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On soliton equations and soliton interactions

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Based on:

- V. S. Gerdjikov. *Algebraic and Analytic Aspects of N-wave Type Equations.* Contemporary Mathematics **301**, 35-68 (2002).

- V. S. Gerdjikov, D. J. Kaup, N. A. Kostov, T. I. Valchev. *On classification of soliton solutions of multicomponent nonlinear evolution equations.*
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- Nikolay Kostov, Vladimir Gerdjikov. *Reductions of multicomponent mKdV equations on symmetric spaces of* **DIII**-*type* . SIGMA **4** (2008), paper 029, 30 pages; **ArXiv:0803.1651**.

- V. S. Gerdjikov. *Selected Aspects of Soliton Theory. Constant boundary conditions.* In: Prof. G. Manev's Legacy in Contemporary Aspects of Astronomy, Gravitational and Theoretical Physics Eds.: V. Gerdjikov, M. Tsvetkov, Heron Press Ltd, Sofia, 2005. pp. 277-290. **nlin.SI/0604004**

Integrable MNLS and Lax representations

A.III-type MNLS or the vector NLS (the Manakov model) – Manakov, 1974::

$$H_{\rm A.III} = \int_{-\infty}^{\infty} dx \, \left((\vec{q}_x{}^\dagger, \vec{q}_x) - (\vec{q}^{\,\dagger}, \vec{q})^2 \right),$$

$$i\vec{q}_t + \bar{q}_{xx} + 2(\vec{q}^{\,\dagger}, \vec{q})\vec{q}(x,t) = 0,$$

BD.I-type MNLS:

$$H_{\rm BD.I} = \int_{-\infty}^{\infty} dx \, \left((\vec{q}_x{}^\dagger, \vec{q}_x) - (\vec{q}^{\,\dagger}, \vec{q})^2 + \frac{1}{2}|(\vec{q}, s_0\vec{q})|^2 \right)$$

$$i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}^{\,\dagger}, \vec{q})\vec{q}(x,t) - (\vec{q}, s_0\vec{q})s_0\vec{q}^{\,*}(x,t) = 0, \qquad s_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & \end{pmatrix}$$

$$[L(\lambda), M(\lambda)] = 0, \qquad \text{identically w.r.t. to } \lambda:$$

A.III-type MNLS

$$L\psi(x,\lambda) \equiv i\frac{d\psi}{dx} + q(x)\psi(x,\lambda) - \lambda J\psi(x,\lambda) = 0, \qquad (1)$$

$$Q(x,t) = \begin{pmatrix} 0 & \vec{q}^T(x,t) \\ -\vec{q}^*(x,t) & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1}_n \end{pmatrix}.$$

$$\begin{aligned} M\psi &\equiv i\frac{d\psi}{dt} + \left(V_0(x,t) - V_{0,+} + 2\lambda Q(x,t) - 2\lambda^2 J\right)\psi(x,t,\lambda) \\ &= \psi(x,t,\lambda)C(\lambda), \\ V_0(x,t) &= \left[\mathrm{ad}_J^{-1}Q, Q(x,t)\right] + 2i\,\mathrm{ad}_J^{-1}Q_x, \end{aligned}$$

BD.I-type MNLS

$$
\begin{aligned}
L\psi(x,t,\lambda) &\equiv i\partial_x\psi + (Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0. \\
M\psi(x,t,\lambda) &\equiv i\partial_t\psi + (V_0(x,t) + \lambda V_1(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0, \\
V_1(x,t) &= Q(x,t), \qquad V_0(x,t) = i\mathrm{ad}\,_J^{-1}\frac{dQ}{dx} + \frac{1}{2}\left[\mathrm{ad}\,_J^{-1}Q, Q(x,t)\right].
\end{aligned}
$$

where $J = \mathrm{diag}(1, 0, \dots 0, -1)$ and

$$
Q = \begin{pmatrix} 0 & \vec{q}^T & 0 \\ \vec{p} & 0 & s_0\vec{q} \\ 0 & \vec{p}^T s_0 & 0 \end{pmatrix}, \quad S_0 = \sum_{k=1}^{2r+1} (-1)^{k+1} E_{k,2r+2-k} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -s_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{2}
$$

Here $(E_{kn})_{ij} = \delta_{ik}\delta_{nj}$ and the $2r-1$-vectors $\vec{q}$ and $\vec{p} = \vec{q}^*$ take the form

$$
\vec{q} = (q_1, \dots, q_{r-1}, q_0, q_{-1}, \dots, q_{-r+1})^T,
$$

BEC with hyperfine structure

$^{23}\mathrm{Na} \Leftrightarrow F = 1 \qquad ^{87}\mathrm{Rb} \Leftrightarrow F = 2$

see Wadati et al (2004), (2006), (2007); Ohmi & Machida (1998); Kuwamoto et al (2004); Gerdjikov et al (2007), (2008)

The assembly of atoms in the hyperfine state of spin F is described by a normalized spinor wave vector with $2F+1$ components

$$\Phi(x,t) = (\Phi_F(x,t), \dots, \Phi_0(x,t), \dots, \Phi_{-F}(x,t))^T$$

Ginzburg-Pitaevsky equation in the one-dimensional approximation:

$$i\frac{\partial \Phi}{\partial t} = \frac{\delta E_{\mathrm{GP}}[\Phi]}{\delta \Phi^*}. \tag{3}$$

where for $F=1$ the energy functional is given by:

$$E_{\mathrm{GP}} = \int dx \left\{ \frac{\hbar^2}{2m} |\partial_x \Phi|^2 + \frac{\bar{c}_0 + \bar{c}_2}{2} \left[(\Phi^\dagger, \Phi)^2 - \frac{\bar{c}_0}{2} \left| 2\Phi_1 \Phi_{-1} - \Phi_0^2 \right|^2 \right] \right. .$$

the effective 1D couplings $\bar{c}_{0,2}$ are represented by

$$\bar{c}_0 = c_0/2a_\perp^2, \quad \bar{c}_2 = c_2/2a_\perp^2, \tag{4}$$

where $a_\perp$ is the size of the transverse ground state. In this expression,

$$c_0 = \pi\hbar^2(a_0 + 2a_2)/3m, \qquad c_2 = \pi\hbar^2(a_2 - a_0)/3m, \tag{5}$$

where a_f – s-wave scattering lengths; m is the mass of the atom.

Special (integrable) choice for the coupling constants $\bar{c}_0 = \bar{c}_2 \equiv -c < 0$, equivalently scattering lengths $2a_0 = -a_2 > 0$. In the dimensionless form: $\Phi \to \{\Phi_1, \Phi_0, \Phi_{-1}\}^T$ the corresponding GPE take the form:

$$\begin{aligned}
&i\partial_t\Phi_1 + \partial_x^2\Phi_1 + 2(|\Phi_1|^2 + 2|\Phi_0|^2)\Phi_1 + 2\Phi_{-1}^*\Phi_0^2 = 0,\\
&i\partial_t\Phi_0 + \partial_x^2\Phi_0 + 2(|\Phi_{-1}|^2 + |\Phi_0|^2 + |\Phi_1|^2)\Phi_0 + 2\Phi_0^*\Phi_1\Phi_{-1} = 0,\\
&i\partial_t\Phi_{-1} + \partial_x^2\Phi_{-1} + 2(|\Phi_{-1}|^2 + 2|\Phi_0|^2)\Phi_{-1} + 2\Phi_1^*\Phi_0^2 = 0.
\end{aligned} \tag{6}$$

$F = 2$ hyperfine state is described by a 5-component spinor wave vector

$$\Phi(x,t) = (\Phi_2(x,t), \Phi_1(x,t), \Phi_0(x,t), \Phi_{-1}(x,t), \Phi_{-2}(x,t))^T, \tag{7}$$

$$E_{\rm GP}[\Phi] = \int_{-\infty}^{\infty} dx \left(\frac{\hbar^2}{2m}|\partial_x\Phi|^2 + \frac{\epsilon c_0}{2}n^2 + \frac{c_2}{2}\mathbf{f}^2 + \frac{\epsilon c_4}{2}|\Theta|^2 \right), \tag{8}$$

$$\epsilon = \pm 1, \qquad n = (\vec{\Phi}^{\,\dagger}, \vec{\Phi}) = \sum_{\alpha=-2}^{2} \Phi_\alpha \Phi_\alpha^*,$$

$$\Theta = (\vec{\Phi}, s_0 \vec{\Phi}) = 2\Phi_2\Phi_{-2} - 2\Phi_1\Phi_{-1} + \Phi_0^2.$$

Choosing $c_2 = 0$, $c_4 = 1$ and $c_0 = -2$ we obtain

$$i\partial_t \Phi_{\pm 2} + \partial_{xx}\Phi_{\pm 2} = -2\epsilon(\vec{\Phi}, \vec{\Phi^*})\Phi_{\pm 2} + \epsilon(2\Phi_2\Phi_{-2} - 2\Phi_1\Phi_{-1} + \Phi_0^2)\Phi_{\mp 2}^*,$$

$$i\partial_t \Phi_{\pm 1} + \partial_{xx}\Phi_{\pm 1} = -2\epsilon(\vec{\Phi}, \vec{\Phi^*})\Phi_{\pm 1} - \epsilon(2\Phi_2\Phi_{-2} - 2\Phi_1\Phi_{-1} + \Phi_0^2)\Phi_{\mp 1}^*,$$

$$i\partial_t \Phi_0 + \partial_{xx}\Phi_0 = -2\epsilon(\vec{\Phi}, \vec{\Phi^*})\Phi_{\pm 0} + \epsilon(2\Phi_2\Phi_{-2} - 2\Phi_1\Phi_{-1} + \Phi_0^2)\Phi_0^*.$$

which is integrable by the inverse scattering method.

Lax pairs for systems on symmetric spaces – Fordy, Kulish (1983)
For our system we have **BD**.**I**-type symmetric spaces:

$$\simeq \mathrm{SO(n+2)/SO(2) \times SO(n)}$$

with $n = 3$ and $n = 5$ respectively.

Symmetric and homogeneous spaces

Symmetric space: $\mathcal{M}$ is globally symmetric if each its point p is isolated invariant point under an involutive isometry:

$$\mathcal{K}(\mathcal{M}) \equiv K\mathcal{M}K^{-1} = \mathcal{M}, \qquad \mathcal{K}^2 = \mathbb{1}.$$

Cartan has classified all such involutions.
$\mathcal{M} \equiv \mathfrak{G}/\mathcal{H}$ where $\mathfrak{G}$ is simple and $\mathcal{H}$ is semisimple. Normally

$$\mathcal{H} \equiv \{K \in \mathfrak{G}, \quad \text{such that} \quad KJK^{-1} = J, \quad J \in \mathcal{H}\}.$$

Local coordinates:

$$Q(x) = [J, Q'(x)].$$

Typically $\mathcal{H}$ is simple:

$$J = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad Q(x) = \begin{pmatrix} 0 & Q^+(x) \\ Q^-(x) & 0 \end{pmatrix},$$

But for BD.I-type symmetric spaces $\mathcal{H}$ is semi-simple: $\mathcal{H} \simeq SO(2) \otimes SO(n)$

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & \vec{q}^T & 0 \\ \vec{p} & 0 & s_0 \vec{q} \\ 0 & \vec{p}^T s_0 & 0 \end{pmatrix},$$

Effectively it is enough to properly specify $\mathfrak{G}$ and J in order to determine $\mathcal{M}$. The corresponding Lie algebra $\mathfrak{g}$ acquires $\mathbb{Z}_2$-grading:

$$\mathfrak{g} = \mathfrak{g}^{(0)} + \mathfrak{g}^{(1)},$$

$$\mathfrak{g}^{(0)} \equiv \{X : X \in \mathfrak{g} \qquad \mathcal{K}(X) = X\}, \qquad \mathfrak{g}^{(1)} \equiv \{Y : Y \in \mathfrak{g} \qquad \mathcal{K}(Y) = -Y\},$$

The grading property:

$$[\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}] \in \mathfrak{g}^{(0)}, \qquad [\mathfrak{g}^{(0)}, \mathfrak{g}^{(1)}] \in \mathfrak{g}^{(1)}, \qquad [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] \in \mathfrak{g}^{(0)}$$

The set of positive roots Δ^+ also splits into two subsets:

$$\Delta^+ = \Delta_0^+ \cup \Delta_1^+,$$

$$\Delta_0^+ \equiv \{\alpha : \quad \alpha(J) = 0\} \qquad \Delta_1^+ \equiv \{\alpha : \quad \alpha(J) = a > 0\}$$

Inverse scattering method and reconstruction of potential from minimal scattering data

Solving the direct and the inverse scattering problem (ISP) for L uses the Jost solutions

$$\lim_{x\to-\infty} \phi(x,t,\lambda)e^{i\lambda Jx} = \mathbb{1}, \qquad \lim_{x\to\infty} \psi(x,t,\lambda)e^{i\lambda Jx} = \mathbb{1} \tag{9}$$

and the scattering matrix $T(\lambda,t) \equiv \psi^{-1}\phi(x,t,\lambda)$. We use the following block-matrix structure of $T(\lambda,t)$

$$T(\lambda,t) = \begin{pmatrix} m_1^+ & -\vec{b}^{-T} & c_1^- \\ \vec{b}^+ & \mathbf{T}_{22} & -s_0\vec{B}^- \\ c_1^+ & \vec{B}^{+T}s_0 & m_1^- \end{pmatrix}, \tag{10}$$

Theorem. If $Q(x,t)$ evolves according to (6) then the scattering matrix and its elements satisfy the following linear evolution equations

$$\begin{aligned} & i\frac{d\vec{b}^{\pm}}{dt} \pm \lambda^2 \vec{b}^{\pm}(t,\lambda) = 0, \qquad i\frac{d\vec{B}^{\pm}}{dt} \pm \lambda^2 \vec{B}^{\pm}(t,\lambda) = 0, \\ & i\frac{dm_1^{\pm}}{dt} = 0, \qquad i\frac{d\mathbf{T}_{22}^{\pm}}{dt} = 0, \end{aligned} \tag{11}$$

Consequence: MNLS have **infinite** number of integrals of motion. Indeed $m_1^{\pm}(\lambda)$ are generating functionals of the integrals of motion.

Solving MNLS by the Inverse scattering method:

$$\begin{array}{ccccc} \vec{q}(x,t=0) \longrightarrow & L_0 & & L|_{t>0} & \longrightarrow \vec{q}(x,t) \\ & \mathrm{I}\Big\downarrow & & \Big\uparrow\mathrm{III} & \\ & T(0,\lambda) & \xrightarrow{\mathrm{II}} & T(t,\lambda) & \end{array} \tag{12}$$

Important: All steps reduce to **linear** integral equations.

The ISP is reduced to a Riemann-Hilbert problem (RHP) for the fundamental analytic solution (FAS) $\chi^{\pm}(x,t,\lambda)$. Their construction is based on the generalized Gauss decomposition of $T(\lambda, t)$

$$T(\lambda) = T_J^-(\lambda) D_J^+(\lambda) \hat{S}_J^+(\lambda) = T_J^+(\lambda) D_J^-(\lambda) \hat{S}_J^-(\lambda), \qquad (13)$$

Here $S_J^\pm$, $T_J^\pm$ upper- and lower-block-triangular matrices, while $D_J^\pm(\lambda)$ are block-diagonal matrices with the same block structure as $T(\lambda, t)$ above. The explicit expressions of the Gauss factors in terms of the matrix elements of $T(\lambda, t)$ is

$$S_J^+(t,\lambda) = \begin{pmatrix} 1 & \vec{\tau}^{\,+,T} & c_1^+ \\ 0 & \mathbb{1} & s_0 \vec{\tau}^{\,+} \\ 0 & 0 & 1 \end{pmatrix}, \quad S_J^-(t,\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ -\vec{\tau}^{\,-} & \mathbb{1} & 0 \\ c_1^- & -\vec{\tau}^{\,-,T} s_0 & 1 \end{pmatrix},$$

$$\tau^+ = \frac{b^-}{m_1^+}, \qquad \tau^- = \frac{B_1^+}{m_1^-}, \qquad \rho^+ = \frac{b^+}{m_1^+}, \qquad \rho^- = \frac{B_1^-}{m_1^-},$$

$$T_J^+(t,\lambda) = \begin{pmatrix} 1 & -\vec{\rho}^{\,-,T} & c_1^{\prime,-} \\ 0 & \mathbb{1} & -s_0 \vec{\rho}^{\,-} \\ 0 & 0 & 1 \end{pmatrix}, \quad T_J^-(t,\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ \vec{\rho}^{\,+} & \mathbb{1} & 0 \\ c_1^{\prime,+} & \vec{\rho}^{\,+,T} s_0 & 1 \end{pmatrix},$$

$$D_J^+ = \begin{pmatrix} m_1^+ & 0 & 0 \\ 0 & \mathbf{m}_2^+ & 0 \\ 0 & 0 & 1/m_1^+ \end{pmatrix}, \qquad D_J^- = \begin{pmatrix} 1/m_1^- & 0 & 0 \\ 0 & \mathbf{m}_2^- & 0 \\ 0 & 0 & m_1^- \end{pmatrix},$$

and

$$\mathbf{m}_2^+ = \mathbf{T}_{22} + \frac{\vec{b}^+ \vec{b}^{-T}}{m_1^+}, \qquad \mathbf{m}_2^- = \mathbf{T}_{22} + \frac{s_0 \vec{b}^- \vec{b}^{+T} s_0}{m_1^-}.$$

Then the FAS can be defined as:

$$\chi^\pm(x,t,\lambda) = \phi(x,t,\lambda) S_J^\pm(t,\lambda) = \psi(x,t,\lambda) T_J^\mp(t,\lambda) D_J^\pm(\lambda). \tag{14}$$

The FAS for real λ are linearly related

$$\chi^+(x,t,\lambda) = \chi^-(x,t,\lambda) G_J(\lambda,t), \qquad G_{0,J}(\lambda,t) = S_J^-(\lambda,t) S_J^+(\lambda,t). \tag{15}$$

One can rewrite eq. (15) in an equivalent form for the FAS $\xi^\pm(x,t,\lambda) = \chi^\pm(x,t,\lambda) e^{i\lambda J x}$ which satisfy also the relation

$$\lim_{\lambda\to\infty} \xi^\pm(x,t,\lambda) = \mathbb{1}. \tag{16}$$

Then these FAS satisfy

$$\xi^+(x,t,\lambda) = \xi^-(x,t,\lambda) G_J(x,\lambda,t), \qquad G_J(x,\lambda,t) = e^{-i\lambda Jx} G_{0,J}^-(\lambda,t) e^{i\lambda Jx}. \tag{17}$$

Obviously the sewing function $G_j(x,\lambda,t)$ is uniquely determined by the Gauss factors $S_J^{\pm}(\lambda,t)$.

Given the solution $\xi^{\pm}(x,t,\lambda)$ one recovers $Q(x,t)$ via the formula

$$Q(x,t) = \lim_{\lambda\to\infty} \lambda \left(J - \xi^{\pm} J \widehat{\xi}^{\pm}(x,t,\lambda) \right). \tag{18}$$

We impose also the standard reduction:

$$Q(x,t) = \epsilon Q^\dagger(x,t) \qquad \Leftrightarrow \qquad p_k = \epsilon q_k^*.$$

As a consequence we have

$$\vec{\rho}^{\,-}(\lambda,t) = \epsilon \vec{\rho}^{\,+,*}(\lambda,t), \qquad \vec{\tau}^{\,-}(\lambda,t) = \epsilon \vec{\tau}^{\,+,*}(\lambda,t).$$

Zakharov-Shabat dressing method and soliton solutions

Starting from a regular solution $\chi_0^{\pm}(x,t,\lambda)$ of $L_0(\lambda)$ with potential $Q_{(0)}(x,t)$ construct new singular solutions $\chi_1^{\pm}(x,t,\lambda)$ of L with a potential $Q_{(1)}(x,t)$ with two additional singularities located at prescribed positions $\lambda_1^{\pm}$; the reduction $\vec{p} = \vec{q}^*$ ensures that $\lambda_1^- = (\lambda_1^+)^*$. It is related to the regular one by a dressing factor $u(x,t,\lambda)$

$$\chi_1^{\pm}(x,t,\lambda) = u(x,\lambda)\chi_0^{\pm}(x,t,\lambda)u_-^{-1}(\lambda). \qquad u_-(\lambda) = \lim_{x\to-\infty} u(x,\lambda) \quad (19)$$

Note that $u_-(\lambda)$ is a block-diagonal matrix. $u(x,\lambda)$ must satisfy

$$i\partial_x u + Q_{(1)}(x)u - uQ_{(0)}(x) - \lambda[J, u(x,\lambda)] = 0, \quad (20)$$

and the normalization condition $\lim_{\lambda\to\infty} u(x,\lambda) = \mathbb{1}$.

The construction of $u(x,\lambda)$ is based on an appropriate anzats specifying explicitly the form of its λ-dependence:

$$u(x,\lambda) = \mathbb{1} + (c(\lambda)-1)P(x,t) + \left(\frac{1}{c(\lambda)} - 1\right)\overline{P}(x,t), \qquad \overline{P} = S_0^{-1}P^T S_0, \tag{21}$$

where $P(x,t)$ and $\overline{P}(x,t)$ are projectors which satisfy $P\overline{P}(x,t) = 0$.

$$P(x,t) = \frac{|n_1(x,t)\rangle\langle n_1^\dagger(x,t)|}{\langle n_1^\dagger(x,t)|n_1(x,t)\rangle},$$

$$|n_1(x,t)\rangle = \chi_0^+(x,t,\lambda_1^+)|n_{0,1}\rangle, \quad c(\lambda) = \frac{\lambda-\lambda_1^+}{\lambda-\lambda_1^-}, \quad \langle n_{0,1}|S_0|n_{0,1}\rangle = 0. \tag{22}$$

Taking the limit $\lambda \to \infty$ in eq. (28) we get that

$$Q_{(1)}(x,t) - Q_{(0)}(x,t) = (\lambda_1^- - \lambda_1^+)[J, P(x,t) - \overline{P}(x,t)].$$

If $Q_{(0)} = 0$ and put $\lambda_1^\pm = \mu \pm i\nu$, $\chi_0^+(x,t,\lambda) = e^{-i\lambda Jx}$:

$$q_k^{(1s)}(x,t) = -2i\nu\left(P_{1k}(x,t) + (-1)^k P_{\bar{k},2r+1}(x,t)\right), \tag{23}$$

where $\bar{k} = 2r + 2 - k$.

The one-soliton solution reads

$$q_k = \frac{-i\nu e^{-i\mu(x-vt-\delta_0)}}{\cosh 2z + \Delta_0^2} \left(\alpha_k e^{z-i\phi_k} + (-1)^k \alpha_{\bar{k}} e^{-z+i\phi_{\bar{k}}}\right),$$

$$v = \frac{\nu^2 - \mu^2}{\mu}, \qquad u = -2\mu, \qquad z(x,t) = \nu(x - ut - \xi_0), \tag{24}$$

$$\xi_0 = \frac{1}{2\nu} \ln \frac{|n_{0,2r+1}|}{|n_{0,1}|}, \qquad \alpha_k = \frac{|n_{0,k}|}{\sqrt{|n_{0,1}||n_{0,2r+1}|}}, \qquad \Delta_0^2 = \frac{\sum_{k=2}^{2r} |n_{0,k}|^2}{2|n_{0,1} n_{0,2r+1}|},$$

and $\delta_0 = \arg n_{0,1}/\mu = -\arg n_{0,2r+1}/\mu$, $\phi_k = \arg n_{0,k}$. The polarization vectors satisfy the following relation

$$\sum_{k=1}^{r} 2(-1)^{k+1} n_{0,k} n_{0,\bar{k}} + (-1)^r n_{0,r+1}^2 = 0. \tag{25}$$

Thus for $r = 2$ we identify $\Phi_1 = q_2$, $\Phi_0 = q_3/\sqrt{2}$ and $\Phi_3 = q_4$ and we obtain the following solutions for the equation (6)

$$\Phi_{\pm 1} = -\frac{2i\nu\sqrt{\alpha_2\alpha_4}e^{-i\mu(x-vt-\delta_{\pm 1})}}{\cosh 2z + \Delta_0^2}\left(\cos\phi_{\pm 1}\cosh z_{\pm 1} - i\sin\phi_{\pm 1}\sinh z_{\pm 1}\right),$$

$$\delta_{\pm 1} = \delta_0 \mp \frac{\phi_2 - \phi_4}{2\mu}, \qquad \phi_{\pm 1} = \frac{\phi_2 + \phi_4}{2} \qquad z_{\pm 1} = z \mp \frac{1}{2}\ln\frac{\alpha_4}{\alpha_2},$$

$$\Phi_0 = -\frac{\sqrt{2}i\nu\alpha_3 e^{-i\mu(x-vt-\delta_0)}}{\cosh 2z + \Delta_0^2}\left(\cos\phi_3\sinh z - i\sin\phi_3\cosh z\right).$$

For $r = 3$ we identify $\Phi_2 = q_2$, $\Phi_1 = q_3$, $\Phi_0 = q_4$, $\Phi_{-1} = q_5$ and $\Phi_{-2} = q_6$, so that the one-soliton solution for equation (**??**) reads

$$\Phi_{\pm 2} = -\frac{2i\nu\sqrt{\alpha_2\alpha_6}e^{-i\mu(x-vt-\delta_{\pm 2})}}{\cosh 2z + \Delta_0^2}\left(\cos\phi_{\pm 2}\cosh z_{\pm 2} - i\sin\phi_{\pm 2}\sinh z_{\pm 2}\right),$$

$$\Phi_{\pm 1} = -\frac{2i\nu\sqrt{\alpha_3\alpha_5}e^{-i\mu(x-vt-\delta_{\pm 1})}}{\cosh 2z + \Delta_0^2}\left(\cos\phi_{\pm 1}\sinh z_{\pm 1} - i\sin\phi_{\pm 1}\cosh z_{\pm 1}\right),$$

$$\delta_{\pm 2} = \delta_0 \mp \frac{\phi_2 - \phi_6}{2\mu}, \qquad \phi_{\pm 2} = \frac{\phi_2 + \phi_6}{2} \qquad z_{\pm 2} = z \mp \frac{1}{2}\ln\frac{\alpha_6}{\alpha_2},$$

$$\delta_{\pm 1} = \delta_0 \mp \frac{\phi_3 - \phi_5}{2\mu}, \qquad \phi_{\pm 1} = \frac{\phi_3 + \phi_5}{2}, \qquad z_{\pm 1} = z \mp \frac{1}{2} \ln \frac{\alpha_5}{\alpha_3},$$

$$\Phi_0 = -\frac{2i\nu\alpha_4 e^{-i\mu(x - vt - \delta_0)}}{\cosh 2z + \Delta_0^2} (\cos\phi_4 \cosh z - i \sin\phi_4 \sinh z) .$$

Choosing appropriately the polarization vectors $|n\rangle$ we are able to reproduce the soliton solutions obtained by Wadati et al. both for $F = 1$ and $F = 2$ BEC.

Alternative methods and N-soliton solutions

In order to obtain N-soliton solutions one has to apply dressing procedure with a $2N$-poles dressing factor of the form

$$u(x, \lambda) = \mathbb{1} + \sum_{k=1}^{N} \left(\frac{A_k(x)}{\lambda - \lambda_k^+} + \frac{B_k(x)}{\lambda - \lambda_k^-} \right) . \qquad (26)$$

The N-soliton solution itself can be generated via the following formula

$$Q_{N,\mathrm{s}}(x) = \sum_{k=1}^{N} [J, A_k(x) + B_k(x)]. \tag{27}$$

The dressing factor $u(x,\lambda)$ must satisfy the equation

$$i\partial_x u + Q_{N,\mathrm{s}} u - \lambda[J,u] = 0 \tag{28}$$

and the normalization condition $\lim_{\lambda\to\infty} u(x,\lambda) = \mathbb{1}$. The construction of $u(x,\lambda) \in SO(n+2)$ is based on an appropriate anzatz specifying the form of its λ-dependence [?, ?]

The residues of u admit the following decomposition

$$A_k(x) = X_k(x) F_k^T(x), \qquad B_k(x) = Y_k(x) G_k^T(x).$$

where all matrices involved are supposed to be rectangular and of maximal rank s. By comparing the coefficients before the same powers of $\lambda - \lambda_k^{\pm}$ in (28) we convince ourselves that the factors F_k and G_k can be expressed by the fundamental analytic solutions $\chi_0^{\pm}(x,\lambda)$ as follows

$$F_k^T(x) = F_{k,0}^T [\chi_0^+(x,\lambda_k^+)]^{-1}, \qquad G_k^T(x) = G_{k,0}^T [\chi_0^-(x,\lambda_k^-)]^{-1}.$$

The constant rectangular matrices $F_{k,0}$ and $G_{k,0}$ obey the algebraic relations

$$F_{k,0}^T S_0 F_{k,0} = 0, \qquad G_{k,0}^T S_0 G_{k,0} = 0.$$

The other two types of factors X_k and Y_k are solutions to the algebraic system

$$\begin{aligned} S_0 F_k &= X_k \alpha_k + \sum_{l \neq k} \frac{X_l F_l^T S_0 F_k}{\lambda_l^+ - \lambda_k^+} + \sum_{l} \frac{Y_l G_l^T S_0 F_k}{\lambda_l^- - \lambda_k^+}, \\ S_0 G_k &= \sum_{l} \frac{X_l F_l^T S_0 G_k}{\lambda_l^+ - \lambda_k^-} + Y_k \beta_k + \sum_{l \neq k} \frac{Y_l G_l^T S_0 G_k}{\lambda_l^- - \lambda_k^-}. \end{aligned} \tag{29}$$

The square $s \times s$ matrices $\alpha_k(x)$ and $\beta_k(x)$ introduced above depend on χ_0^+ and χ_0^- and their derivatives by λ as follows

$$\begin{aligned} \alpha_k(x) &= -F_{0,k}^T [\chi_0^+(x, \lambda_k^+)]^{-1} \partial_\lambda \chi_0^+(x, \lambda_k^+) S_0 F_{0,k} + \alpha_{0,k}, \\ \beta_k(x) &= -G_{0,k}^T [\chi_0^-(x, \lambda_k^-)]^{-1} \partial_\lambda \chi_0^-(x, \lambda_k^-) S_0 G_{0,k} + \beta_{0,k}. \end{aligned} \tag{30}$$

Below for simplicity we will choose F_k and G_k to be $2r+1$-component vectors. Then one can show that $\alpha_k = \beta_k = 0$ which simplifies the system

(29). We also introduce the following more convenient parametrization for F_k and G_k, namely (see eq. (32)):

$$F_k(x,t) = S_0|n_k(x,t)\rangle = \begin{pmatrix} e^{-z_k+i\phi_k} \\ -\sqrt{2}s_0\vec{\nu}_{0k} \\ e^{z_k-i\phi_k} \end{pmatrix}, \qquad G_k(x,t) = |n_k^*(x,t)\rangle = \begin{pmatrix} e^{z_k+i\phi_k} \\ \sqrt{2}\vec{\nu}_{0k}{}^* \\ e^{-z_k-i\phi_k} \end{pmatrix}, \tag{31}$$

where $\vec{\nu}_{0k}$ are constant $2r-1$-component polarization vectors and

$$\begin{aligned} z_j &= \nu_j(x + 2\mu_j t) + \xi_{00}, \qquad \phi_j = \mu_j x + (\mu_j^2 - \nu_j^2)t + \delta_{00}, \\ \langle n_j^T(x,t)|S_0|n_j(x,t)\rangle &= 0, \qquad \text{or} \qquad (\vec{\nu}_{0,j}s_0\vec{\nu}_{0,j}) = 1. \end{aligned} \tag{32}$$

The polarization vectors automatically satisfy $\langle n_j(x,t)|S_0|n_j(x,t)\rangle = 0$.

Thus for $N = 1$ we get the system:

$$|Y_1\rangle = -\frac{(\lambda_1^+ - \lambda_1^-)|n_1\rangle}{\langle n_1^\dagger|n_1\rangle}, \qquad |X_1\rangle = \frac{(\lambda_1^+ - \lambda_1^-)S_0|n_1^*\rangle}{\langle n_1^\dagger|n_1\rangle}, \tag{33}$$

which is easily solved. As a result for the one-soliton solution we get:

$$\vec{q}_{1\mathrm{s}} = -\frac{i\sqrt{2}(\lambda_1^+ - \lambda_1^-)e^{-i\phi_1}}{\Delta_1} \left(e^{-z_1} s_0 |\vec{\nu}_{01}\rangle + e^{z_1} |\vec{\nu}_{01}^*\rangle \right), \qquad \Delta_1 = \cosh(2z_1) + \langle \vec{\nu}_{01}^\dagger | \vec{\nu}_{01} \rangle. \tag{34}$$

For $n = 3$ we put $\nu_{0k} = |\nu_{0k}| e^{\alpha_{0k}}$ get:

$$\begin{aligned} \Phi_{1\mathrm{s};\pm 1} &= -\frac{\sqrt{2|\nu_{01;1}\nu_{01;3}|}(\lambda_1^+ - \lambda_1^-)}{\Delta_1} e^{-i\phi_1 \pm i\beta_{13}} \\ &\quad \times \left(\cosh(z_1 \mp \zeta_{01}) \cos(\alpha_{13}) - i \sinh(z_1 \mp \zeta_{01}) \sin(\alpha_{13}) \right), \\ \Phi_{1\mathrm{s};0} &= -\frac{\sqrt{2}|\nu_{01;2}|(\lambda_1^+ - \lambda_1^-)}{\Delta_1} e^{-i\phi_1} \\ &\quad \times \left(\sinh z_1 \cos(\alpha_{02}) + i \cosh z_1 \sin(\alpha_{02}) \right), \\ \beta_{13} &= \frac{1}{2}(\alpha_{03} - \alpha_{01}), \qquad \zeta_{01} = \frac{1}{2} \ln \frac{|\nu_{01;3}|}{|\nu_{01;1}|}, \qquad \alpha_{13} = \frac{1}{2}(\alpha_{03} + \alpha_{01}), \end{aligned} \tag{35}$$

Note that the 'center of mass' of $\Phi_{1\mathrm{s};1}$ (resp. of $\Phi_{1\mathrm{s};-1}$) is shifted with respect to the one of $\Phi_{1\mathrm{s};0}$ by ζ_{01} to the right (resp to the left); besides

$|\Phi_{1s;1}| = |\Phi_{1s;-1}|$, i.e. they have the same amplitudes.

For $n=5$ we put $\nu_{0k} = |\nu_{0k}|e^{\alpha_{0k}}$ and get analogously:

$$\Phi_{1s;\pm 2} = -\frac{\sqrt{2|\nu_{01;1}\nu_{01;5}|}(\lambda_1^+ - \lambda_1^-)}{\Delta_1} e^{-i\phi_1 \pm i\beta_{15}}$$
$$\times \left(\cosh(z_1 \mp \zeta_{01})\cos(\alpha_{15}) - i\sinh(z_1 \mp \zeta_{01})\sin(\alpha_{15})\right),$$
$$\Phi_{1s;\pm 1} = \frac{\sqrt{2|\nu_{01;2}\nu_{01;4}|}(\lambda_1^+ - \lambda_1^-)}{\Delta_1} e^{-i\phi_1 \pm i\beta_{24}}$$
$$\times \left(\cosh(z_1 \mp \zeta_{02})\cos(\alpha_{24}) - i\sinh(z_1 \mp \zeta_{01})\sin(\alpha_{24})\right),$$
$$\Phi_{1s;0} = -\frac{\sqrt{2}|\nu_{01;3}|(\lambda_1^+ - \lambda_1^-)}{\Delta_1} e^{-i\phi_1} \left(\cosh z_1 \cos(\alpha_{03}) - i\sinh z_1 \sin(\alpha_{03})\right),$$
$$\beta_{15} = \frac{1}{2}(\alpha_{05} - \alpha_{01}), \qquad \zeta_{01} = \frac{1}{2}\ln\frac{|\nu_{01;5}|}{|\nu_{01;1}|}, \qquad \alpha_{15} = \frac{1}{2}(\alpha_{05} + \alpha_{01}),$$
$$\beta_{24} = \frac{1}{2}(\alpha_{04} - \alpha_{02}), \qquad \zeta_{02} = \frac{1}{2}\ln\frac{|\nu_{01;4}|}{|\nu_{01;2}|}, \qquad \alpha_{24} = \frac{1}{2}(\alpha_{04} + \alpha_{02}), \tag{36}$$

Similarly the ‘center of mass‘ of $\Phi_{1s;2}$ and $\Phi_{1s;1}$ (resp. of $\Phi_{1s;-2}$ and

$\Phi_{1s;-1}$) are shifted with respect to the one of $\Phi_{1s;0}$ by ζ_{01} and ζ_{02} to the right (resp to the left); besides $|\Phi_{1s;2}| = |\Phi_{1s;-2}|$ and $|\Phi_{1s;1}| = |\Phi_{1s;-1}|$.

For $N = 2$ we get:

$$
\begin{aligned}
|n_1(x,t)\rangle &= \frac{X_2(x,t)f_{21}}{\lambda_2^+ - \lambda_1^+} + \frac{Y_1(x,t)\kappa_{11}}{\lambda_1^- - \lambda_1^+} + \frac{Y_2(x,t)\kappa_{21}}{\lambda_2^- - \lambda_1^+}, \\
|n_2(x,t)\rangle &= \frac{X_1(x,t)f_{12}}{\lambda_1^+ - \lambda_2^+} + \frac{Y_1(x,t)\kappa_{12}}{\lambda_1^- - \lambda_2^+} + \frac{Y_2(x,t)\kappa_{22}}{\lambda_2^- - \lambda_2^+}, \\
S_0|n_1^*(x,t)\rangle &= \frac{X_1(x,t)\kappa_{11}}{\lambda_2^+ - \lambda_1^+} + \frac{X_2(x,t)\kappa_{11}}{\lambda_2^+ - \lambda_1^-} + \frac{Y_2(x,t)f_{21}^*}{\lambda_2^- - \lambda_1^-}, \\
S_0|n_2^*(x,t)\rangle &= \frac{X_1(x,t)\kappa_{21}}{\lambda_1^+ - \lambda_2^-} + \frac{X_2(x,t)\kappa_{22}}{\lambda_2^+ - \lambda_2^-} + \frac{Y_1(x,t)f_{12}^*}{\lambda_1^- - \lambda_2^-},
\end{aligned}
\tag{37}
$$

where

$$
\begin{aligned}
\kappa_{kj}(x,t) &= e^{z_k + z_j + i(\phi_k - \phi_j)} + e^{-z_k - z_j - i(\phi_k - \phi_j)} + 2\left(\vec{\nu}^{\,\dagger}_{0k}, \vec{\nu}_{0j}\right), \\
f_{kj}(x,t) &= e^{z_k - z_j - i(\phi_k - \phi_j)} + e^{z_j - z_k + i(\phi_k - \phi_j)} - 2\left(\vec{\nu}_{0k}^T s_0 \vec{\nu}_{0j}\right),
\end{aligned}
\tag{38}
$$

In other words:

$$\mathcal{M}\vec{X} \equiv \begin{pmatrix} 0 & \frac{f_{21}}{\lambda_2^+ - \lambda_1^+} & \frac{\kappa_{11}}{\lambda_1^- - \lambda_1^+} & \frac{\kappa_{21}}{\lambda_2^- - \lambda_1^+} \\ \frac{f_{12}}{\lambda_1^+ - \lambda_2^+} & 0 & \frac{\kappa_{12}}{\lambda_1^- - \lambda_2^+} & \frac{\kappa_{22}}{\lambda_2^- - \lambda_2^+} \\ \frac{\kappa_{11}}{\lambda_1^+ - \lambda_1^-} & \frac{\kappa_{12}}{\lambda_2^+ - \lambda_1^-} & 0 & \frac{f_{21}^*}{\lambda_2^- - \lambda_1^-} \\ \frac{\kappa_{21}}{\lambda_1^+ - \lambda_2^-} & \frac{\kappa_{22}}{\lambda_2^+ - \lambda_2^-} & \frac{f_{12}^*}{\lambda_1^- - \lambda_2^-} & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} |n_1\rangle \\ |n_2\rangle \\ S_0|n_1^*\rangle \\ S_0|n_2^*\rangle \end{pmatrix}. \tag{39}$$

We can rewrite $\mathcal{M}$ in block-matrix form:

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix}, \qquad \mathcal{M}_{22} = \mathcal{M}_{11}^*, \qquad \mathcal{M}_{21} = -\mathcal{M}_{12}^T,$$
$$\mathcal{M}_{11} = \frac{f_{12}}{\lambda_2^+ - \lambda_1^+} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{M}_{12} = \begin{pmatrix} \frac{\kappa_{11}}{\lambda_1^- - \lambda_1^+} & \frac{\kappa_{21}}{\lambda_2^- - \lambda_1^+} \\ \frac{\kappa_{12}}{\lambda_1^- - \lambda_2^+} & \frac{\kappa_{22}}{\lambda_2^- - \lambda_2^+} \end{pmatrix}. \tag{40}$$

The inverse of $\mathcal{M}$ is given by:

$$\mathcal{M}^{-1} = \begin{pmatrix} (\mathcal{M}_{11} - \mathcal{M}_{12}\hat{\mathcal{M}}_{11}^*\mathcal{M}_{21})^{-1} & -(\mathcal{M}_{11} - \mathcal{M}_{12}\hat{\mathcal{M}}_{11}^*\mathcal{M}_{21})^{-1}\mathcal{M}_{12}\hat{\mathcal{M}}_{11}^* \\ -(\mathcal{M}_{11}^* - \mathcal{M}_{21}\hat{\mathcal{M}}_{11}\mathcal{M}_{12})^{-1}\mathcal{M}_{21}\hat{\mathcal{M}}_{11} & (\mathcal{M}_{11}^* - \mathcal{M}_{21}\hat{\mathcal{M}}_{11}\mathcal{M}_{12})^{-1} \end{pmatrix}, \tag{41}$$

One can check by direct calculation that:

$$\begin{aligned}
\mathcal{M}_{11} - \mathcal{M}_{12}\hat{\mathcal{M}}_{11}^{*}\mathcal{M}_{21} &= \frac{f_{12}^{*}}{\lambda_2^- - \lambda_1^-} Z \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
\mathcal{M}_{11}^{*} - \mathcal{M}_{21}\hat{\mathcal{M}}_{11}\mathcal{M}_{12} &= \frac{f_{12}}{\lambda_2^+ - \lambda_1^+} Z \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
Z &= \left(\frac{|f_{12}|^2}{|\lambda_2^+ - \lambda_1^+|^2} - \frac{\kappa_{12}\kappa_{21}}{|\lambda_2^+ - \lambda_1^-|^2} + \frac{\kappa_{11}\kappa_{22}}{4\nu_1\nu_2} \right),
\end{aligned} \tag{42}$$

Finally we get:

$$\mathcal{M}^{-1} = \frac{1}{Z} \begin{pmatrix} 0 & \frac{f_{12}^{*}}{\lambda_1^- - \lambda_2^-} & -\frac{\kappa_{22}}{\lambda_2^+ - \lambda_2^-} & \frac{\kappa_{12}}{\lambda_2^+ - \lambda_1^-} \\ -\frac{f_{12}^{*}}{\lambda_1^- - \lambda_2^-} & 0 & \frac{\kappa_{21}}{\lambda_1^+ - \lambda_2^-} & -\frac{\kappa_{11}}{\lambda_1^+ - \lambda_1^-} \\ \frac{\kappa_{22}}{\lambda_2^+ - \lambda_2^-} & -\frac{\kappa_{21}}{\lambda_1^+ - \lambda_2^-} & 0 & -\frac{f_{12}}{\lambda_1^+ - \lambda_2^+} \\ -\frac{\kappa_{12}}{\lambda_2^+ - \lambda_1^-} & \frac{\kappa_{11}}{\lambda_1^+ - \lambda_1^-} & \frac{f_{12}}{\lambda_2^+ - \lambda_1^+} & 0 \end{pmatrix}, \tag{43}$$

From eqs. (39) and (43) we obtain:

$$
\begin{aligned}
|X_1\rangle &= \frac{1}{Z}\left(\frac{f_{12}^*}{\lambda_1^- - \lambda_2^-}|n_2\rangle - \frac{\kappa_{22}}{\lambda_2^+ - \lambda_2^-}S_0|n_1^*\rangle + \frac{\kappa_{12}}{\lambda_2^+ - \lambda_1^-}S_0|n_2^*\rangle\right), \\
|X_2\rangle &= \frac{1}{Z}\left(-\frac{f_{12}^*}{\lambda_1^- - \lambda_2^-}|n_1\rangle + \frac{\kappa_{21}}{\lambda_1^+ - \lambda_2^-}S_0|n_1^*\rangle - \frac{\kappa_{11}}{\lambda_1^+ - \lambda_1^-}S_0|n_2^*\rangle\right), \\
|Y_1\rangle &= \frac{1}{Z}\left(\frac{\kappa_{22}}{\lambda_2^+ - \lambda_2^-}|n_1\rangle - \frac{\kappa_{21}}{\lambda_1^+ - \lambda_2^-}|n_2\rangle - \frac{f_{12}}{\lambda_1^+ - \lambda_2^+}S_0|n_2^*\rangle\right), \\
|Y_2\rangle &= \frac{1}{Z}\left(-\frac{\kappa_{12}}{\lambda_2^+ - \lambda_1^-}|n_1\rangle + \frac{\kappa_{11}}{\lambda_1^+ - \lambda_1^-}|n_2\rangle + \frac{f_{12}}{\lambda_2^+ - \lambda_1^+}S_0|n_1^*\rangle\right),
\end{aligned}
\tag{44}
$$

Inserting this result into eq. (27) we obtain the following expression for the 2-soliton solution of the MNLS:

$$Q_{2\mathrm{s}}(x,t) = [J, A_1 + B_1 + A_2 + B_2] = \frac{1}{Z}[J, C(x,t) - S_0 C^T(x,t) S_0],$$
$$C(x,t) = \frac{\kappa_{22}}{\lambda_2^+ - \lambda_2^-}|n_1\rangle\langle n_1^\dagger| - \frac{\kappa_{12}}{\lambda_2^+ - \lambda_1^-}|n_1\rangle\langle n_2^\dagger| - \frac{\kappa_{21}}{\lambda_1^+ - \lambda_2^-}|n_2\rangle\langle n_1^\dagger|$$
$$+ \frac{\kappa_{11}}{\lambda_1^+ - \lambda_1^-}|n_2\rangle\langle n_2^\dagger| - \frac{f_{12}^*}{\lambda_1^- - \lambda_2^-}|n_1\rangle\langle n_2|S_0 - \frac{f_{12}}{\lambda_1^+ - \lambda_2^+}S_0|n_2^*\rangle\langle n_1^\dagger|. \quad (45)$$

Two Soliton interactions

$$\kappa_{22} = \begin{cases} e^{2\tau} \exp(\nu_2 z_1/\nu_1) + 2\mathcal{C}_1, & \text{for} \quad \tau \to \infty, \\ e^{-2\tau} \exp(-\nu_2 z_1/\nu_1) + 2\mathcal{C}_1, & \text{for} \quad \tau \to -\infty, \end{cases}$$

$$\kappa_{12} = \begin{cases} e^{\tau} \exp((1+\nu_2/\nu_1) z_1 + i(\phi_1 - \phi_2)) + \mathcal{O}(1), & \text{for} \quad \tau \to \infty, \\ e^{-\tau} \exp(-(1+\nu_2/\nu_1) z_1 - i(\phi_1 - \phi_2)) + \mathcal{O}(1), & \text{for} \quad \tau \to -\infty, \end{cases}$$

$$\kappa_{21} = \begin{cases} e^{\tau} \exp((1+\nu_2/\nu_1) z_1 - i(\phi_1 - \phi_2)) + \mathcal{O}(1), & \text{for} \quad \tau \to \infty, \\ e^{-\tau} \exp(-(1+\nu_2/\nu_1) z_1 + i(\phi_1 - \phi_2)) + \mathcal{O}(1), & \text{for} \quad \tau \to -\infty, \end{cases}$$

$$f_{12} = \begin{cases} e^{\tau} \exp(-(1-\nu_2/\nu_1) z_1 + i(\phi_1 - \phi_2)) + \mathcal{O}(1), & \text{for} \quad \tau \to \infty, \\ e^{-\tau} \exp((1-\nu_2/\nu_1) z_1 - i(\phi_1 - \phi_2)) + \mathcal{O}(1), & \text{for} \quad \tau \to -\infty, \end{cases} \tag{46}$$

After somewhat lengthy calculations we get:

$$\begin{aligned}
\lim_{\tau\to\infty} \vec{q}_{2\mathrm{s}}(x,t) &= -\frac{i\sqrt{2}\nu_1 e^{-i(\phi_1-\alpha_+)}\left(e^{-z_1-r_+} s_0|\vec{\nu}_{01}\rangle + e^{z_1+r_+}|\vec{\nu}^*_{01}\rangle\right)}{\cosh(2(z_1+r_+)) + (\vec{\nu}^\dagger_{01}, \vec{\nu}_{01})}, \\
&= \vec{q}^{(1)}_{1\mathrm{s}}(z_1+r_+, \phi_1-\alpha_+) \\
\lim_{\tau\to-\infty} \vec{q}_{2\mathrm{s}}(x,t) &= \frac{i\sqrt{2}\nu_1 e^{-i(\phi_1+\alpha_+)}\left(e^{-z_1+r_+} s_0|\vec{\nu}_{01}\rangle + e^{z_1-r_+}|\vec{\nu}^*_{01}\rangle\right)}{\cosh(2(z_1-r_+)) + (\vec{\nu}^\dagger_{01}, \vec{\nu}_{01})} \\
&= \vec{q}^{(1)}_{1\mathrm{s}}(z_1-r_+, \phi_1+\alpha_+).
\end{aligned} \tag{47}$$

where

$$r_+ = \ln\left|\frac{\lambda_1^+ - \lambda_2^+}{\lambda_1^+ - \lambda_2^-}\right|, \qquad \alpha_+ = \arg\frac{\lambda_1^+ - \lambda_2^+}{\lambda_1^+ - \lambda_2^-}.$$

The Generalized Fourier Transforms for Non-regular J

We show that the ISM can be viewed as generalized Fourier transform (GFT). We determine explicitly the proper generalizations of the usual exponents. We also introduce a skew–scalar product on $\mathcal{M}$ which provides it with a symplectic structure.

The Wronskian relations

Along with the Lax operator we consider associated systems:

$$i\frac{d\hat{\psi}}{dx} - \hat{\psi}(x,t,\lambda)U(x,t,\lambda) = 0, \qquad U(x,\lambda) = Q(x) - \lambda J, \tag{48}$$

$$i\frac{d\delta\psi}{dx} + \delta U(x,t,\lambda)\psi(x,t,\lambda) + U(x,t,\lambda)\delta\psi(x,t,\lambda) = 0 \tag{49}$$

$$i\frac{d\dot{\psi}}{dx} - \lambda J\psi(x,t,\lambda) + U(x,t,\lambda)\dot{\psi}(x,t,\lambda) = 0 \tag{50}$$

where $\delta\psi$ corresponds to a given variation $\delta Q(x,t)$ of the potential, while by dot we denote the derivative with respect to the spectral parameter.

We start with the identity:

$$(\hat{\chi} J \chi(x,\lambda) - J)|_{x=-\infty}^{\infty} = i \int_{-\infty}^{\infty} dx\, \hat{\chi} [J, Q(x)] \chi(x,\lambda), \qquad (51)$$

where $\chi(x,\lambda)$ can be any fundamental solution of L.

One can use the asymptotics of $\chi^{\pm}(x,\lambda)$ for $x \to \pm\infty$to express the l.h.sides of the Wronskian relations in terms of the scattering data. Then

$$\begin{aligned}
\langle \left(\hat{\chi}^{\pm} J \chi^{\pm}(x,\lambda) - J \right) E_\beta \rangle \big|_{x=-\infty}^{\infty} &= i \int_{-\infty}^{\infty} dx \,\langle \Big([J, Q(x)] \boldsymbol{e}_\beta^{\pm}(x,\lambda) \Big) \rangle, \\
\langle \left(\hat{\chi}'^{,\pm} J \chi'^{,\pm}(x,\lambda) - J \right) E_\beta \rangle \big|_{x=-\infty}^{\infty} &= i \int_{-\infty}^{\infty} dx \,\langle \Big([J, Q(x)] \boldsymbol{e}'^{,\pm}_\beta(x,\lambda) \Big) \rangle,
\end{aligned} \qquad (52)$$

where

$$e_\beta^{\pm}(x,\lambda) = \chi^{\pm} E_\beta \hat{\chi}^{\pm}(x,\lambda), \qquad \boldsymbol{e}_\beta^{\pm}(x,\lambda) = P_{0J}(\chi^{\pm} E_\beta \hat{\chi}^{\pm}(x,\lambda)),$$
$$e_\beta^{\prime,\pm}(x,\lambda) = \chi^{\prime,\pm} E_\beta \hat{\chi}^{\prime,\pm}(x,\lambda), \qquad \boldsymbol{e}_\beta^{\prime,\pm}(x,\lambda) = P_{0J}(\chi^{\prime,\pm} E_\beta \hat{\chi}^{\prime,\pm}(x,\lambda)), \tag{53}$$

are the natural generalization of the 'squared solutions' introduced first for the $sl(2)$-case. By P_{0J} we have denoted the projector $P_{0J} = \mathrm{ad}_J^{-1} \mathrm{ad}_J$ on the block-off-diagonal part of the corresponding matrix-valued function.

The right hand sides of eq. (53) can be written down with the skew–scalar product:

$$[\![X, Y]\!] = \int_{-\infty}^{\infty} dx \langle X(x), [J, Y(x)] \rangle, \tag{54}$$

where $\langle X, Y \rangle$ is the Killing form; in what follows we assume that the Cartan-Weyl generators satisfy $\langle E_\alpha, E_{-\beta} \rangle = \delta_{\alpha,\beta}$ and $\langle H_j, H_k \rangle = \delta_{jk}$. The product is skew-symmetric $[\![X, Y]\!] = -[\![Y, X]\!]$ and is non-degenerate on the space of allowed potentials $\mathcal{M}$. Thus we find

$$\begin{gathered}
\rho_\beta^+ = -i\,[\![Q(x), \boldsymbol{e}_\beta^{\prime,+}]\!], \qquad \rho_\beta^- = -i\,[\![Q(x), \boldsymbol{e}_{-\beta}^{\prime,-}]\!], \\
\tau_\beta^+ = -i\,[\![Q(x), \boldsymbol{e}_{-\beta}^{+}]\!], \qquad \tau_\beta^- = -i\,[\![Q(x), \boldsymbol{e}_{\beta}^{-}]\!], \\
\vec{\rho}^{\,+} = \frac{\vec{b}^+}{m_1^+}, \qquad \vec{\rho}^{\,-} = \frac{\vec{B}^-}{m_1^-}, \qquad \vec{\tau}^{\,+} = \frac{\vec{b}^-}{m_1^+}, \qquad \vec{\tau}^{\,-} = \frac{\vec{B}^+}{m_1^-}.
\end{gathered} \tag{55}$$

Thus the mappings $\mathfrak{F} : Q(x,t) \to \mathfrak{T}_i$ can be viewed as generalized Fourier transform in which $\boldsymbol{e}_\beta^\pm(x,\lambda)$ and $\boldsymbol{e}_\beta^{\prime,\pm}(x,\lambda)$ can be viewed as generalizations of the standard exponentials.

We apply ideas similar to the ones above and get:

$$\begin{gathered}
\delta\rho_\beta^+ = -i\,[\![\mathrm{ad}_J^{-1}\delta Q(x), \boldsymbol{e}_\beta^{\prime,+}]\!], \qquad \delta\rho_\beta^- = i\,[\![\mathrm{ad}_J^{-1}\delta Q(x), \boldsymbol{e}_{-\beta}^{\prime,-}]\!], \\
\delta\tau_\beta^+ = i\,[\![\mathrm{ad}_J^{-1}\delta Q(x), \boldsymbol{e}_{-\beta}^{+}]\!], \qquad \delta\tau_\beta^- = -i\,[\![\mathrm{ad}_J^{-1}\delta Q(x), \boldsymbol{e}_{\beta}^{-}]\!],
\end{gathered} \tag{56}$$

where $\beta \in \Delta_1^+$.

These relations are basic in the analysis of the related NLEE and their Hamiltonian structures. Assume that

$$\delta Q(x,t) = Q_t\delta t + \mathcal{O}((\delta t)^2). \tag{57}$$

Keeping only the first order terms with respect to δt we find:

$$\begin{aligned} \frac{d\rho_\beta^+}{dt} &= -i[\![\operatorname{ad}_J^{-1} Q_t(x), \boldsymbol{e}'^{,+}_{\beta}]\!], \qquad & \frac{d\rho_\beta^-}{dt} &= i[\![\operatorname{ad}_J^{-1} Q_t(x), \boldsymbol{e}'^{,-}_{-\beta}]\!], \\ \frac{d\tau_\beta^+}{dt} &= i[\![\operatorname{ad}_J^{-1} Q_t(x), \boldsymbol{e}^{+}_{-\beta}]\!], \qquad & \frac{d\tau_\beta^-}{dt} &= -i[\![\operatorname{ad}_J^{-1} Q_t(x), \boldsymbol{e}^{-}_{\beta}]\!], \end{aligned} \tag{58}$$

Completeness of the 'squared solutions'

Let us introduce the sets of 'squared solutions'

$$\{\boldsymbol{\Psi}\} = \{\boldsymbol{\Psi}\}_{\rm c} \cup \{\boldsymbol{\Psi}\}_{\rm d}, \qquad \{\boldsymbol{\Phi}\} = \{\boldsymbol{\Phi}\}_{\rm c} \cup \{\boldsymbol{\Phi}\}_{\rm d}, \tag{59}$$

$$\begin{aligned} \{\boldsymbol{\Psi}\}_{\rm c} &\equiv \left\{ \boldsymbol{e}^{+}_{-\alpha}(x,\lambda), \quad \boldsymbol{e}^{-}_{\alpha}(x,\lambda), \quad \lambda \in \mathbb{R}, \quad \alpha \in \Delta_1^+ \right\}, \\ \{\boldsymbol{\Psi}\}_{\rm d} &\equiv \left\{ \boldsymbol{e}^{\pm}_{\mp\alpha;j}(x), \quad \dot{\boldsymbol{e}}^{\pm}_{\mp\alpha;j}(x), \quad \alpha \in \Delta_1^+, \right\}, \end{aligned} \tag{60}$$

$$\begin{aligned} \{\boldsymbol{\Phi}\}_{\rm c} &\equiv \left\{ \boldsymbol{e}^{+}_{\alpha}(x,\lambda), \quad \boldsymbol{e}^{-}_{-\alpha}(x,\lambda), \quad \lambda \in \mathbb{R}, \quad \alpha \in \Delta_1^+ \right\}, \\ \{\boldsymbol{\Phi}\}_{\rm d} &\equiv \left\{ \boldsymbol{e}^{\pm}_{\pm\alpha;j}(x), \quad \dot{\boldsymbol{e}}^{\pm}_{\pm\alpha;j}(x), \quad \alpha \in \Delta_1^+, \right\}, \end{aligned} \tag{61}$$

where $j = 1, \dots, N$ and the subscripts 'c' and 'd' refer to the continuous and discrete spectrum of L, the latter consisting of $2N$ discrete eigenvalues $\lambda_j^\pm \in \mathbb{C}_\pm$.

Theorem 1 (see V.S.G. (1998))**.** *The sets* $\{\boldsymbol{\Psi}\}$ *and* $\{\boldsymbol{\Phi}\}$ *form complete sets of functions in* $\mathcal{M}_J$*. The completeness relation has the form:*

$$\begin{aligned}\delta(x-y)\Pi_{0J} &= \frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda (G_1^+(x,y,\lambda) - G_1^-(x,y,\lambda)) \\ &\quad - 2i\sum_{j=1}^{N} (G_{1,j}^+(x,y) + G_{1,j}^-(x,y)),\end{aligned} \tag{62}$$

$$\begin{aligned}\Pi_{0J} &= \sum_{\alpha\in\Delta_1^+} (E_\alpha \otimes E_{-\alpha} - E_{-\alpha}\otimes E_\alpha), \\ G_1^\pm(x,y,\lambda) &= \sum_{\alpha\in\Delta_1^+} \boldsymbol{e}_{\pm\alpha}^\pm(x,\lambda)\otimes \boldsymbol{e}_{\mp\alpha}^+(y,\lambda),\end{aligned} \tag{63}$$

$$G_{1,j}^\pm(x,y) = \sum_{\alpha\in\Delta_1^+} (\dot{\boldsymbol{e}}_{\pm\alpha;j}^\pm(x)\otimes \boldsymbol{e}_{\mp\alpha;j}^\pm(y) + \boldsymbol{e}_{\pm\alpha;j}^\pm(x)\otimes \dot{\boldsymbol{e}}_{\mp\alpha;j}^\pm(y). \tag{64}$$

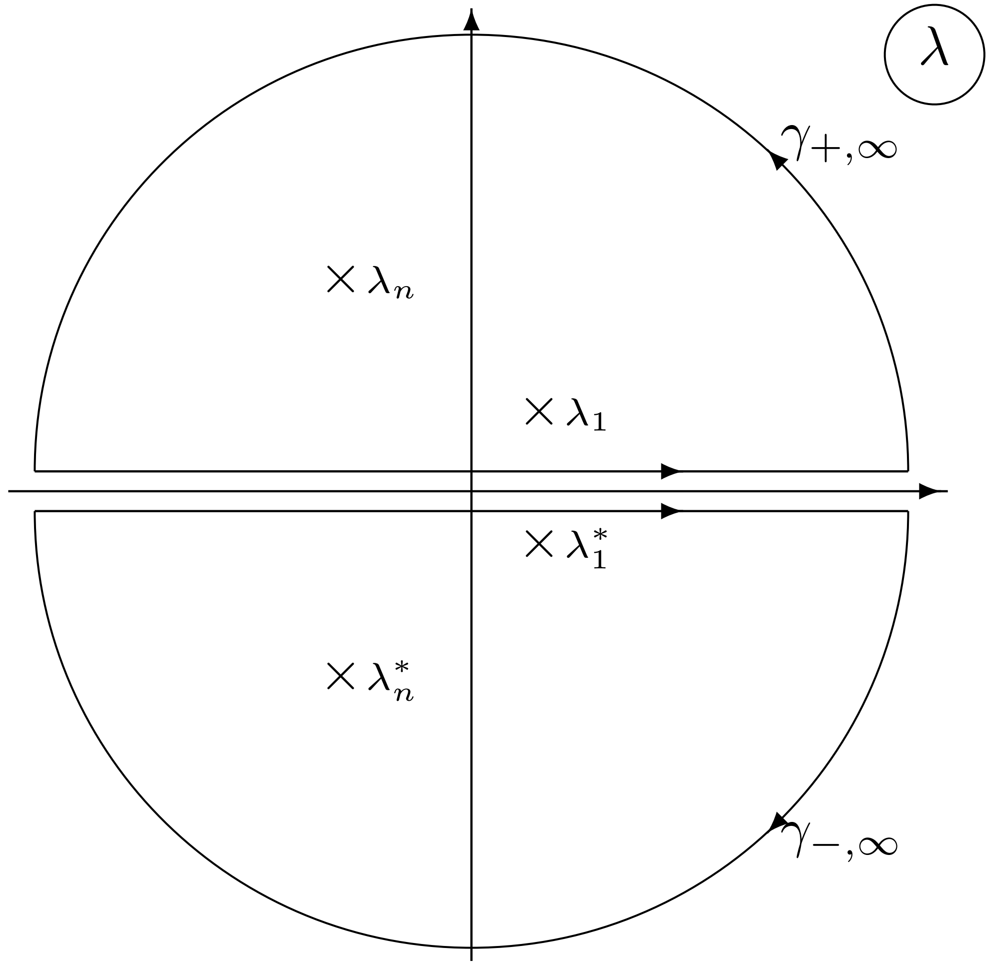
Idea of the proof. Apply the contour integration method to the function

$$
\begin{aligned}
G^{\pm}(x,y,\lambda) &= G_1^{\pm}(x,y,\lambda)\theta(y-x) - G_2^{\pm}(x,y,\lambda)\theta(x-y),\\
G_1^{\pm}(x,y,\lambda) &= \sum_{\alpha\in\Delta_1^+} \boldsymbol{e}_{\pm\alpha}^{\pm}(x,\lambda)\otimes \boldsymbol{e}_{\mp\alpha}^{\pm}(y,\lambda),\\
G_2^{\pm}(x,y,\lambda) &= \sum_{\alpha\in\Delta_0\cup\Delta_1^-} \boldsymbol{e}_{\pm\alpha}^{-}(x,\lambda)\otimes \boldsymbol{e}_{\mp\alpha}^{-}(y,\lambda) + \sum_{j=1}^{r} \boldsymbol{h}_j^{\pm}(x,\lambda)\otimes \boldsymbol{h}_j^{\pm}(y,\lambda),\\
\boldsymbol{h}_j^{\pm}(x,\lambda) &= \chi^{\pm}(x,\lambda)H_j\hat{\chi}^{\pm}(x,\lambda),
\end{aligned}
\tag{65}
$$

and calculate the integral

$$
\mathcal{J}_G(x,y) = \frac{1}{2\pi i}\oint_{\gamma_+} d\lambda G^{+}(x,y,\lambda) - \frac{1}{2\pi i}\oint_{\gamma_-} d\lambda G^{-}(x,y,\lambda), \tag{66}
$$

in two ways: i) via the Cauchy residue theorem and ii) integrating along the contours. □



Фигура 1: The contours $\gamma_{\pm} = \mathbb{R} \cup \gamma_{\pm\infty}$.

Remark 1. *There is a dual completeness relation for the 'squared solutions' obtained by replacing all* $e^{\pm}_{\alpha}(x,\lambda)$ *with* $e'^{,\pm}_{\alpha}(x,\lambda)$.

Expansions of $Q(x)$ and $\mathrm{ad}_J^{-1}\delta Q(x)$.

$$\begin{aligned} Q(x) = &-\frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \left(\tau^+_{\alpha}(\lambda)\boldsymbol{e}^+_{\alpha}(x,\lambda) - \tau^-_{\alpha}(\lambda)\boldsymbol{e}^-_{-\alpha}(x,\lambda)\right) \\ &- 2\sum_{j=1}^{N}\sum_{\alpha\in\Delta_1^+}\left(\operatorname*{Res}_{\lambda=\lambda_j^+} \tau^+_{\alpha}\boldsymbol{e}^+_{\alpha}(x,\lambda) + \operatorname*{Res}_{\lambda=\lambda_j^-} \tau^-_{\alpha}\boldsymbol{e}^-_{-\alpha}(x,\lambda)\right), \end{aligned} \tag{67}$$

$$\begin{aligned} Q(x) = &\frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \left(\rho^+_{\alpha}(\lambda)\boldsymbol{e}'^{,+}_{-\alpha}(x,\lambda) - \rho^-_{\alpha}(\lambda)\boldsymbol{e}'^{,-}_{\alpha}(x,\lambda)\right) \\ &+ 2\sum_{j=1}^{N}\sum_{\alpha\in\Delta_1^+}\left(\operatorname*{Res}_{\lambda=\lambda_j^+} \rho^+_{\alpha}\boldsymbol{e}'^{,+}_{\alpha}(x,\lambda) + \operatorname*{Res}_{\lambda=\lambda_j^-} \rho^-_{\alpha}\boldsymbol{e}'^{,-}_{\alpha}(x,\lambda)\right), \end{aligned} \tag{68}$$

$$
\begin{aligned}
\mathrm{ad}\,_J^{-1}\delta Q(x) &= \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \left(\delta\tau_\alpha^+(\lambda)\boldsymbol{e}_\alpha^+(x,\lambda) + \delta\tau_\alpha^-(\lambda)\boldsymbol{e}_{-\alpha}^-(x,\lambda)\right) \\
&\quad + 2\sum_{j=1}^{N}\sum_{\alpha\in\Delta_1^+}\left(\operatorname*{Res}_{\lambda=\lambda_j^+}\delta\tau_\alpha^+\boldsymbol{e}_\alpha^+(x,\lambda) - \operatorname*{Res}_{\lambda=\lambda_j^-}\delta\tau_\alpha^-\boldsymbol{e}_{-\alpha}^-(x,\lambda)\right),
\end{aligned} \tag{69}
$$

$$
\begin{aligned}
\mathrm{ad}\,_J^{-1}\delta Q(x) &= \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \left(\delta\rho_\alpha^+(\lambda)\boldsymbol{e}_{-\alpha}^{\prime,+}(x,\lambda) + \delta\rho_\alpha^-(\lambda)\boldsymbol{e}_\alpha^{\prime,-}(x,\lambda)\right) \\
&\quad - 2\sum_{j=1}^{N}\sum_{\alpha\in\Delta_1^+}\left(\operatorname*{Res}_{\lambda=\lambda_j^+}\delta\rho_\alpha^+\boldsymbol{e}_{-\alpha}^{\prime,+}(x,\lambda) - \operatorname*{Res}_{\lambda=\lambda_j^-}\delta\rho_\alpha^-\boldsymbol{e}_\alpha^{\prime,-}(x,\lambda)\right).
\end{aligned} \tag{70}
$$

These expansions combined with the proposition above give another way to establish the one-to-one correspondence between $Q(x)$ and each of the minimal sets of scattering data $\mathcal{T}_1$ and $\mathcal{T}_2$.

$$
\begin{aligned}
\operatorname{ad}_J^{-1}\frac{dQ}{dt} &= \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+}\left(\frac{d\tau_\alpha^+}{dt}\boldsymbol{e}_\alpha^+(x,\lambda) + \frac{d\tau_\alpha^-}{dt}\boldsymbol{e}_{-\alpha}^-(x,\lambda)\right) \\
&\quad + 2\sum_{j=1}^{N}\sum_{\alpha\in\Delta_1^+}\left(\operatorname*{Res}_{\lambda=\lambda_j^+}\frac{d\tau_\alpha^+}{dt}\boldsymbol{e}_\alpha^+(x,\lambda) - \operatorname*{Res}_{\lambda=\lambda_j^-}\frac{d\tau_\alpha^-}{dt}\boldsymbol{e}_{-\alpha}^-(x,\lambda)\right),
\end{aligned}
\tag{71}
$$

$$
\begin{aligned}
\operatorname{ad}_J^{-1}\frac{dQ}{dt} &= \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+}\left(\frac{d\rho_\alpha^+}{dt}\boldsymbol{e}_{-\alpha}^{\prime,+}(x,\lambda) + \frac{d\rho_\alpha^-}{dt}\boldsymbol{e}_{\alpha}^{\prime,-}(x,\lambda)\right) \\
&\quad - 2\sum_{j=1}^{N}\sum_{\alpha\in\Delta_1^+}\left(\operatorname*{Res}_{\lambda=\lambda_j^+}\frac{d\rho_\alpha^+}{dt}\boldsymbol{e}_{-\alpha}^{\prime,+}(x,\lambda) - \operatorname*{Res}_{\lambda=\lambda_j^-}\frac{d\rho_\alpha^-}{dt}\boldsymbol{e}_{\alpha}^{\prime,-}(x,\lambda)\right).
\end{aligned}
\tag{72}
$$

The generating operators

Introduce the generating operators $\Lambda_\pm$ through:

$$\begin{aligned} (\Lambda_+ - \lambda)\boldsymbol{e}^+_{-\alpha}(x,\lambda) &= 0, \qquad (\Lambda_+ - \lambda)\boldsymbol{e}^-_{\alpha}(x,\lambda) = 0, \\ (\Lambda_- - \lambda)\boldsymbol{e}^+_{\alpha}(x,\lambda) &= 0, \qquad (\Lambda_- - \lambda)\boldsymbol{e}^-_{-\alpha}(x,\lambda) = 0. \end{aligned} \tag{73}$$

Their derivation starts by introducing the splitting:

$$e^\pm_\alpha(x,\lambda) = e^{\mathrm{d},\pm}_\alpha(x,\lambda) + \boldsymbol{e}^\pm_\alpha(x,\lambda), \qquad e^{\mathrm{d},\pm}_\alpha(x,\lambda) = (\mathbb{1} - P_{0J})e^\pm_\alpha(x,\lambda), \tag{74}$$

into the equation

$$i\frac{de_\alpha}{dx} + [Q(x) - \lambda J, e_\alpha(x,\lambda)] = 0. \tag{75}$$

which is obviously satisfied by the 'squared solutions'. Then eq. (75) splits into:

$$i\frac{de^{\mathrm{d},\pm}_\alpha}{dx} + [Q(x), \boldsymbol{e}^\pm_\alpha(x,\lambda)] \;=\; 0, \tag{76}$$

$$i\frac{d\boldsymbol{e}_{\alpha}^{\pm}}{dx} + [Q(x), e_{\alpha}^{\mathrm{d},\pm}(x,\lambda)] = \lambda[J, \boldsymbol{e}_{\alpha}^{\pm}(x,\lambda)], \tag{77}$$

Eq. (76) can be integrated formally with the result

$$e_{\alpha}^{\mathrm{d},\pm}(x,\lambda) = C_{\alpha;\epsilon}^{\mathrm{d},\pm}(\lambda) + i\int_{\epsilon\infty}^{x} dy\,[Q(y), \boldsymbol{e}_{\alpha}^{\pm}(y,\lambda)], \tag{78}$$

$$C_{\alpha;\epsilon}^{\mathrm{d},\pm}(\lambda) = \lim_{y\to\epsilon\infty} e_{\alpha}^{\mathrm{d},\pm}(y,\lambda), \qquad \epsilon = \pm 1. \tag{79}$$

Next insert (78) into (77) and act on both sides by ad_J^{-1}. This gives us:

$$(\Lambda_{\pm} - \lambda)\boldsymbol{e}_{\alpha}^{\pm}(x,\lambda) = i[C_{\alpha;\epsilon}^{\mathrm{d},\pm}(\lambda), \mathrm{ad}_J^{-1} Q(x)], \tag{80}$$

where the generating operators $\Lambda_{\pm}$ are given by:

$$\Lambda_{\pm} X(x) \equiv \mathrm{ad}_J^{-1}\left(i\frac{dX}{dx} + i\left[Q(x), \int_{\pm\infty}^{x} dy\,[Q(y), X(y)]\right]\right). \tag{81}$$

$$(\Lambda_{+} - \lambda)\boldsymbol{e}_{-\alpha}^{+}(x,\lambda) = 0, \qquad (\Lambda_{+} - \lambda)\boldsymbol{e}_{\alpha}^{-}(x,\lambda) = 0, \tag{82}$$

$$(\Lambda_- - \lambda)\boldsymbol{e}^+_\alpha(x,\lambda) = 0, \qquad (\Lambda_- - \lambda)\boldsymbol{e}^-_{-\alpha}(x,\lambda) = 0, \tag{83}$$

Thus the sets $\{\Psi\}$ and $\{\Phi\}$ are the complete sets of eigen- and adjoint functions of Λ_+ and Λ_-.

Fundamental properties of the MNLS equations

The principal class of NLEE

By principle class of NLEE we mean the ones whose dispersion laws take the form:

$$F(\lambda) = f(\lambda)J, \tag{84}$$

where $f(\lambda)$ may be rational functions of λ whose poles lie outside the spectrum of L. The corresponding NLEE is

$$i\mathrm{ad}\,_J^{-1} Q_t + f(\Lambda_\pm)Q(x,t) = 0. \tag{85}$$

Theorem 2. *The NLEE (85) are equivalent to: i) the equations (11) and ii) to the following evolution equations for the generalized Gauss factors of $T(\lambda)$:*

$$i\frac{dS_J^+}{dt} + [F(\lambda), S_J^+] = 0, \qquad i\frac{dT_J^-}{dt} + [F(\lambda), T_J^-] = 0, \tag{86}$$

and

$$i\frac{dS_J^-}{dt} + [F(\lambda), S_J^-] = 0, \qquad i\frac{dT_J^+}{dt} + [F(\lambda), T_J^+] = 0. \tag{87}$$

The integrals of motion Hamiltonian properties of the MNLS eqs.

The block-diagonal Gauss factors $D_J^\pm(\lambda)$ are generating functionals of the integrals of motion. The principal series of integrals is generated by $m_1^\pm(\lambda)$:

$$\pm \ln m_1^\pm = \sum_{k=1}^{\infty} I_k \lambda^{-k}. \tag{88}$$

Let us outline a way to calculate their densities as functionals of $Q(x,t)$. Use a third type of Wronskian identities involving $\dot{\chi}^{\pm}(x,\lambda)$. They have the form:

$$\left(\hat{\chi}^{\pm}\dot{\chi}^{\pm}(x,\lambda) + iJx\right)\Big|_{x=-\infty}^{\infty} = -i\int_{-\infty}^{\infty} dx\ \left(\hat{\chi}J\chi(x,\lambda) - J\right), \qquad (89)$$

which gives

$$\pm\frac{d}{d\lambda}\ln m_1^{\pm}(\lambda) = -i\int_{-\infty}^{\infty} dx\,(\langle\chi(x,\lambda)J\hat{\chi}J\rangle - 1). \qquad (90)$$

Note that in the integrand of the above equation we have in fact $\langle h_1^{\pm}(x,\lambda)J\rangle$. Splitting $h_1^{\pm}(x,\lambda) = h_1^{d,\pm}(x,\lambda) + \boldsymbol{h}_1^{\pm}(x,\lambda)$ into 'block-diagonal' and 'block-off-diagonal' parts we get

$$\begin{aligned}(\Lambda_+ - \lambda)\boldsymbol{h}_1^{\pm}(x,\lambda) &= i\left[\lim_{y\to\pm\infty} h_1^{d,\pm}(x,\lambda), \operatorname{ad}_J^{-1}Q(x)\right] \\ &= i[J, \operatorname{ad}_J^{-1}Q(x)] \equiv Q(x),\end{aligned} \qquad (91)$$

i.e.

$$\begin{aligned} (\Lambda_\pm - \lambda)\boldsymbol{h}_1^\pm(x,\lambda) &= Q(x), \\ h_1^{d,\pm}(x,\lambda) &= J + \int_{\pm\infty}^x dy\,[Q(y), \boldsymbol{h}_1^\pm(x,\lambda)]. \end{aligned} \tag{92}$$

Using eq. (92) and inverting formally the operator $(\Lambda_\pm - \lambda)$ we obtain the relations:

$$\begin{aligned} \pm\frac{d}{d\lambda}\ln m_1^\pm(\lambda) &= -i\int_{-\infty}^{\infty} dx\,\left(\left\langle J + \int_{\pm\infty}^x dy\,[Q(y), \boldsymbol{h}_1^\pm(x,\lambda)], J\right\rangle - 1\right) \\ &= -i\int_{-\infty}^{\infty} dx\,\int_{\pm\infty}^x dy\,\left\langle [J, Q(y)], \boldsymbol{h}_1^\pm(x,\lambda)\right\rangle \\ &= -i\int_{-\infty}^{\infty} dx\,\int_{\pm\infty}^x dy\,\left\langle [J, Q(y)], (\Lambda_\pm - \lambda)^{-1}Q(x)\right\rangle. \end{aligned} \tag{93}$$

This procedure allows us to express the integrals of motion as functionals of $Q(x)$ in compact form:

$$I_s = \frac{1}{s} \int_{-\infty}^{\infty} dx \int_{\pm\infty}^{x} dy \, \left\langle [J, Q(y)], \Lambda_{\pm}^{s} Q(x) \right\rangle . \tag{94}$$

Note: the operators Λ_+ and Λ_- produce the same integrals of motion.

Using the explicit form of $\Lambda_\pm$ we find that:

$$\begin{aligned}
\Lambda_\pm Q &= i\mathrm{ad}\,_J^{-1} \frac{dQ}{dx} = i\frac{dQ^+}{dx} - i\frac{dQ^-}{dx}, \\
\Lambda_\pm^2 Q &= -\frac{d^2 Q}{dx^2} + \left[Q^+ - Q^-, [Q^+, Q^-] \right], \\
\Lambda_\pm^3 Q &= -i\frac{d^3 Q^+}{dx^3} + i\frac{d^3 Q^-}{dx^3} + 3i\left[Q^+, [Q_x^+, Q^-] \right] + 3i\left[Q^-, [Q^+, Q_x^-] \right],
\end{aligned} \tag{95}$$

$$Q^+(x,t) = (\vec{q}(x,t) \cdot \vec{E}_1^+), \qquad Q^-(x,t) = (\vec{p}(x,t) \cdot \vec{E}_1^-).$$

Thus for the first three integrals of motion we get:

$$
\begin{aligned}
I_1 &= -i\int_{-\infty}^{\infty} dx\, \langle Q^+(x), Q^-(x)\rangle, \\
I_2 &= \frac{1}{2}\int_{-\infty}^{\infty} dx\, \left(\langle Q^+_x(x), Q^-(x)\rangle - \langle Q^+(x), Q^-_x(x)\rangle\right), \\
I_3 &= i\int_{-\infty}^{\infty} dx\, \left(-\langle Q^+_x(x), Q^-_x(x)\rangle + \frac{1}{2}\langle [Q^+(x), Q^-(x)], [Q^+(x), Q^-(x)]\rangle\right).
\end{aligned}
\tag{96}
$$

iI_1 – is the density of the particles, I_2 is the momentum and $-iI_3$ is the Hamiltonian of the MNLS equations. Indeed, taking $H_{(0)} = -iI_3$ with the Poissson brackets

$$
\{q_k(y,t), p_j(x,t)\} = i\delta_{kj}\delta(x-y), \tag{97}
$$

coincide with the MNLS equations (). The above Poisson brackets are dual to the canonical symplectic form:

$$
\Omega_0 = i\int_{-\infty}^{\infty} dx\, \mathrm{tr}\,\left(\delta\vec{p}(x)\wedge\delta\vec{q}(x)\right)
$$

$$= \frac{1}{i} \int_{-\infty}^{\infty} dx \, \mathrm{tr} \, \left(\mathrm{ad}_J^{-1} \delta Q(x) \wedge [J, \mathrm{ad}_J^{-1} \delta Q(x)] \right) \qquad (98)$$

$$= \frac{1}{i} [\![\mathrm{ad}_J^{-1} \delta Q(x) \underset{,}{\wedge} \mathrm{ad}_J^{-1} \delta Q(x)]\!], \qquad (99)$$

The last expression for Ω_0 is preferable to us because it makes obvious the interpretation of $\delta Q(x,t)$ as local coordinate on the co-adjoint orbit passing through J. It can be evaluated in terms of the scattering data variations.

$$\Omega_0 = \frac{1}{\pi i} \int_{-\infty}^{\infty} d\lambda \, \left(\Omega_0^+(\lambda) - \Omega_0^-(\lambda) \right) - 2 \sum_{j=1}^{N} \left(\operatorname*{Res}_{\lambda=\lambda_j^+} \Omega_0^+(\lambda) + \operatorname*{Res}_{\lambda=\lambda_j^-} \Omega_0^-(\lambda) \right),$$

$$\Omega_0^{\pm}(\lambda) = \sum_{\alpha,\gamma \in \Delta_1^+} \delta\tau^{\pm}(\lambda) D_{\alpha,\gamma}^{\pm} \wedge \delta\rho_{\gamma}^{\pm}, \qquad D_{\alpha,\gamma}^{\pm} = \left\langle \hat{D}^{\pm} E_{\mp\gamma} D^{\pm}(\lambda) E_{\pm\alpha} \right\rangle,$$

Hierarchy of Hamiltonian formulations of MNLS:

$$\Omega_k = \frac{1}{i} [\![\mathrm{ad}_J^{-1} \delta Q \underset{,}{\wedge} \Lambda^k \mathrm{ad}_J^{-1} \delta Q]\!], \qquad \Lambda = \frac{1}{2} (\Lambda_+ + \Lambda_-), \qquad (100)$$

$$H_k = i^{k+3} I_{k+3}. \tag{101}$$

We can also calculate Ω_k in terms of the scattering data variations. Doing this we will need also eqs. (82) and (83). The answer is

$$\Omega_k = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\lambda \, \lambda^k \left(\Omega_0^+(\lambda) - \Omega_0^-(\lambda) \right) - i \sum_{j=1}^{N} \left(\Omega_{k,j}^+ + \Omega_{k;j}^- \right), \tag{102}$$

$$\Omega_{k,j}^{\pm} = \operatorname*{Res}_{\lambda=\lambda_j^{\pm}} \lambda^k \Omega_0^{\pm}(\lambda). \tag{103}$$

This allows one to prove that if we are able to cast Ω_0 in canonical form then all Ω_k will also be cast in canonical form and will be pair-wise equivalent.

Modeling Soliton Interactions of the perturbed vector nonlinear Schrödinger equation

The idea of the adiabatic approximation to the soliton interactions - Karpman (1980)
Modeling of the N-soliton trains of the perturbed NLS eq.:

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u(x,t) = iR[u]. \qquad (104)$$

N-soliton train

$$\begin{aligned} u(x,t=0) &= \sum_{k=1}^{N} \vec{u}_k(x,t=0), & u_k(x,t) &= \frac{2\nu_k e^{i\phi_k}}{\cosh(z_k)}, \\ z_k &= 2\nu_k(x-\xi_k(t)), & \xi_k(t) &= 2\mu_k t + \xi_{k,0}, \\ \phi_k &= \frac{\mu_k}{\nu_k} z_k + \delta_k(t), & \delta_k(t) &= 2(\mu_k^2+\nu_k^2)t + \delta_{k,0}. \end{aligned} \qquad (105)$$

Adiabatic approximation holds true if:

$$|\nu_k - \nu_0| \ll \nu_0, \quad |\mu_k - \mu_0| \ll \mu_0, \quad |\nu_k - \nu_0||\xi_{k+1}$$

$$\nu_0 = \frac{1}{N}\sum_{k=1}^{N} \nu_k, \qquad \mu_0 = \frac{1}{N}\sum_{k=1}^{N} \mu_k$$

Two different scales:

$$|\nu_k - \nu_0| \simeq \varepsilon_0^{1/2}, \qquad |\mu_k - \mu_0| \simeq \varepsilon_0^{1/2}, \qquad |\xi_{k+1,0} - \xi_{k,0}| \simeq \varepsilon_0^{-1}.$$

Consider perturbation by external potentials:

$$iR[u] = (V_2x^2 + V_1x + V_0 + A\cos(\Omega x + \Omega_0))u(x,t), \qquad V_2 > 0. \quad (108)$$

Perturbed CTC model (VSG et al (1996)):

$$\begin{aligned} \frac{d\lambda_k}{dt} &= -4\nu_0\left(e^{Q_{k+1}-Q_k} - e^{Q_k - Q_{k-1}}\right) + M_k + iN_k, \\ \frac{dQ_k}{dt} &= -4\nu_0\lambda_k + 2i(\mu_0 + i\nu_0)\Xi_k - iX_k, \\ \lambda_k &= \mu_k + i\nu_k, \qquad X_k = 2\mu_k\Xi_k + D_k \end{aligned} \quad (109)$$

$$Q_k = -2\nu_0\xi_k + k\ln 4\nu_0^2 - i(\delta_k + \delta_0 + k\pi - 2\mu_0\xi_k),$$
$$\nu_0 = \frac{1}{N}\sum_{s=1}^{N}\nu_s, \qquad \mu_0 = \frac{1}{N}\sum_{s=1}^{N}\mu_s, \qquad \delta_0 = \frac{1}{N}\sum_{s=1}^{N}\delta_s. \tag{110}$$

$$N_k = 0, \quad M_k = -V_2\xi_k - \frac{V_1}{2} + \frac{\pi A\Omega^2}{8\nu_k \sinh Z_k}\sin(\Omega\xi_k + \Omega_0), \quad \Xi_k = 0,$$
$$D_k = V_2\left(\frac{\pi^2}{48\nu_k^2} - \xi_k^2\right) - V_1\xi_k - V_0 - \frac{\pi^2 A\Omega^2}{16\nu_k^2}\frac{\cosh Z_k}{\sinh^2 Z_k}\cos(\Omega\xi_k + \Omega_0),$$
$$Z_k = \Omega\pi/(4\nu_k). \tag{111}$$

Perturbed vector NLS:

$$i\vec{u}_t + \frac{1}{2}\vec{u}_{xx} + (\vec{u}^\dagger, \vec{u})\vec{u}(x,t) = iR[\vec{u}]. \tag{112}$$

Vector N-soliton train:

$$\vec{u}(x,t=0)=\sum_{k=1}^{N}\vec{u}_k(x,t=0), \qquad \vec{u}_k(x,t)=\frac{2\nu_k e^{i\phi_k}}{\cosh(z_k)}\vec{n}_k,$$
$$z_k=2\nu_k(x-\xi_k(t)), \qquad \xi_k(t)=2\mu_k t+\xi_{k,0},$$
$$\phi_k=\frac{\mu_k}{\nu_k}z_k+\delta_k(t), \qquad \delta_k(t)=2(\mu_k^2+\nu_k^2)t+\delta_{k,0}. \tag{113}$$

$$(\vec{n}_k^\dagger,\vec{n}_k)=1, \qquad \sum_{s=1}^{n}\arg\vec{n}_{k;s}=0.$$

Variational approach and PCTC for PVNLS and generalized CTC

$$\begin{aligned} \mathcal{L}[\vec{u}] &= \int_{-\infty}^{\infty} dt \, \frac{i}{2} \left[(\vec{u}\dagger, \vec{u}_t) - (\vec{u}_t^\dagger, \vec{u}) \right] - H, \\ H[\vec{u}] &= \int_{-\infty}^{\infty} dx \, \left[-\frac{1}{2} (\vec{u}_x^\dagger, \vec{u}_x) + \frac{1}{2} (\vec{u}^\dagger, \vec{u})^2 - (\vec{u}^\dagger, \vec{u}) V(x) \right]. \end{aligned} \tag{114}$$

Then the Lagrange equations of motion:

$$\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \vec{u}_t^\dagger} - \frac{\delta \mathcal{L}}{\delta \vec{u}\dagger} = 0, \tag{115}$$

coincide with the vector NLS with external potential $V(x)$.

Insert $\vec{u}(x,t) = \sum_{k=1}^{N} \vec{u}_k(x,t)$ and integrate over x neglecting all terms of order ϵ and higher. Assume that at $t = 0$

$$\xi_1 < \xi_2 < \cdots < \xi_N$$

$$\int_{-\infty}^{\infty} dx\ (\vec{u}^{\dagger}_{k,x}, \vec{u}_{p,x}), \qquad \int_{-\infty}^{\infty} dx\ (\vec{u}^{\dagger}_{k}, \vec{u}_{p}), \qquad \int_{-\infty}^{\infty} dx\ (\vec{u}^{\dagger}_{k}, \vec{u}_{p}) V(x), \tag{116}$$

with $|p-k| \geq 2$ can be neglected. The same holds true also for the integrals

$$\int_{-\infty}^{\infty} dx\ (\vec{u}^{\dagger}_{k}, \vec{u}_{p})(\vec{u}^{\dagger}_{s}, \vec{u}_{l}),$$

where at least three of the indices k, p, s, l have different values.

Thus after long calculations we obtain:

$$\begin{aligned}
&\mathcal{L} = \sum_{k=1}^{N} \mathcal{L}_k + \sum_{k=1}^{N} \sum_{n=k\pm 1} \widetilde{\mathcal{L}}_{k,n}, \qquad && \mathcal{L}_{k,n} = 16\nu_0^3 e^{-\Delta_{k,n}} (R_{k,n} + R^*_{k,n}), \\
&R_{k,n} = e^{i(\widetilde{\delta}_n - \widetilde{\delta}_k)} (\vec{n}^{\dagger}_k \vec{n}_n), && \widetilde{\delta}_k = \delta_k - 2\mu_0 \xi_k, \\
&\Delta_{k,n} = 2 s_{k,n} \nu_0 (\xi_k - \xi_n) \gg 1, && s_{k,k+1} = -1, \qquad s_{k,k-1} = 1.
\end{aligned} \tag{117}$$

$$
\begin{aligned}
\mathcal{L}_k = & -2i\nu_k \left((\vec{n}^{\dagger}_{k,t}, \vec{n}_k) - (\vec{n}^{\dagger}_k, \vec{n}_{k,t}) \right) + 8\mu_k \nu_k \frac{d\xi_k}{dt} \\
& - 4\nu_k \frac{d\delta_k}{dt} - 8\mu_k^2 \nu_k + \frac{8\nu_k^3}{3} + 2\pi \nu_k V_0 + \frac{\pi^3}{8\nu_k} V_2 + \frac{\pi A \cos(\Omega_0)}{2\cosh(Z_k)}
\end{aligned} \tag{118}
$$

The equations of motion are given by:

$$
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta p_{k,t}} - \frac{\delta \mathcal{L}}{\delta p_k} = 0, \tag{119}
$$

where p_k stands for one of the soliton parameters: δ_k, ξ_k, μ_k, ν_k and $\vec{n}^{\dagger}_k$. The corresponding system is a generalization of CTC:

$$
\begin{aligned}
\frac{d\lambda_k}{dt} &= -4\nu_0 \left(e^{Q_{k+1}-Q_k} (\vec{n}^{\dagger}_{k+1}, \vec{n}_k) - e^{Q_k - Q_{k-1}} (\vec{n}^{\dagger}_k, \vec{n}_{k-1}) \right) + M_k + iN_k, \\
\frac{dQ_k}{dt} &= -4\nu_0 \lambda_k + 2i(\mu_0 + i\nu_0)\Xi_k - iX_k, \qquad \frac{d\vec{n}_k}{dt} = \mathcal{O}(\epsilon),
\end{aligned} \tag{120}
$$

Additional equations describing the evolution of the polarization vectors. But we can replace $(\vec{n}^{\dagger}_{k+1}, \vec{n}_k)$ by their initial values

$$
(\vec{n}^{\dagger}_{k+1}, \vec{n}_k)\Big|_{t=0} = m_{0k}^2 e^{2i\phi_{0k}}, \qquad k = 1, \dots, N-1 \tag{121}
$$

Effects of the polarization vectors on the soliton interaction

The CTC is completely integrable model; it allows Lax representation $L_t = [A.L]$, where:

$$L = \sum_{s=1}^{N} (b_s E_{ss} + a_s(E_{s,s+1} + E_{s+1,s})) , \quad A = \sum_{s=1}^{N} (a_s(E_{s,s+1} - E_{s+1,s})) ,$$
$$a_s = \exp((Q_{s+1} - Q_s)/2), \qquad b_s = \mu_{s,t} + i\nu_{s,t}, \qquad (E_{ks})_{pj} = \delta_{kp}\delta_{sj} \tag{122}$$

The eigenvalues of L $\zeta_s = \kappa_s + i\eta_s$ are integrals of motion and κ_s determine the asymptotic velocities of CTC.

The GCTC is also a completely integrable model; its allows Lax representation $\tilde{L}_t = [\tilde{A}.\tilde{L}]$, where:

$$\tilde{L} = \sum_{s=1}^{N} \left(\tilde{b}_s E_{ss} + \tilde{a}_s (E_{s,s+1} + E_{s+1,s}) \right), \quad A = \sum_{s=1}^{N} \left(\tilde{a}_s (E_{s,s+1} - E_{s+1,s}) \right),$$
$$\tilde{a}_s = m_{0k}^2 e^{2i\phi_{0k}} a_s, \qquad b_s = \mu_{s,t} + i\nu_{s,t} \tag{123}$$

The eigenvalues of $\tilde{L}$ $\tilde{\zeta}_s = \tilde{\kappa} + i\tilde{\eta}_s$ are integrals of motion and $\tilde{\kappa}_s$ determine the asymptotic velocities for the soliton train described by GCTC.

Thus, starting from the set of initial soliton parameters we can calculate $L|_{t=0}$ (resp. $\tilde{L}|_{t=0}$), evaluate the real parts of their eigenvalues and thus determine the asymptotic regime of the soliton train.

Regime (i) $\kappa_k \neq \kappa_j$ (resp. $\tilde{\kappa}_k \neq \tilde{\kappa}_j$) for $k \neq j$, i.e. the asymptotic velocities are all different. Then we have asymptotically separating, free solitons, see also [**?**, **?**, **?**]

Regime (ii) $\kappa_1 = \kappa_2 = \cdots = \kappa_N = 0$ (resp. $\tilde{\kappa}_1 = \tilde{\kappa}_2 = \cdots = \tilde{\kappa}_N = 0$), i.e. all N solitons move with the same mean asymptotic velocity, and form a “bound state".

Regime (iii) a variety of intermediate situations when one group (or several groups) of particles move with the same mean asymptotic velocity; then they would form one (or several) bound state(s) and the rest of the particles will have free asymptotic motion.

Remark 2. *The sets of eigenvalues of L and $\tilde{L}$ are generically different. Thus varying only the polarization vectors one can change the asymptotic regime of the soliton train.*

Several particular cases.

Case 1 $\vec{n}_1 = \cdots = \vec{n}_N$. Since the vector $\vec{n}_1$ is normalized, then all coefficients $m_{ok} = 1$ and $\phi_{0k} = 0$. Then the interactions of the vector and scalar solitons are identical.

Case 2 $(\vec{n}^\dagger_{s+1}, \vec{n}_s) = 0$. Then the GCTC splits into two unrelated GCTC: one for the solitons $\{1, 2, \dots, s\}$ and another for $\{s+1, s+2, \dots .N\}$. If the two sets of soliton parameters are such that both groups of solitons are in bound state regimes, then these two bound states

Case 3 $\langle n^{\dagger}_{k+1}|\vec{n}_k\rangle = m_0^2 e^{2i\varphi_0}$ – effective change of distance and phases of solitons. Rewrite

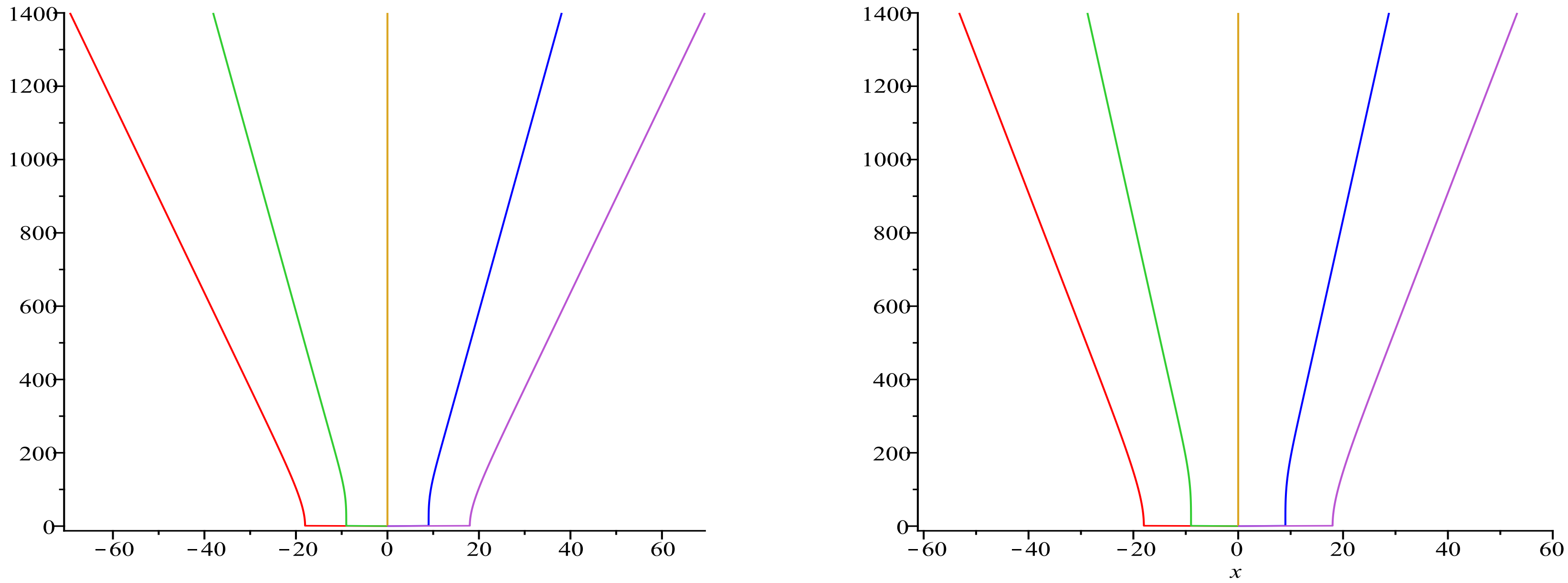
$$\tilde{a}_s = \exp((\tilde{Q}_{s+1} - \tilde{Q}_s)/2), \qquad \tilde{Q}_{s+1} - \tilde{Q}_s = Q_{s+1} - Q_s + \ln m_0 + i\varphi_0,$$

i.e. the distance between any two neighboring vector solitons has changed by $\ln m_0/(2\nu_0)$; similarly the phases

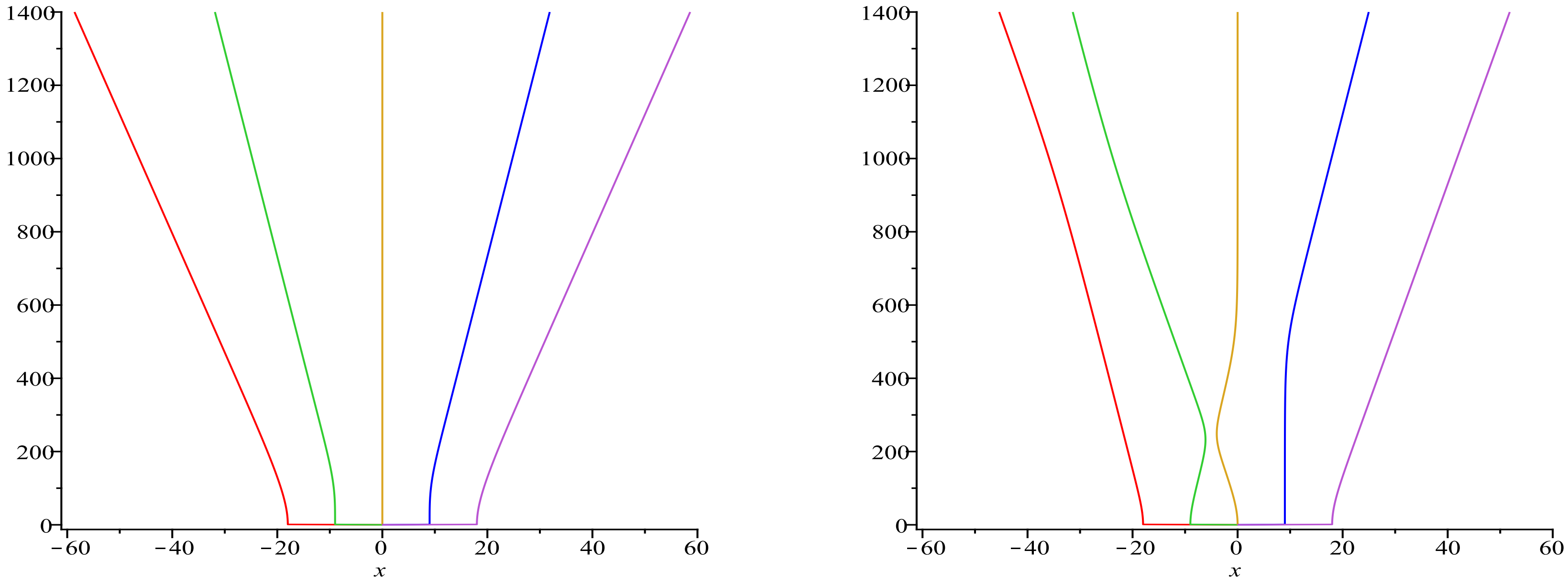
Initial parameters of the solitons:

$$\nu_k(0) = 0.5, \qquad \phi_k(0) = k\pi, \qquad \xi_{k+1}(0) - \xi_k(0) = r_0, \qquad \mu_k = 0. \tag{124}$$

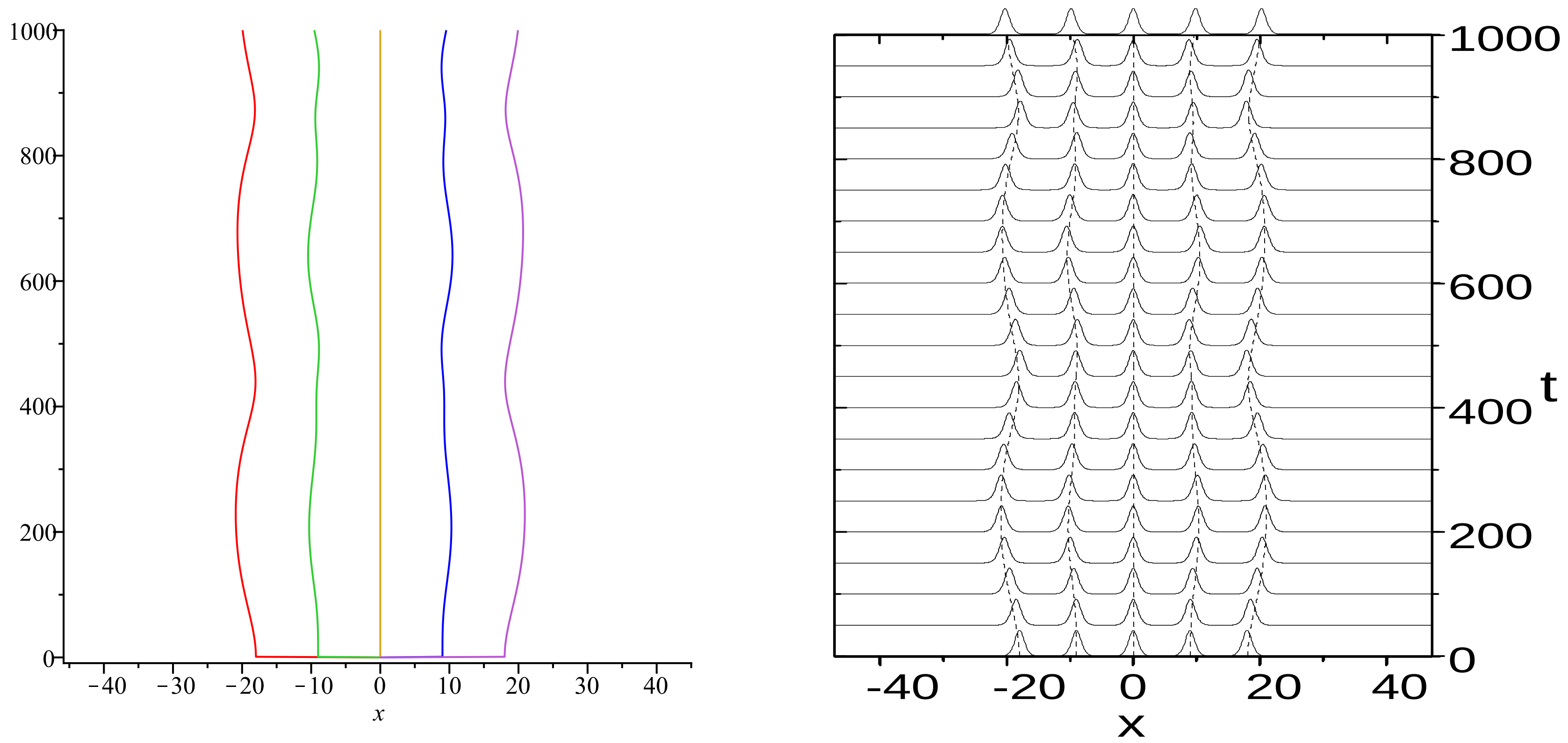
Effects of external potentials



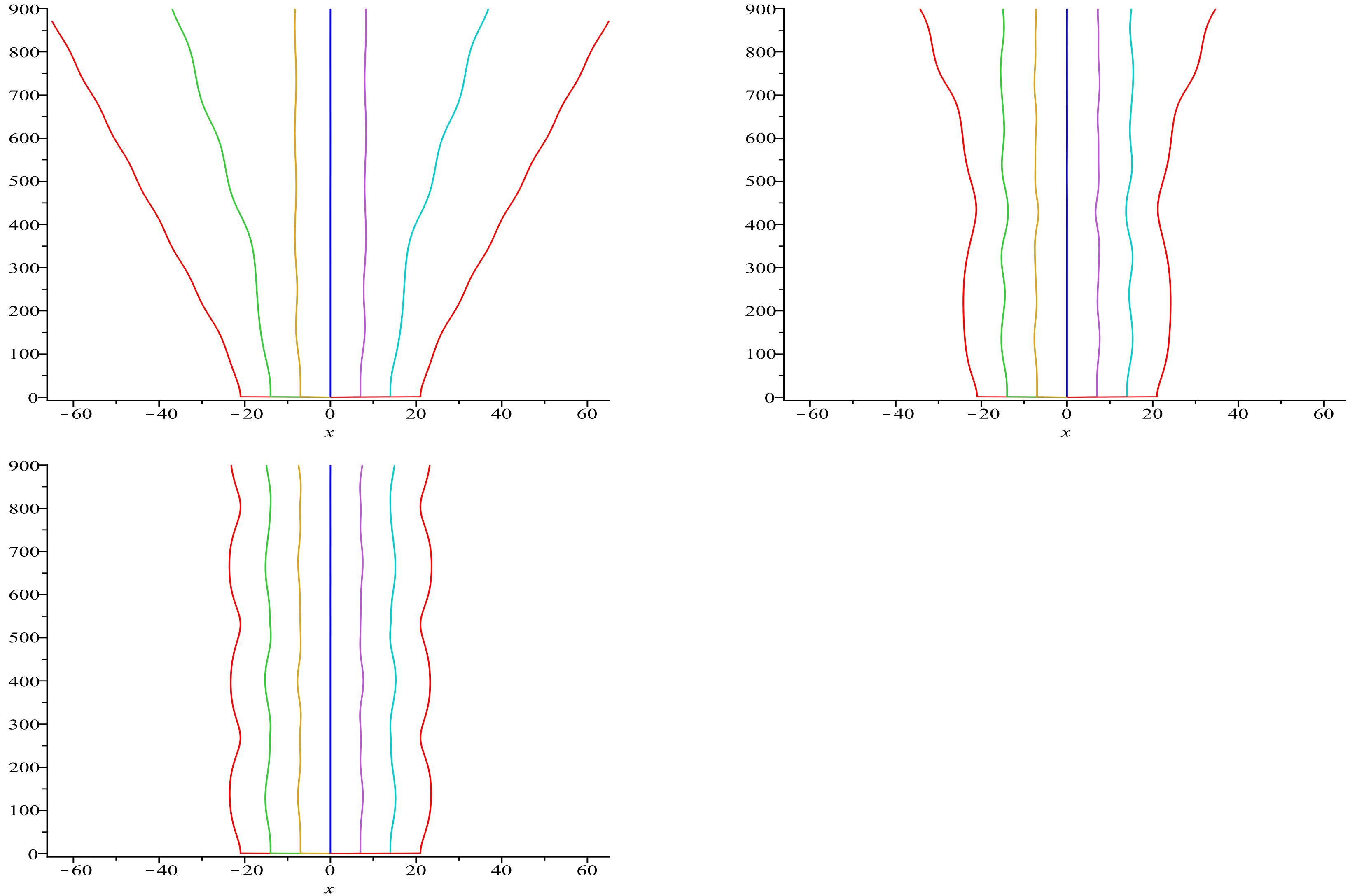
Фигура 2: The initial soliton parameters as like in (125) with $r_0 = 9$. Left panel: scalar soliton train; Right panel: vector soliton train with $r_0 = 9$ and $m_{0s} = 0.7$.



Фигура 3: Left panel: vector soliton train with $m_{0s} = 0.8$; Right panel: vector soliton train with $m_{01} = m_{03} = m_{04} = 0.8$ and $m_{02} = 0.031$.



Фигура 4: Oscillations of the 5-soliton train (see (125) in a moderately weak periodic potential, $A = 0.0005$, $\Omega = 2\pi/9$, $r_0 = 9$. Left panel: the trajectories as described by the CTC. Right panel: the numerical solution of the NLS eq.



Фигура 5: The effect of the periodic potential on 7-soliton trains (125) with $r_0 = 7$ and subcritical intensities. UL: $V_2 = -0.00075$; UR: $V_2 = -0.0012$; Below: critical intensity: $V_2 = -0.0013$.

Conclusions

- The ISM for solving soliton equations can be viewed as Generalized Fourier transform
- The recursion operators generate all fundamental properties of the soliton equations
- The GCTC models the soliton interaction in adiabatic approximation for the vector NLS