The Geometry of Monopoles: New and Old II

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Curve results with T.P. Northover.
Monopole Results in collaboration with V.Z. Enolski, A.D'Avanzo.

Recall

- Lax Pair $[\frac{d}{ds} + M(\zeta), L(\zeta)] = 0$ leads to the study of a curve

$$\mathcal{C}: \; 0 = \det(\eta 1_n + L(\zeta)) := P(\eta, \zeta)$$

- The flows (via M) are governed by meromorphic differentials γ_∞ on $\mathcal{C}$.

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- Transcendental constraints.
  1. $\mathcal{C}$ constrained by requiring periods of a given meromorphic differential to be specified. $2\mathbf{U} \in \Lambda$
     - BPS Monopoles
     - Sigma Model reductions in AdS/CFT
     - Harmonic Maps
  2. Flows and Theta Divisor. $s\mathbf{U} + \mathbf{C} \notin \Theta$

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$\mathcal{C} \subset \mathcal{S}$

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$$P(\eta, \zeta) = \eta^n + a_1(\zeta)\eta^{n-1} + \ldots + a_n(\zeta)$$

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- Symmetry: $\mathcal{C} \subset \mathbb{P}^{a,b,c} \quad [X, Y, Z] \sim [\lambda^a X, \lambda^b Y, \lambda^c Z], \; \lambda \in \mathbb{C}^*$

$\mathcal{C} \subset \mathcal{S} \Rightarrow$

$\mathcal{C}$ often comes with an antiholomorphic involution or real structure

- Reverse orientation of lines $(\eta, \zeta) \to (-\bar{\eta}/\bar{\zeta}^2, -1/\bar{\zeta})$

  $a_r(\zeta) = (-1)^r \zeta^{2r} \overline{a_r(-\frac{1}{\bar{\zeta}})} \Longrightarrow$

  $a_r(\zeta) = \chi_r \left[\prod_{l=1}^r \left(\frac{\bar{\alpha}_l}{\alpha_l}\right)^{1/2}\right] \prod_{k=1}^r (\zeta - \alpha_r)(\zeta + \frac{1}{\bar{\alpha}_r})$

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  $\alpha_r \in \mathbb{C}, \ \chi \in \mathbb{R}$ $a_r(\zeta)$ given by $2r + 1$ (real) parameters
- reality constrains the form of the period matrix
- there may be between 0 and $g + 1$ ovals of fixed points of the antiholomorphic involution.
- Imposing reality can be one of the hardest steps.

- $SO(3)$ induces an action on $T\mathbb{P}^1$ via $PSU(2)$

$$\begin{pmatrix} p & q \\ -\bar{q} & \bar{p} \end{pmatrix} \in PSU(2), \qquad |p|^2 + |q|^2 = 1,$$

$$\zeta \to \frac{\bar{p}\,\zeta - \bar{q}}{q\,\zeta + p}, \qquad \eta \to \frac{\eta}{(q\,\zeta + p)^2}$$

- corresponds to a rotation by θ around $\mathbf{n} \in S^2$

$$n_1 \sin(\theta/2) = \mathrm{Im}\, q, \qquad n_2 \sin(\theta/2) = -\mathrm{Re}\, q,$$
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- Invariant curves yield symmetric monopoles.

Spectral Curves

Basic Quantities

- Homology basis $\{\gamma_i\}_{i=1}^{2g} = \{\mathfrak{a}_i, \mathfrak{b}_i\}_{i=1}^{g}$
  - algorithm for branched covers of $\mathbb{P}^1$ (Tretkoff & Tretkoff)
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- K_Q $\qquad \mathcal{K}_{\mathcal{C}} \equiv 2\Delta, \qquad \deg\Delta = g-1$
  $-K_Q = \phi_*(\Delta - (g-1)Q) = \phi_Q(\Delta)$

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Curves with lots of symmetries: evaluate τ via character theory

$$w^2 = z^{2g+2} - 1\ (D_{2g+2}),\ \ w^2 = z(z^{2g+1} - 1)\ (C_{2g+1})$$

Example: $y^2 = x^6 + bx^3 + 1$

- D_3 symmetry $s : (x, y) \to (\rho x, y)$ $r : (x, y) \to (1/x, y/x^3)$
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$$\int_{\mathfrak{a}_1} u_1 = \int_{\mathfrak{a}+r\mathfrak{a}} u_1 = \int_{\mathfrak{a}} (u_1 + r^* u_1) = 2\int_{\mathfrak{a}} u_1$$
$$\int_{\mathfrak{a}_2} u_1 = \int_{\mathfrak{b}-r\mathfrak{b}} u_1 = \int_{\mathfrak{b}} (u_1 - r^* u_1) = 0$$

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$$\Pi = \begin{pmatrix} 2\int_{\mathfrak{a}} u_1 & 0 \\ 0 & 2\int_{\mathfrak{b}} u_2 \\ \int_{\mathfrak{b}} u_1 & \int_{\mathfrak{b}} u_2 \\ \int_{\mathfrak{a}} u_1 & -\int_{\mathfrak{a}} u_2 \end{pmatrix} = \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix}, \quad \tau = \begin{pmatrix} \dfrac{\int_{\mathfrak{b}} u_1}{2\int_{\mathfrak{a}} u_1} & \dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{-\int_{\mathfrak{a}} u_2}{2\int_{\mathfrak{b}} u_2} \end{pmatrix}$$

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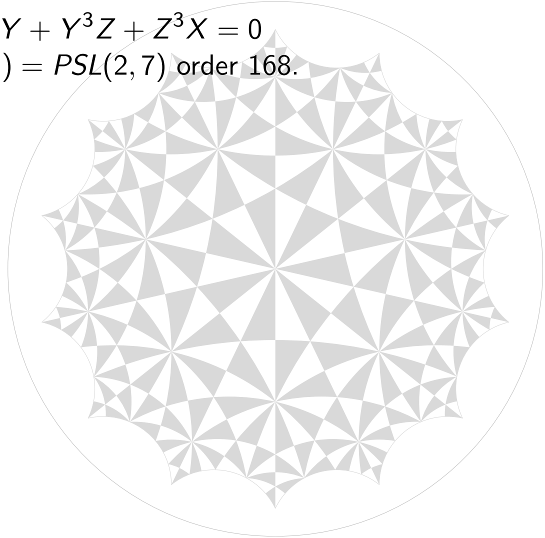
  D_3: choose $\mathfrak{b} = s\mathfrak{a}$ so $s\mathfrak{b} = s^2\mathfrak{a} = -\mathfrak{a} - \mathfrak{b}$

$$\tau = \begin{pmatrix} \lambda_1 & 1/2 \\ 1/2 & \lambda_2 \end{pmatrix} \qquad 12\lambda_1\lambda_2 + 1 = 0$$

- $\mathcal{C}$: $X^3Y + Y^3Z + Z^3X = 0$
- $\mathrm{Aut}(\mathcal{C}) = PSL(2,7)$ order 168.



Calculation

Example: Klein's Curve and Problems

- $\mathcal{C}$: $X^3Y + Y^3Z + Z^3X = 0$
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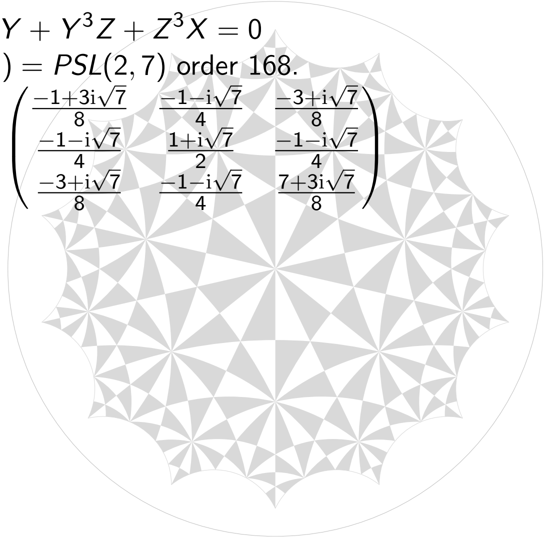

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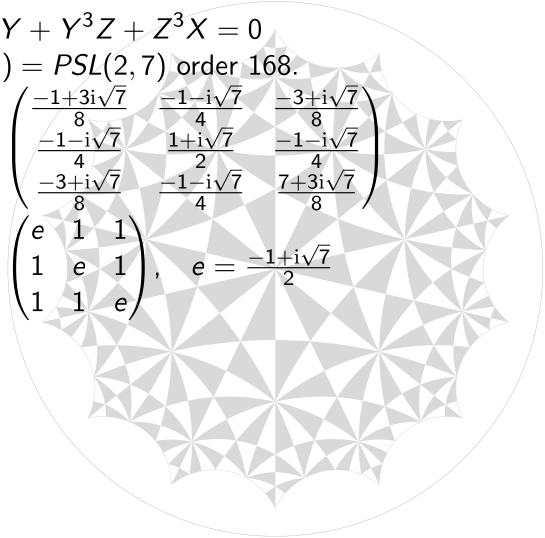

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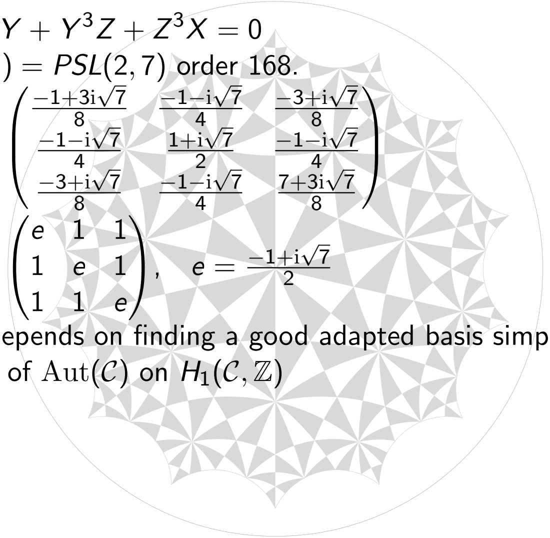

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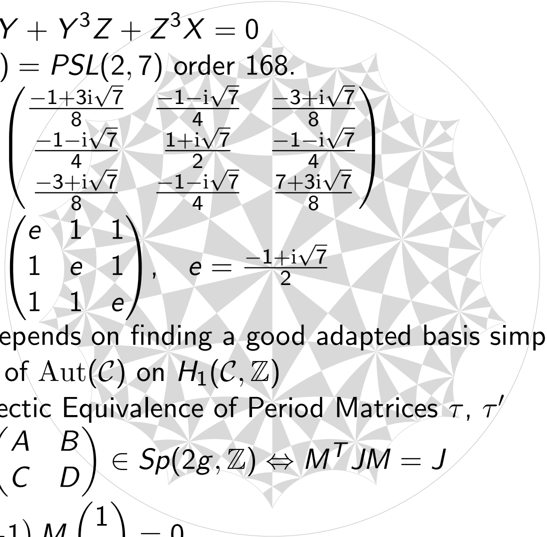
- This depends on finding a good adapted basis simplifying the action of $\mathrm{Aut}(\mathcal{C})$ on $H_1(\mathcal{C}, \mathbb{Z})$



- $\mathcal{C}$: $X^3Y + Y^3Z + Z^3X = 0$
- $\mathrm{Aut}(\mathcal{C}) = PSL(2,7)$ order 168.
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- Symplectic Equivalence of Period Matrices τ, τ'

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g,\mathbb{Z}) \Leftrightarrow M^T JM = J$$

$$\begin{pmatrix} \tau' & -1 \end{pmatrix} M \begin{pmatrix} 1 \\ \tau \end{pmatrix} = 0$$

$$\mathcal{C}:\ w^7 = (z-1)(z-\rho)^2(z-\rho^2)^4,\ \ \rho = \exp(2\pi \mathrm{i}/3)$$

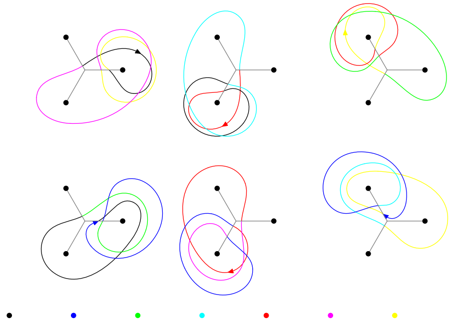

Figure: Homology basis in (z, w) coordinates

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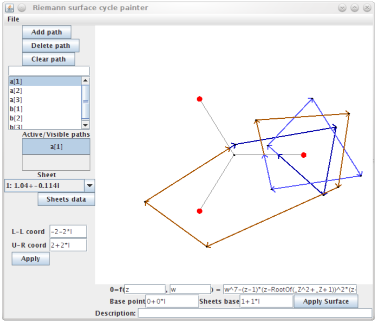

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  Example (Fay): $\phi : \hat{\mathcal{C}} \to \hat{\mathcal{C}},\ \phi^2 = \mathrm{Id},\quad \pi : \hat{\mathcal{C}} \to \mathcal{C} := \hat{\mathcal{C}}/ < \phi >$
  $2k$ fixed points. $\hat{g} = 2g + k - 1$

$$\mathfrak{a}_1, \mathfrak{b}_1, \ldots \mathfrak{a}_g, \mathfrak{b}_g, \mathfrak{a}_{g+1}, \mathfrak{b}_{g+1}, \ldots \mathfrak{a}_{g+k+1}, \mathfrak{b}_{g+k+1}, \mathfrak{a}_{1'}, \mathfrak{b}_{1'}, \ldots \mathfrak{a}_{g'}, \mathfrak{b}_{g'}$$

where $\mathfrak{a}_{1'}, \mathfrak{b}_{1'}, \ldots, \mathfrak{a}_{g'}, \mathfrak{b}_{g'}$ a basis of $H_1(\mathcal{C}, \mathbb{Z})$ and

$$\mathfrak{a}_{\alpha'} + \phi(\mathfrak{a}_\alpha) = 0 = \mathfrak{b}_{\alpha'} + \phi(\mathfrak{b}_\alpha), \qquad 1 \le \alpha \le g$$
$$\mathfrak{a}_i + \phi(\mathfrak{a}_i) = 0 = \mathfrak{b}_i + \phi(\mathfrak{b}_i), \qquad g+1 \le i \le g+k-1$$

Calculation

Symmetry and K_Q

$$-2K_Q = \phi_*\left(2\Delta - 2(g-1)Q\right) = \int_*^{2\Delta} \omega - 2(g-1)\int_*^{Q} \omega$$

$$-2K_Q.L = \int_*^{2\Delta} \sigma^*\omega - 2(g-1)\int_*^{Q} \sigma^*\omega$$

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$\sigma^N = Id$. If $L-1$ is invertible and Q a fixed point of σ then K_Q is a $2N$-torsion point.

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Corollary

Lemma$+\psi \in \mathrm{Aut}(\mathcal{C})$. Then $\int_Q^{\psi(Q)} \omega$ is a $2N(g-1)$-torsion point.

Symmetry+Fixed point $\Rightarrow$ K_Q a torsion point.
Suppose $\exists$ l, $m \in \mathbb{Z}^{2g}$ such that $m\Pi = l\Pi\,[L-1] = l[M-1]\Pi$.
Then $(-2K_Q + l\Pi)\,[L-1] = (n+m)\Pi$ in $\mathbb{C}$

Idea: Use Smith Normal Form of $M-1$ to choose l, $l(M-1) = m$ so as to make $n+m$ as simple as possible.

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Order 4 Automorphism $\Rightarrow k = 3$. Thus $-2K_Q$ fixed. Final half-period done numerically. $K_0 = \dfrac{i}{\sqrt{7}}(3,-1,5)$

- Ercolani-Sinha Constraints
  1. $\mathbf{U} = \frac{1}{2\pi\imath}\left(\oint_{\mathfrak{b}_1}\gamma_\infty, \ldots, \oint_{\mathfrak{b}_g}\gamma_\infty\right)^T = \frac{1}{2}\mathbf{n} + \frac{1}{2}\tau\mathbf{m}.$
  2. $\Omega = \dfrac{\beta_0\eta^{n-2} + \beta_1(\zeta)\eta^{n-3} + \ldots + \beta_{n-2}(\zeta)}{\frac{\partial\mathcal{P}}{\partial\eta}}\, d\zeta, \ \oint_{\mathfrak{es}}\Omega = -2\beta_0$

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- $\dfrac{\eta^{n-2}\, d\zeta}{\frac{\partial\mathcal{P}}{\partial\eta}}$ invariant under $\sigma \in \mathrm{Aut}(\mathcal{C}) \Longleftrightarrow$

$$(\mathbf{n}, \mathbf{m}) = (\mathbf{n}, \mathbf{m})M = (\mathbf{n}, \mathbf{m})\begin{pmatrix}A & B\\ C & D\end{pmatrix}$$

- $H < G = \mathrm{Aut}(\mathcal{C})$

$$\begin{array}{ccc} t^{-1}\cdot x & \xrightarrow{\;t\;} & x \\ {\scriptstyle H}\big\downarrow & & {\scriptstyle H}\big\downarrow \\ H\cdot t^{-1}x & & H\cdot x \end{array}$$

Spectral Curves

Intermediate Quotients

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- Can we explicitly determine the curve $\mathcal{C}/H$?
  $\iota : \mathcal{C} \hookrightarrow \mathbb{P}^{\mathbf{a}}$ the vanishing of a hom poly. $\mathcal{L} = \iota^*(\mathcal{O}(1))$
  $\exists\ k\ \mathcal{L}^k$ ample

$$R = \oplus_{n\geq 0} H^0(X, (\mathcal{L}^k)^{\otimes n}), \qquad \mathcal{C} \equiv \mathrm{Proj}(R).$$

Provided G commutes with the $\mathbb{C}^*$ action

$$\mathbb{P}^{a_0,a_1,a_2}//G = \mathrm{Proj}\,\mathbb{C}[x_0,x_1,x_2]^G, \quad \deg x_i = a_i$$

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deg.	1	2	3
invt.	Z	$(X-\rho Y)(X-\rho^2 Y),\ Z^2$	$\alpha, \beta, \gamma, \delta$

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  $\delta = i\sqrt{3}\,(X-\rho Y)(X-\rho^2 Y)Z$
- $\beta = \alpha + \gamma, \qquad 0 = \delta^3 + \alpha\gamma(\alpha+\gamma)$

  Ring of invariants for quotient curve $\mathcal{R} = \dfrac{\mathbb{C}[\alpha,\delta,\gamma]}{\delta^3 + \alpha\gamma(\alpha+\gamma)}$
- $\mathrm{Proj}(\mathcal{R}) = \mathcal{E} \quad$ 3 : 1 unbranched covering.

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  1. Calculation of matrix of periods and τ
  2. Calculation of K_Q
  3. Calculation of ES vector $\mathbf{U}$
- We will see how symmetry simplifies $\theta\left(s\mathbf{U}+\mathbf{C}|\tau\right)$
- Have yet to solve any of the transcendental constraints on $\mathcal{C}$
  1. ES constraints: $2\mathbf{U}\in\Lambda\Longleftrightarrow\mathbf{U}$
  2. Flows and Theta Divisor: $s\mathbf{U}+\mathbf{C}\notin\Theta$