Two-symmetric Lorentzian manifolds

Anton Galaev

Masaryk University, Brno, Czech Republic

Joint work with D.V. Alekseevsky

Definition

A pseudo-Riemannian manifold (M, g) *satisfying the condition*

$$\nabla^k R = 0, \quad \nabla^{k-1} R \neq 0, \quad k \geq 1$$

is called k*-symmetric.*

1-symmetric manifolds are the same as locally symmetric.

For Riemannian manifolds, the condition $\nabla^k R = 0$ implies $\nabla R = 0$ (Tanno 1972).

On there other hand, there exist pseudo-Riemannian k-symmetric spaces with $k \geq 2$ (e.g. Kaigorodov 1985).

We find the local form of all 2-symmetric Lorentzian manifolds.

Review of different equations on the covariant derivatives of the curvature tensor of Lorentzian manifolds:
V. R. Kaigorodov, *Structure of the curvature of space-time.* Journal of Soviet Math. 28 (1985) no. 2, 256–273.

First detailed investigation of two-symmetric Lorentzian spaces:
J. M. Senovilla, *Second-order symmetric Lorentzian manifolds. I. Characterization and general results,* Classical Quantum Gravity 25 (2008), no. 24, 245011, 25 pp.

It is proven that any two-symmetric Lorentzian space admits a parallel null vector field.

A classification of four-dimensional two-symmetric Lorentzian spaces is obtained in the paper
O. F. Blanco, M. Sánchez, J. M. Senovilla, *Complete classifcation of second-order symmetric spacetimes.* Journal of Physics: Conference Series 229 (2010), 012021, 5pp.

This result of this talk was send to arXiv on November 15, 2010:

D.V. Alekseevsky, A.S. Galaev, *Two-symmetric Lorentzian manifolds*, arXiv:1011.3439 (11 pages).

Later, on January 28, 2011 appeared another much more complicated proof:

O.F. Blanco, M. Sánchez, J.M.M. Senovilla, *Structure of second-order symmetric Lorentzian manifolds,* arXiv:1101.5503 (30 pages).

Reduction:

By the Wu theorem, any Lorentzian manifold is locally a product of a Riemannian manifold (M_1, g_1) and of a locally indecomposable Lorentzian manifold (M_2, g_2).

If (M, g) is 2-symmetric, then (M_1, g_1) is locally symmetric and (M_2, g_2) is 2-symmetric.

Thus we may assume that (M, g) is locally indecomposable.

This implies that the holonomy algebra $\mathfrak{g} \subset \mathfrak{so}(1, d-1)$ of (M, g) is weakly irreducible, i.e. it does not preserve any non-degenerate subspace of the tangent space.

Definition

A Lorentzian manifold (M, g) (of dimension $n+2$) is called a pp-wave if locally there exist coordinates $v, x^1, ..., x^n, u$ such that

$$g = 2dvdu + \sum_{i=1}^{n}(dx^i)^2 + H(du)^2,$$

where H is a function of $x^1, ..., x^n, u$.

Simply connected Lorentzian symmetric space are exhausted by de Sitter space, anti de Sitter space, and by the Cahen-Wallach spaces.

Cahen-Wallach spaces are pp-waves with $H = \sum \lambda_i (x^i)^2$.

Theorem

Let (M, g) be a locally indecomposable Lorentzian manifold of dimension $n+2$. Then (M, g) is two-symmetric if and only if locally there exist coordinates $v, x^1, ..., x^n, u$ such that

$$g = 2dvdu + \sum_{i=1}^{n}(dx^i)^2 + (H_{ij}u + F_{ij})x^i x^j (du)^2,$$

where H_{ij} is a diagonal matrix with the diagonal elements $\lambda_1 \leq \cdots \leq \lambda_n$ that are simultaneously non-zero real numbers, F_{ij} is a symmetric real matrix.

Any other metric of this form isometric to g is given by the same H_{ij} and by $\tilde{F}_{ij} = cH_{ij} + F_{kl}a_i^k a_j^l$, where $c \in \mathbb{R}$ and a_i^j is an orthogonal matrix such that $H_{kl}a_i^k a_j^l = H_{ij}$.

For the proof, we used the methods of the theory of holonomy: the tensor ∇R is parallel, hence its value at any point must be annihilated by the holonomy algebra.
Using the classification of Lorentzian holonomy algebras and the description of the corresponding curvature tensors, we show that the holonomy algebra must be commutative, i.e. the space is a pp-wave. Then it is not hard to check which pp-waves are 2-symmetric.
We also use the result by A. Derdzinski and W. Roter about Lorentzian manifolds with parallel Weyl conformal tensor.

Sketch of the proof.

Lorentzian holonomy algebras.

Let (M, g) be a locally indecomposable Lorentzian manifold of dimension $n+2$ and $\mathfrak{g} \subset \mathfrak{so}(1, n+1)$ be its holonomy algebra, which is weakly irreducible.

If $\mathfrak{g} \subset \mathfrak{so}(1, n+1)$ is irreducible, then $\mathfrak{g} = \mathfrak{so}(1, n+1)$.
Any other weakly irreducible holonomy algebra $\mathfrak{g} \subset \mathfrak{so}(1, n+1)$ preserves an isotropic line of the tangent space $\mathbb{R}^{1,n+1}$.

Fix two isotropic vectors $p, q \in \mathbb{R}^{1,n+1}$ such that $g(p,q) = 1$. Let $E \subset \mathbb{R}^{1,n+1}$ be the orthogonal complement to $\mathbb{R}p \oplus \mathbb{R}q$. Then

$$\mathbb{R}^{1,n+1} = \mathbb{R}p \oplus E \oplus \mathbb{R}q.$$

Denote by $\mathfrak{sim}(n)$ the maximal subalgebra of $\mathfrak{so}(1, n+1)$ preserving $\mathbb{R}p$.
In the matrix form:

$$\mathfrak{sim}(n) = \left\{ \left. \begin{pmatrix} a & X^t & 0 \\ 0 & A & -X \\ 0 & 0 & -a \end{pmatrix} \right| \begin{array}{c} a \in \mathbb{R}, \\ X \in \mathbb{R}^n, \\ A \in \mathfrak{so}(n) \end{array} \right\}.$$

Identify $\mathfrak{so}(1, n+1)$ with $\Lambda^2\mathbb{R}^{1,n+1}$ in such a way that $(u \wedge v)w = g(u, w)v - g(v, w)u$, then

$$\mathfrak{sim}(n) = \mathbb{R}p \wedge q + \mathfrak{so}(E) + p \wedge E.$$

The Lorentzian holonomy algebras $\mathfrak{g} \subset \mathfrak{sim}(n)$ are the following :

(type I) $\mathbb{R}p \wedge q + \mathfrak{h} + p \wedge E$,

(type II) $\mathfrak{h} + p \wedge E$,

(type III) $\{\varphi(A)p \wedge q + A | A \in \mathfrak{h}\} + p \wedge E$,

(type IV) $\{A + p \wedge \psi(A) | A \in \mathfrak{h}\} + p \wedge E_1$,

where $\mathfrak{h} \subset \mathfrak{so}(E)$ is a Riemannian holonomy algebra; $\varphi : \mathfrak{h} \to \mathbb{R}$ is a non-zero linear map, $\varphi|_{[\mathfrak{h},\mathfrak{h}]} = 0$; for the last algebra $E = E_1 \oplus E_2$, $\mathfrak{h} \subset \mathfrak{so}(E_1)$, and $\psi : \mathfrak{h} \to E_2$ is a surjective linear $\psi|_{[\mathfrak{h},\mathfrak{h}]}$.
A locally indecomposable simply connected Lorentzian manifold admits a parallel null vector field if and only if its holonomy group is of type II or IV.

The holonomy group of a two-symmetric Lorentzian manifold

Theorem

(Senovilla 2008) Any two-symmetric Lorentzian manifold (M, g) admits a parallel null vector field.

This implies that the holonomy group can be only of type II or IV. The corner stone of the paper is the following statement.

Theorem

The holonomy algebra $\mathfrak{g}$ of an $(n+2)$-dimensional locally indecomposable two-symmetric Lorentzian manifold (M, g) is $p \wedge E$.

It is known that any $(n+2)$-dimensional Lorentzian manifold with the holonomy algebra $p \wedge E$ is a pp-wave!

Algebraic curvature tensors

For a subalgebra $\mathfrak{g} \subset \mathfrak{so}(1, n+1)$ define *the space of algebraic curvature tensors of type* $\mathfrak{g}$,

$$\mathcal{R}(\mathfrak{g}) = \{R \in \Lambda^2(\mathbb{R}^{1,n+1})^* \otimes \mathfrak{g} \,|\, R(u,v)w + R(v,w)u + R(w,u)v = 0\}.$$

The spaces $\mathcal{R}(\mathfrak{g})$ for holonomy algebras of Lorentzian manifolds are known. For example, let $\mathfrak{g} = \mathbb{R}p \wedge q + \mathfrak{h} + p \wedge E$. For the subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ define the space

$$\mathcal{P}(\mathfrak{h}) = \{P \in E^* \otimes \mathfrak{h} \,|\, g(P(x)y, z) + g(P(y)z, x) + g(P(z)x, y) = 0\}.$$

Any $R \in \mathcal{R}(\mathfrak{g})$ is uniquely determined by the date (λ, e, P, R^0, T), where

$$\lambda \in \mathbb{R},\ e \in E,\ P \in \mathcal{P}(\mathfrak{h}),\ R^0 \in \mathcal{R}(\mathfrak{h}),\ T \in S^2 E,$$

$$\begin{aligned} R(p,q) &= -\lambda p \wedge q - p \wedge e, \\ R(X,Y) &= R^0(X,Y) - p \wedge (P(Y)X - P(X)Y),\ \forall X, Y \in E \\ R(X,q) &= -g(e,X) p \wedge q + P(X) - p \wedge T(X),\ R(p,X) = 0. \end{aligned}$$

We will write $R = R^{(\lambda, e, P, R^0, T)}$.

If R is defined only by T, then we write $R = R^T$. Note that

$$R^T = \sum_{i,j} T_{ij} p \wedge e_i \vee p \wedge e_j,$$

$$T_{ij} = g(Te_i, e_j), \qquad \mathcal{R}(p \wedge E) = \{R^T |\, T \in S^2 E\} \simeq S^2 E.$$

Define the space of covariant derivatives of the curvature tensor

$$\nabla\mathcal{R}(\mathfrak{g})=\{S\in\mathrm{Hom}(\mathbb{R}^{1,n+1},\mathcal{R}(\mathfrak{g}))|\,S_u(v,w)+S_v(w,u)+S_w(u,v)=0\}.$$

It is not difficult to find the space $\nabla\mathcal{R}(\mathfrak{g})$ for each Lorentzian holonomy algebra $\mathfrak{g}\subset\mathfrak{sim}(n)$. It consists of tensors

$$S\in\mathrm{Hom}(V,\mathcal{R}(\mathfrak{g})),\quad S:u\in V\mapsto S_u=R^{(\lambda_u,e_u,P_u,R^0_u,T_u)}\in\mathcal{R}(\mathfrak{g})$$

satisfying the second Bianchi identity. For example,

$$\nabla\mathcal{R}(p\wedge E)=\{S=q'\otimes R^T\,|\,T\in S^2E\}\oplus\{S=R^{Q\cdot}\,|\,Q\in S^3E\},$$

here $q'=g(p,\cdot)$ is the 1-form g-dual to p, the tensor $S=R^{Q\cdot}$ is defined by $S_p=S_q=0$, $S_x=R^{Q_x}$, $x\in E$, $Q_x\in S^2E$ (since $Q\in S^3E$).

Adapted coordinates and reduction lemma

Let (M, g) be an $(n+2)$-dimensional locally indecomposable two-symmetric Lorentz manifold, i.e. the tensor ∇R is not zero, parallel and annihilated by the holonomy algebra. The space $\nabla\mathcal{R}(\mathfrak{so}(1, n+1))$ does not contain non-zero elements annihilated by $\mathfrak{so}(1, n+1)$, hence $\mathfrak{g} \subset \mathfrak{sim}(n)$.

Then (M, g) admits a parallel distribution of null lines. Locally there exist so called Walker coordinates $v, x^1, ..., x^n, u$ such that the metric g has the form

$$g = 2dvdu + h + 2Adu + H(du)^2, \qquad (0.1)$$

where $h = h_{ij}(x^1, ..., x^n, u)dx^i dx^j$ is an u-dependent family of Riemannian metrics, $A = A_i(x^1, \ldots, x^n, u)dx^i$ is an u-dependent family of one-forms, and $H = H(v, x^1, ..., x^n, u)$ is a local function on M.

Consider the local frame

$$p = \partial_v, \quad X_i = \partial_i - A_i \partial_v, \quad q = \partial_u - \frac{1}{2} H \partial_v.$$

Let E be the distribution generated by the vector fields $X_1, \ldots, X_n$. Clearly, the vector fields p, q are isotropic, $g(p, q) = 1$, the restriction of g to E is positive definite, and E is orthogonal to p and q. The vector field p defines the parallel distribution of null lines and it is recurrent, i.e. $\nabla p = \theta \otimes p$.

Let $\mathfrak{g} \subset \mathfrak{sim}(n)$ be the holonomy algebra of the Lorentzian manifold (M, g) and $\mathfrak{h} \subset \mathfrak{so}(E)$ be its orthogonal part. Then there exist the decompositions

$$E = E_0 \oplus E_1 \oplus \cdots \oplus E_r, \quad \mathfrak{h} = \{0\} \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r \qquad (0.2)$$

such that $\mathfrak{h}$ annihilates E_0, $\mathfrak{h}_i(E_j) = 0$ for $i \neq j$, and $\mathfrak{h}_i \subset \mathfrak{so}(E_i)$ is an irreducible subalgebra for $1 \leq i \leq s$.

Ch. Boubel proved that there exist Walker coordinates

$$v,\ x_0 = (x_0^1, \dots, x_0^{n_0}), \dots, x_r = (x_r^1, ..., x_r^{n_r}),\ u$$

adapted to the decomposition (0.2). This means that

$$h = h_0 + h_1 + \cdots + h_r, \quad h_0 = \sum_{i=1}^{n_0} (dx_0^i)^2, \quad h_\alpha = \sum_{i,j=1}^{n_\alpha} h_{\alpha ij} dx_\alpha^i dx_\alpha^j, \tag{0.3}$$

$$A = \sum_{\alpha=1}^{r} A_\alpha, \quad A_0 = 0, \quad A_\alpha = \sum_{k=1}^{n_\alpha} A_k^\alpha dx_\alpha^k,$$

and it holds

$$\frac{\partial}{\partial x_\beta^k} h_{\alpha ij} = \frac{\partial}{\partial x_\beta^k} A_i^\alpha = 0, \qquad \text{if } \beta \neq \alpha. \tag{0.4}$$

The coordinates can be chosen such that $A = 0$ (Galaev,Leistner 2010). Thus we will assume that g is given by (0.1) with $A = 0$, and with h satisfying (0.3) and (0.4).

For $\alpha = 0, ..., r$, consider the submanifolds $M_\alpha \subset M$ defined by $x_\beta = c_\beta$, $\alpha \neq \beta$, where c_β are constant vectors. Then the induced metric is given by

$$g_\alpha = 2dvdu + h_\alpha + H_\alpha (du)^2.$$

Lemma

The submanifold $M_\alpha \subset M$ is totally geodesic. The orthogonal part of the holonomy algebra $\mathfrak{g}_\alpha$ of the metrics g_α coincides with $\mathfrak{h}_\alpha \subset \mathfrak{so}(E_\alpha)$, which is irreducible for $\alpha = 1, ..., r$. If the metric g is two-symmetric, then the curvature tensor of each metric g_α satisfies $\nabla^2 R = 0$.

Now it is enough to consider two cases: $\mathfrak{g} = p \wedge E$ and $\mathfrak{g} = \mathfrak{h} + p \wedge E$, where $\mathfrak{h} \subset \mathfrak{so}(E)$ is irreducible.
We will prove that the second case is impossible.

Lemma

Let $\mathfrak{g} = \mathfrak{h} + p \wedge E$, where $\mathfrak{h} \subset \mathfrak{so}(E)$ is irreducible. Then the subspace $\nabla\mathcal{R}(\mathfrak{g})^0 \subset \nabla\mathcal{R}(\mathfrak{g})$ of $\mathfrak{g}$-annihilated tensors is the one-dimensional subspace given by

$$\nabla\mathcal{R}(\mathfrak{g})^0 = \mathbb{R}S, \quad S = q' \otimes R^{\mathrm{Id}_E}, \quad q' = g(p, \cdot).$$

Let (M,g) be 2-symmetric with the holonomy algebra $\mathfrak{h}+p\wedge E$, where $\mathfrak{h}\subset\mathfrak{so}(E)$ is irreducible.
By Lemma 2, ∇R has the form

$$\nabla_U R = fg(p,U)R^{\mathrm{Id}_E},\quad \forall U\in TM, \tag{0.5}$$

for some smooth function f. It is clear that

$$R^{\mathrm{Id}_E}(U_1,U_2)=p\wedge((U_1\wedge U_2)p),\quad \forall U_1,U_2\in TM.$$

Lemma

Under the above assumptions, the conformal Weyl curvature tensor W is parallel, i.e. $\nabla W=0$.

Proof of the lemma. It is known that

$$W = R + L \wedge g,$$

where

$$L = \frac{1}{d-2}\left(\mathrm{Ric} - \frac{s}{2(d-1)}\mathrm{Id}\right)$$

is the Schouten tensor, Ric is the Ricci operator, and s is the scalar curvature. Recall that by definition,

$$(L \wedge g)(U_1, U_2) = LU_1 \wedge U_2 + U_1 \wedge LU_2, \quad U_1, U_2 \in TM.$$

For any vector field U it holds

$$\nabla_U W = \nabla_U R + (\nabla_U L) \wedge g.$$

The covariant derivative of the Ricci operator is given by

$$(\nabla_{U_1}\mathrm{Ric})U_2 = -nfg(p, U_1)g(p, U_2)p.$$

Hence, $\mathrm{grad} s = 0$ and $(\nabla_{U_1} L)U_2 = -fg(p, U_1)g(p, U_2)p$.
Consequently, $(\nabla_{U_1} L)U_2 \wedge U_3 + U_2 \wedge (\nabla_{U_1} L)U_3 = -\nabla_{U_1} R(U_2, U_3)$.
Thus, $(\nabla_{U_1} L) \wedge g = -\nabla_{U_1} R$ and $\nabla W = 0$. □

Where the lemma comes from:

The well-known decomposition with respect to the pseudo-orthogonal Lie group

$$R = W + \mathrm{Ric}_0 + s,$$

has the following analogy:

$$\nabla R = (\nabla W, C) + C + \mathrm{grad} s + \mathrm{sym}\nabla\left(\mathrm{Ric} - \frac{2}{d+2} sg\right).$$

The formulas for ∇R and $\nabla \mathrm{Ric}$ imply
$C = 0$, $\mathrm{grad} s = 0$, $\nabla R = \mathrm{sym}\nabla\left(\mathrm{Ric} - \frac{2}{d+2} sg\right)$,
consequently $\nabla W = 0$.

Lorentzian manifolds (M, g) with $\nabla W = 0$ are classified by A. Derdzinski and W. Roter:
it holds that either $\nabla R = 0$, or $W = 0$, or (M, g) is a pp-wave.

The condition $W = 0$ under the above assumptions implies that (M, g) is a pp-wave.
This gives a contradiction (we assumed that $\mathfrak{g} = \mathfrak{h} + p \wedge E$, $\mathfrak{h} \subset \mathfrak{so}(E)$ is irreducible).

Thus, $\mathfrak{g} = p \wedge E$, i.e. (M, g) is a pp-wave.

Theorem

Let (M, g) be a locally indecomposable Lorentzian manifold of dimension $n+2$. Then (M, g) is two-symmetric if and only if locally there exist coordinates $v, x^1, ..., x^n, u$ such that

$$g = 2dvdu + \sum_{i=1}^{n}(dx^i)^2 + (H_{ij}u + F_{ij})x^i x^j (du)^2,$$

where H_{ij} is a diagonal matrix with the diagonal elements $\lambda_1 \leq \cdots \leq \lambda_n$ that are simultaneously non-zero real numbers, F_{ij} is a symmetric real matrix.
Any other metric of this form isometric to g is given by the same H_{ij} and by $\tilde{F}_{ij} = cH_{ij} + F_{kl}a_i^k a_j^l$, where $c \in \mathbb{R}$ and a_i^j is an orthogonal matrix such that $H_{kl}a_i^k a_j^l = H_{ij}$.