$\Gamma_1(\Lambda, \Gamma) = \mathcal{W}^1\Gamma_0(\Lambda_1, \Gamma_0) + \Psi_1 \,.$$

On the other hand, by Main theorem,

$$\Gamma_1(\Lambda, \Gamma) = \mathcal{W}^1\Gamma(\Lambda, \Gamma) + (\Phi^{\widetilde{\Lambda},\Gamma} \otimes \mathrm{id}_{T^*M})(\Phi_1, \Psi_0, \Psi_1)$$

and if we compare these two expressions in coordinates we get

$$\begin{aligned}(\Phi^{\widetilde{\Lambda},\Gamma} \otimes \mathrm{id}_{T^*M})(\Phi_1, \Psi_0, \Psi_1) = \big(& (\Phi_1)^\lambda_{\mu\nu} \,,\, (\Psi_0)^a{}_\lambda \,, \\ & \partial_\mu (\Psi_0)^a{}_\lambda + (\Psi_0)^a{}_\rho \, \Lambda^\rho_{\mu\lambda} + \Gamma^a{}_\rho \, (\Phi_1)^\rho_{\mu\lambda} + (\Psi_0)^a{}_\rho \, (\Phi_1)^\rho_{\mu\lambda} + (\Psi_1)^a{}_{\mu\lambda} \big) \,.\end{aligned}$$

5. Classification of natural principal connections on $W^r P$: the case of linear gauge group $GL(n)$

For the linear gauge group $GL(n)$ we can describe explicitly Ad-invariant elements in $\mathfrak{gl}(n)$ and Ad-invariant linear mappings $\varphi\colon \mathfrak{gl}(n) \to \mathfrak{gl}(n)$.

Lemma: Any Ad-invariant element in $\mathfrak{gl}(n)$ is of the form

$$A^i_j = a\,\delta^i_j\,, \qquad a \in \mathbb{R}\,. \qquad \clubsuit$$

Lemma: Any Ad-invariant linear mapping $\varphi\colon \mathfrak{gl}(n) \to \mathfrak{gl}(n)$ is of the form

$$\varphi = a\ \mathrm{id}_{\mathfrak{gl}(n)} + B\,tr\,,$$

where $a \in \mathbb{R}$, tr is the trace of (n, n) matrices and $B \in \mathfrak{gl}(n)$ is an Ad-invariant element. $\clubsuit$

5. Classification of natural principal connections on $W^r P$: the case of linear gauge group $GL(n)$

Let $E \to M$ be a vector bundle with n-dimensional fibres and let us denote by $PE \to M$ the frame bundle of E, i.e. PE is the principal bundle with the structure group $GL(n)$.

Theorem: (Vondra, 2008) **1.** All natural operators transforming a classical connection Λ on M and a principal connection K on PE into principal connections $\Gamma_0(\Lambda, K)$ on PE are of the maximal order 0 and form a 1-parameter family.

2. All natural operators transforming a classical connection Λ on M and a principal connection K on PE into principal connections $\Gamma_1(\Lambda, K)$ on $W^1 PE$ are of the maximal order 1 and form a 14-parameter family.

3. All natural operators transforming a classical connection Λ on M and a principal connection K on PE into principal connections $\Gamma_2(\Lambda, K)$ on $W^2 PE$ are of the maximal order 2 form a 65-parameter family.

♣