# Tau Functions and Convolution Symmetries*

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# Outline

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   - Hilbert space Grassmannian and linear group actions
   - The $\tau$ function as a determinant
   - Examples of $\tau$ functions
     - Schur functions
     - Orthogonal polynomials
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2. **Convolution symmetries**
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   - Fock space representation of convolution symmetries
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# KP $\tau$ functions.

A KP **tau function** $\tau(\mathbf{t})$ is a function of an infinite set of flow variables $\mathbf{t} = (t_1, t_2, \dots)$, satisfying an infinite set of bilinear equations, the **Hirota Bilinear equations:**

$$\mathrm{res}_{z=0}\left(\psi^+(z,\mathbf{t})\psi^-(z,\mathbf{t}+\mathbf{s})\right) = 0,$$

(identically in $\mathbf{s} := (s_1, s_2, \dots)$), where the **Baker-Akhiezer function** $\psi^+(z,\mathbf{t})$ and its dual $\psi^-(z,\mathbf{t})$ are defined by the **Sato formula**:

$$\psi^\pm(z,\mathbf{t}) := e^{\pm\sum_{i=1}^\infty t_i z^i} \times \frac{\tau(\mathbf{t} \mp [z^{-1}])}{\tau(\mathbf{t})}$$

$$[z^{-1}] := \left(\frac{1}{z}, \frac{1}{2z^2} \dots\right)$$

**Question**: How to construct such $\tau$ functions? What do they mean?

# Hilbert Space Grassmannians

## Model for Hilbert space

$$\mathcal{H} := L^2(S^1) = \mathcal{H}_+ + \mathcal{H}_-,$$
$$\mathcal{H}_+ = \mathrm{span}\{z^i\}_{i\in\mathbf{N}}, \quad \mathcal{H}_- = \mathrm{span}\{z^{-i}\}_{i\in\mathbf{N}^+},$$

The Sato-Segal-Wilson Grassmannian is defined as

$$Gr_{\mathcal{H}_+}(\mathcal{H}) = \{\text{closed subspaces } w \subset \mathcal{H} \text{ ``commensurable'' with } \mathcal{H}_+\}$$

i.e., such that orthogonal projection to $\mathcal{H}_+$ along $\mathcal{H}_-$

$$\pi^\perp : w \rightarrow \mathcal{H}_+$$

is a Fredholm map and orthogonal projection to $\mathcal{H}_-$

$$\pi^\perp : w \rightarrow \mathcal{H}_-$$

is "small" (e.g., Hilbert-Schmidt).    ($\mathcal{H}_+ \in Gr_{\mathcal{H}_+}(\mathcal{H})$ is the "origin".)

## Basis labelling and frames

**Orthonormal basis for $\mathcal{H}$:**

$$\{e_i := z^{-i-1}\}_{i\in \mathbf{Z}},$$

In terms of **frames**, let

$$w = \mathrm{span}\{w_1, w_2, \dots\},$$

and expand the basis vectors $w_i$ in the orthonormal basis $\{e_j\}$

$$w_i := \sum_{j\in \mathbf{Z}} W_{ji} e_j.$$

Define doubly $\infty$ column vectors $\{\mathbf{W}_i\}_{i=1,2\dots}$ with components

$$(\mathbf{W}_i)_j := W_{ji}$$

and the rectangular $2\infty \times \infty$ matrix $W$ with columns $\{\mathbf{W}_i\}_{i=1,2\dots}$

$$W := (\mathbf{W}_1, \mathbf{W}_2, \cdots)$$

## Linear and abelian group actions

Abelian group actions: $\Gamma_\pm \times \mathcal{H} \to \mathcal{H}$:

$$\Gamma_\pm := \{\gamma_\pm(\mathbf{t}) := e^{\sum_{i=1}^\infty t_i z^{\pm i}}\}$$

$$(\gamma_\pm(\mathbf{t}), f \in L^2(S^1) \mapsto \gamma_\pm(\mathbf{t}) f$$

This induces an action on frames $W$, for $w \in Gr_{\mathcal{H}_+}(\mathcal{H})$

$$\gamma_\pm(\mathbf{t}) \times W \mapsto W(\mathbf{t}) := e^{\sum_{i=1}^\infty t_i \Lambda^{\pm i}} W$$

where

$$\Lambda(e_i) = e_{i-1}$$

More generally, we have the **general linear group action**:

$$GL(\mathcal{H}) \times Gr_{\mathcal{H}_+}(\mathcal{H}) \to Gr_{\mathcal{H}_+}(\mathcal{H})$$
$$(g \in GL(\mathcal{H}), W) \to gW$$

represented by doubly infinite, invertible matrices

$$g = e^A, \quad A \in \mathfrak{gl}(\infty). \quad A = (A_{ij})|_{i,j,\in \mathbf{Z}}$$

## Sato-Segal-Wilson definition of KP $\tau$ functions

For $w =\in Gr_{\mathcal{H}_+}(\mathcal{H})$, the KP- $\tau$ function $\tau_w(\mathbf{t})$ is obtained as the Fredholm determinant of the orthogonal projection of $W(\mathbf{t})$ to $\mathcal{H}_+$:

### KP $\tau$-function

$$\tau_w(\mathbf{t}) = \det(\pi^\perp : w(\mathbf{t}) \rightarrow \mathcal{H}_+), \quad \mathbf{t} = (t_1, t_2, \dots)$$

or, equivalently if

$$W(\mathbf{t}) = \begin{pmatrix} W_+(\mathbf{t}) \\ W_-(\mathbf{t}) \end{pmatrix}$$

then

$$\tau_w(\mathbf{t}) = \det W_+(\mathbf{t})).$$

## Example: 1. Schur functions ("elementary building blocks")

Consider **Partitions**:

$$\lambda = (\lambda_1, \dots \lambda_{\ell(\lambda)}), \quad \lambda_1 \geq \cdots \geq \lambda_{\ell(\lambda}, \quad \lambda_i \in \mathbf{N}^+$$

of length $\ell(\lambda)$ and weight $|\lambda| := \sum_{i=1}^{\ell(\lambda)} \lambda_i$

Define $w_\lambda \in Gr_{\mathcal{H}_+}(\mathcal{H})$ as

$$w_\lambda := \mathrm{span}\{e_{\lambda_i - i}\}$$

Then

$$\tau_{w_\lambda}(\mathbf{t}) = s_\lambda(\mathbf{t})$$

where the **Schur function**

$$s_\lambda(\mathbf{t}) := \mathrm{tr}(\rho_\lambda(g)), \quad g \in GL(N)$$
$$\mathbf{t} := (t_1, t_2, \cdots), \quad t_i := \tfrac{1}{i}\mathrm{tr}(g^i), \quad g \in GL(N)$$

is the character of the irreducible representation

$$\rho_\lambda : GL(N) \longrightarrow \mathrm{End}(T^{(\lambda)} \subset (\mathbf{C}^N)^{\otimes|\lambda|})$$

obtained by restricting to tensors of symmetry type $\lambda$.

## Example: 2. Orthogonal polynomials and Random Matrix integrals

Let

$$w_{d\mu} = \mathrm{span}\{\frac{1}{z^N} p_{N+i}\}_{i=0,1,2,\dots} \in Gr_{\mathcal{H}_+}(\mathcal{H})$$

where $\{p_i(z)\}_{i\in\mathbf{N}}$ are **orthogonal polynomials** with respect to a measure $d\mu(z)$ on some set of curve segments $\Gamma$ in the complex plane (e.g., the real line $\mathbf{R}$)

$$\int p_i(z)p_j(z)d\mu(z) = \delta_{ij}$$

Then

$$\tau_{w_{d\mu}}(\mathbf{t}) = \prod_{a=1}^{N} \int_\Gamma d\mu(z_a) e^{\sum_{i=1}^\infty t_i z_a^i} \Delta^2(\mathbf{z})$$

where $\Delta(\mathbf{z}) = \prod_{a<b}^N (z_a - z_b)$ ( (Vandermonde determinant)

# Random matrix integrals

By the **Weyl integral formula** on $U(N)$), we have

$$\tau_{w_{d\mu}}(\mathbf{t}) \propto \mathbf{Z}_{N,f}(\mathbf{t}) := \int_{\mathbf{H}^{N\times N}} d\mu_{N,f}(M, \mathbf{t})$$

where

$$d\mu_N(M, \mathbf{t}) := d\mu_N(M) e^{\mathrm{tr}(\sum_{i=1}^{\infty} t_i M^i)}$$

is a deformation family of $U(N)$ conjugation invariant measures on the space $\mathbf{H}^{N\times N}$ of Hermitan $N \times N$ matrices.

$$d\mu_N(UMU^{\dagger}) = d\mu_N(M), \quad \forall U \in U(N), \quad M \in \mathbf{H}^{N\times N}$$

## Fermionic Fock space $\mathcal{F}$

For every partition $\lambda = (\lambda_1, \lambda_2, \dots)$ and integer $N \in \mathbf{Z}$ define the extended semi-infinite sequence

$$\lambda = (\lambda_1, \dots \lambda_{\ell(\lambda)}), 0, 0, \dots)$$

and **"particle positions"**

$$l_j := \lambda_j - j + N$$

The **fermionic Fock space $\mathcal{F}$** is the **exterior space** (orthogonal direct sum of charge $N$ subspaces )

$$\mathcal{F} := \Lambda\mathcal{H} = \bigoplus_{N \in \mathbf{Z}} \mathcal{F}_N.$$

spanned by semi-infinite wedge products (orthonormal basis for $\mathcal{F}_N$)

$$|\lambda, N\rangle := e_{l_1} \wedge e_{l_2} \wedge \cdots$$

Each charge $N$ sector $\mathcal{F}_N$ has a charged **vacuum vector**

$$|0, N\rangle = e_{N-1} \wedge e_{N-2} \wedge \dots,$$

# Fermionic creation and annihilation operators

In terms of the **Orthonormal basis for $\mathcal{H}$**, and **dual basis for $\mathcal{H}^*$**

$$\{e_i := z^{-i-1}\}_{i\in\mathbf{Z}}, \qquad \{\tilde{e}_i\}_{i\in\mathbf{Z}}, \qquad \tilde{e}_i(e_j) = \delta_{ij}$$

define the Fermi **creation and annihilation operators** (exterior and interior muliplication):

$$\psi_i v := e_i \wedge v, \quad \psi_i^\dagger v := i_{\tilde{e}^i} v, \quad v \in \mathcal{H}.$$

These satisfy the usual anti-commutation relations

$$[\psi_i, \psi_j]_+ = [\psi_i^\dagger, \psi_j^\dagger]_+ = 0, \quad [\psi_i, \psi_j^\dagger]_+ = \delta_{ij}.$$

determining the $\infty$ dimensional Clifford algebra of fermionic operators.

# Plücker map and Plücker coordinates

The **Plücker map** $\mathcal{P} : \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}) \rightarrow \mathbf{P}(\mathcal{F})$ into the projectivization of $\mathcal{F}$,

$$\mathcal{P} : \mathrm{span}(w_1, w_2, \dots) \mapsto [w_1 \wedge w_2 \wedge \cdots],$$

embeds $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ in $\mathbf{P}(\mathcal{F})$ as the intersection of an infinite number of quadrics. If orthogonal projection to $\mathcal{H}_+$

$$\pi^{\perp} : w \rightarrow \mathcal{H}_+$$

has Fredholm index $N$, is in the charge $N$ sector $\mathcal{P}(w) \subset \mathcal{F}_N$. Expanding in the standard orthonormal basis,

$$\mathcal{P}(w) = w_1 \wedge w_2 \wedge \cdots = \sum_{\lambda} \pi_\lambda(w, N) |\lambda, N >,$$

the coefficients $\pi_\lambda(w, N)$ are the **Plücker cordinates** of $w$ (which satisfy the infinite set of bilinear **Plücker equations**.)

## Fermionic representation of group actions and flows

The **Plücker map**

$$\mathcal{P} : \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}) \to \mathbf{P}(\mathcal{F})$$

interlaces the action of the abelian groups

$$\Gamma_\pm \times \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}) \to \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$$

with the following representations on $\mathcal{F}$ (and its projectivization)

$$\gamma_\pm(\mathbf{t}) : v \mapsto \hat{\gamma}_\pm(\mathbf{t})v, \quad \hat{\gamma}_\pm(\mathbf{t}) := e^{\sum_{i=1}^\infty t_i J_{\pm i}}, \quad v \in \mathcal{F}$$

where

$$J_i := \sum_{n\in\mathbb{Z}} \psi_n \psi^\dagger_{n+i}, \quad i \in \mathbb{Z}$$

More generally, if $g = e^A \in GL(\mathcal{H})$, $A \in \mathfrak{gl}(\mathcal{H})$ has the fermionic representation

$$\hat{g} := e^{\sum_{i,j\in\mathbb{Z}} A_{ij} :\psi_i \psi_j^\dagger:},$$

## Fermionic representation of KP-chain and 2-Toda $\tau$ function

For $w \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}) = g(\mathcal{H}_+)$, $g \in GL(\mathcal{H})$, with $\mathcal{P}(w) \subset \mathcal{F}_N$ in the charge-$N$ sector, the KP chain $\tau$-function has the **fermionic representation**:

$$\tau_w(\mathbf{t}, N) = \langle N|\hat{\gamma}_+(\mathbf{t})\hat{g}|N\rangle =: \tau_g(\mathbf{t}, N)$$

Similarly, for the **2-Toda $\tau$ function**:

$$\tau_w^{(2)}(\mathbf{t}, \tilde{\mathbf{t}}, N) = \langle N|\hat{\gamma}_+(\mathbf{t})\hat{g}\hat{\gamma}_-(\tilde{\mathbf{t}})|N\rangle := \tau_g^{(2)}(\mathbf{t}, \tilde{\mathbf{t}}, N)$$

## Schur function expansions

It follows that we have the **Schur function expansions**

$$\tau_g(\mathbf{t}, N) = \sum_{\lambda} \pi_\lambda(g(\mathcal{H}_+), N) s_\lambda(\mathbf{t}),$$
$$\tau_g^{(2)}(\mathbf{t}, \tilde{\mathbf{t}}, N) = \sum_{\lambda} \sum_{\mu} B_{\lambda,\mu}(g, N) s_\lambda(\mathbf{t}) s_\mu(\tilde{\mathbf{t}}).$$

where

$$\pi_\lambda(g(\mathcal{H}_+), N) = \langle \lambda, N | \hat{g} | N \rangle$$
$$B_{\lambda,\mu}(g, N) = \langle \lambda, N | \hat{g} | \mu, N \rangle$$

are the Plücker coordinates along the basis direction $|\lambda, N\rangle$ .

## 2. Convolution symmetries

Given an infinite sequence of complex numbers $\mathbf{T} = \{T_i\}_{i\in\mathbb{Z}}$, define

$$\rho_i := e^{T_i}, \qquad r_i := e^{T_i - T_{i-1}}, \quad i \in \mathbb{Z}.$$

Assume the series $\sum_{i=1}^{\infty} T_{-i}$ converges and

$$\lim_{i\to\infty} |r_i| = r \leq 1,$$

The two series

$$\rho_+(z) = \sum_{i=0}^{\infty} \rho_{-i-1} z^i, \quad \rho_-(z) = \sum_{i=1}^{\infty} \rho_{i-1} z^{-i},$$

then define analytic functions $\rho_\pm(z)$ in these regions and

$$R_\rho := \prod_{i=1}^{\infty} \rho_{-i} < \infty$$

## Convolution symmetries (cont'd)

If $w \in L^2(S^1)$ has the Fourier series decomposition

$$w(z) = \sum_{i=-\infty}^{\infty} w_i z^{-i-1} = w_-(z) + w_+(z)$$

$$w_-(z) = \sum_{i=1}^{\infty} w_{i+1} z^{-i}, \quad w_+(z) = \sum_{i=0}^{\infty} w_{-i-1} z^i$$

Define the bounded linear map $C(\mathbf{T}) : L^2(S^1) \rightarrow L^2(S^1)$

$$C(\mathbf{T})(w)(z) = \sum_{i=-\infty}^{\infty} \rho_i w_i z^{-i-1} = \sum_{i=-\infty}^{\infty} \rho_i w_i e_i.$$

so each basis element $e_i$ is mutiplied by $e^{T_i}$.
The group of **Convolution Symmetries** $C(\mathbf{T}) : \mathcal{H} \rightarrow \mathcal{H}$ is represented in the standard monomial basis $\{e_i\}$ by the diagonal matrix

$$C(\mathbf{T}) := \mathrm{diag}\{e^{T_i}\}.$$

# Fock space representation

This abelian subalgebra of $\mathfrak{gl}(\mathcal{H})$ is generated by the operators

$$K_i := :\psi_i\psi_i^\dagger: = \begin{cases} \psi_i\psi_i^\dagger & \text{if} \quad i \geq 0 \\ -\psi_i^\dagger\psi_i & \text{if} \quad i < 0, \end{cases}$$
$$[K_i, K_j] = 0, \quad i, j \in \mathbb{Z}.$$

Define

$$\hat{C}(\mathbf{T}) := e^{\sum_{i=-\infty}^{\infty} T_i K_i}.$$

Then $\hat{C}(\mathbf{T})$ is diagonal in the basis $\{|\lambda, N\rangle\}$,

$$\hat{C}(\mathbf{T})|\lambda, N\rangle = r_\lambda(N, \mathbf{T})|\lambda, N\rangle.$$

with eigenvalues: $\quad r_\lambda(N, \mathbf{T}) := r_0(N, \mathbf{T}) \prod_{(i,j)\in\lambda} r_{N-i+j},$

$$r_0(N, \mathbf{T}) := \begin{cases} e^{\sum_{i=0}^{N-1} T_i} & \text{if} \quad N > 0 \\ 1 & \text{if} \quad N = 0 \\ e^{-\sum_{i=1}^{-N} T_{-i}} & \text{if} \quad N < 0, \end{cases}$$

## Effect of convolution symmetries on $\tau$-functions

**Lemma**

*Convolution actions multiply the coefficients in the Schur function expansions of* $\tau_{C_\rho g}(N, \mathbf{t})$ *and* $\tau^{(2)}_{C_\rho \hat{g} C_{\tilde{\rho}}}(N, \mathbf{t}, \tilde{\mathbf{t}})$ *by the diagonal factors* $r_\lambda(N, \mathbf{T})$ *and* $r_\mu(N, \tilde{\mathbf{T}})$*.*

$$\tau_{C_\rho g}(N, \mathbf{t}) = \sum_\lambda r_\lambda(N, \mathbf{T}) \pi_\lambda(g(\mathcal{H}_+), N) s_\lambda(\mathbf{t}),$$

$$\tau^{(2)}_{C_\rho g C_{\tilde{\rho}}}(N, \mathbf{t}, \tilde{\mathbf{t}}) = \sum_\lambda \sum_\mu r_\lambda(N, \mathbf{T}) B_{\lambda,\mu}(g, N) r_\mu(N, \tilde{\mathbf{T}}) s_\lambda(\mathbf{t}) s_\mu(\tilde{\mathbf{t}}).$$

The Plücker coordinates for the modified Grassmannian elements $C_\rho g(\mathcal{H}_+^N)$ and $C_\rho g C_{\tilde{\rho}}(w_{\mu,N})$ are thus:

$$\pi_\lambda(C_\rho g(\mathcal{H}_+), N) = r_\lambda(N, \mathbf{T}) \pi_{N,g}(\lambda)$$

$$B_{\lambda,\mu}(C_\rho g C_{\tilde{\rho}}, N) = r_\lambda(N, \mathbf{T}) B_{\lambda,\mu}(g, N)) r_\mu(N, \tilde{\mathbf{T}}).$$

# 1. New matrix models as $\tau$ functions. Example 1.

## Example

$$\rho_-(z) = \frac{1}{z} e^{\frac{1}{z}} = \sum_{i=0}^{\infty} \frac{z^{-i-1}}{i!}, \qquad |z| \leq 1$$

$$\rho_+(z) = \frac{1}{1-z} = \sum_{i=1}^{\infty} z^i \quad |z| > 1,$$

$$\rho_i = \begin{cases} \frac{1}{i!} & \text{if} \quad i \geq 0 \\ 1 & \text{if} \quad i \leq -1, \end{cases}$$

$$r_i = \begin{cases} \frac{1}{i} & \text{if} \quad i \geq 1 \\ 1 & \text{if} \quad i \leq 0, \end{cases}$$

$$r_\lambda(N) = \frac{1}{(\prod_{i=1}^{N-1} i!)(N)_\lambda} \quad \text{if} \quad \ell(\lambda) \leq N$$

## New matrix models from old

Hermitian matrix integrals of the form

$$Z_N(\mathbf{t}) = \int_{M\in \mathbf{H}^{N\times N}} d\mu(M)\, e^{\mathrm{tr}\sum_{i=1}^{\infty} t_i M^i}$$
$$= \prod_{a=1}^{N} \int_{\mathbf{R}} d\mu_0(x_a) e^{\sum_{i=1}^{\infty} t_i x_a^i} \Delta^2(X),$$

are KP-Toda $\tau$-functions. The Schur function expansion is

$$Z_N(\mathbf{t}) = \sum_{\ell(\lambda)\leq N} \pi_{N,d\mu}(\lambda) s_\lambda(\mathbf{t})$$

$$\pi_{N,d\mu}(\lambda) = \prod_{a=1}^{N} \left( \int_{\mathbf{R}} d\mu_0(x_a) \right) \Delta^2(X) s_\lambda([X])$$
$$= (-1)^{\frac{1}{2}N(N-1)} N!\, \det(\mathcal{M}_{\lambda_i - i + j + N - 1})|_{1\leq i,j\leq N}$$
$$\mathcal{M}_{ij} := \int_{\mathbf{R}} d\mu_0(x) x^{i+j}$$

## Externally coupled matrix model integral

Now consider the **externally coupled** matrix model integral

$$Z_{N,ext}(A) := \int_{M \in \mathbf{H}^{N\times N}} d\mu(M) e^{\mathrm{tr}(AM)},$$

where $A \in \mathbf{H}^{N\times N}$ is a fixed $N \times N$ Hermitian matrix. Applying the convolution symmetry of Example 1:

**Theorem**

*Applying the convolution symmetry $\tilde{C}_\rho$ to the $\tau$-function $Z_N(\mathbf{t})$, where $\rho_+(z)$ and $\rho_-(z)$ are defined as in Example 1, and choosing the KP flow parameters as $\mathbf{t} = [A]$ gives, within a multiplicative constant, the externally coupled matrix integral*

$$\tilde{C}_\rho(Z_N)([A]) = (\prod_{i=1}^{N-1} i!)^{-1} Z_{N,ext}(A).$$

# Externally coupled two-matrix model integral

## Itzykson-Zuber exponential coupled 2-matrix model

$$Z_N^{(2)}(\mathbf{t}, \tilde{\mathbf{t}}) = \int_{M_1 \in \mathbf{H}^{N\times N}} d\mu(M_1) \int_{M_1 \in \mathbf{H}^{N\times N}} d\tilde{\mu}(M_2)\, e^{\mathrm{tr}\left(\sum_{i=1}^{\infty}\left(t_i M_1^i + \tilde{t}_i M_2^i\right) + M_1 M_2\right)}$$

$$\propto \prod_{a=1}^{N} \left( \int_{\mathbf{R}} d\mu_0(x_a) \int_{\mathbf{R}} d\tilde{\mu}_0(y_a)\, e^{\sum_{i=1}^{\infty}(t_i x_a^i + \tilde{t}_i y_a^i + x_a y_a)} \right) \Delta(X)\Delta(Y)$$

### Theorem

*Applying the convolution symmetry $\tilde{C}_{\rho,\tilde{\rho}}$ to $Z_N^{(2)}$ and evaluating at the parameter values $\mathbf{t} = [A]$, $\tilde{\mathbf{t}} = [B]$ gives the externally coupled matrix integral*

$$\tilde{C}^{(2)}_{\rho,\tilde{\rho}}(Z_N^{(2)})([A],[B])) = Z^{(2)}_{N,\rho,\tilde{\rho}}(A, B)$$

# Convolution flows and the Q operator

## The $Q$-operator

Choose an infinite sequence of constants $\{q_j\}_{j\in\mathbf{Z}}$ with

$$|q_j| > 1 \quad \text{for } j > 0$$

and define the infinite sqquare matrix $Q(\mathbf{q}) \in \mathrm{Mat}^{\mathbf{Z}\times\mathbf{Z}}$ having matrix elements

$$Q_{ij} = (q_j)^i$$

$$\Lambda Q = Q\,\mathrm{diag}(q_i)$$
$$\gamma_+(\mathbf{t})Q = Q\,C(\mathbf{T}(\mathbf{q},\mathbf{t}))$$
$$T_j(\mathbf{q},\mathbf{t}) := \sum_{i=1}^{\infty} t_i (q_j)^i$$

## The $Q$-operator (cont'd)

For suitably chosen values of $(\mathbf{q}, \tilde{\mathbf{q}})$ (see examples below), it is possible to make triangular decompositions

$$Q(\mathbf{q}) = Q_-(\mathbf{q}) Q_0(\mathbf{q}) Q_+(\mathbf{q}),$$

where $Q_0$, is of the form

$$Q_0(\mathbf{q}) = \mathrm{diag}(e^{\phi_j(\mathbf{q})}),$$

for a suitably defined infinite sequence

$$\phi(\mathbf{q}) = \{\phi_j(\mathbf{q})\}, \quad j \in \mathbb{Z},$$

and $Q_\pm(\mathbf{q}), Q_\pm(\tilde{\mathbf{q}})$ are invertible triangular matrices of the form

$$Q_\pm(\mathbf{q}) = e^{A^\pm(\mathbf{q})}, \quad Q_\pm(\tilde{\mathbf{q}}) = e^{A^\pm(\tilde{\mathbf{q}})},$$

where $A^-(\mathbf{q})$ and $A^-(\tilde{\mathbf{q}})$ , $A^+(\mathbf{q})$, $A^+(\tilde{\mathbf{q}})$ are, respectively, strictly lower $(-)$ and strictly upper $(+)$ triangular doubly infinite matrices.

# Fermionic representation of the $Q$-operator

Introduce the fermionic vertex operators

$$\hat{Q}_+(\mathbf{q}) := e^{\sum_{i<j}^{\infty} A^+_{ij}(\mathbf{q})\psi_i\psi_j^\dagger}, \qquad \hat{Q}_-(\mathbf{q}) := e^{\sum_{i>j}^{\infty} A^-_{ij}(\mathbf{q})\psi_i\psi_j^\dagger},$$
$$\hat{\tilde{Q}}_+(\tilde{\mathbf{q}}) := e^{\sum_{i<j}^{\infty} A^-_{ji}(\tilde{\mathbf{q}})\psi_i\psi_j^\dagger}, \qquad \hat{\tilde{Q}}_-(\tilde{\mathbf{q}}) := e^{\sum_{i>j}^{\infty} A^+_{ji}(\tilde{\mathbf{q}})\psi_i\psi_j^\dagger},$$
$$\hat{C}(\phi(\mathbf{q})) := e^{\sum_{i\in\mathbb{Z}} \phi_i(\mathbf{q})K_i}, \qquad \hat{C}(\phi(\tilde{\mathbf{q}})) := e^{\sum_{i\in\mathbb{Z}} \phi_i(\tilde{\mathbf{q}})K_i}.$$

By the equivariance of the Plücker map, we then have

$$\hat{\gamma}_+(\mathbf{t})\hat{Q}_-(\mathbf{q})\hat{C}(\phi(\mathbf{q}))\hat{Q}_+(\mathbf{q}) = \hat{Q}_-(\mathbf{q})\hat{C}(\phi(\mathbf{q}))\hat{Q}_+(\mathbf{q})\hat{C}(\mathbf{T}),$$
$$\hat{\tilde{Q}}_-(\tilde{\mathbf{q}})\hat{C}(\phi(\tilde{\mathbf{q}}))\hat{\tilde{Q}}_+(\tilde{\mathbf{q}})\hat{\gamma}_-(\tilde{\mathbf{t}}) = \hat{C}(\tilde{\mathbf{T}})\hat{\tilde{Q}}_-(\tilde{\mathbf{q}})\hat{C}(\phi(\tilde{\mathbf{q}}))\hat{\tilde{Q}}_+(\tilde{\mathbf{q}}).$$

## Convolution flows and $\tau$ functions

Introduce a new basis for the abelian algebra of convolution flow generators as follows:

$$K_j(\mathbf{q}) := \sum_{i=-\infty}^{\infty} (q_i)^j K_i,$$

and define, correspondingly

$$\hat{C}_{\mathbf{q}}(\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i K_i(\mathbf{q})} = \hat{C}(\mathbf{T}(\mathbf{q}, \mathbf{t})),$$
$$\hat{C}_{\tilde{\mathbf{q}}}(\tilde{\mathbf{t}}) := e^{\sum_{i=1}^{\infty} \tilde{t}_i K_i(\tilde{\mathbf{q}})} = \hat{C}(\mathbf{T}(\tilde{\mathbf{q}}, \tilde{\mathbf{t}})).$$

# Convolution flows and $\tau$ functions (cont'd)

## Theorem

*The fermionic representation of the tau function may be expressed in terms of the corresponding Convolution Symmetry flows as: follows*

$$\tau_{g(\mathbf{q})}(N, \mathbf{t}) = r_0(N, \phi(\mathbf{q}))\langle N|\hat{Q}_+(\mathbf{q})\hat{C}_{\mathbf{q}}(\mathbf{t})\hat{g}|N\rangle$$
$$\tau^{(2)}_{g(\mathbf{q},\tilde{\mathbf{q}})}(N, \mathbf{t}, \tilde{\mathbf{t}}) = r_0(N, \phi(\mathbf{q}) + \phi(\tilde{\mathbf{q}}))\langle N|\hat{Q}_+(\mathbf{q})\hat{C}_{\mathbf{q}}(\mathbf{t})\hat{g}\hat{C}_{\tilde{\mathbf{q}}}(\tilde{\mathbf{t}})\hat{\tilde{Q}}_-(\tilde{\mathbf{q}})|N\rangle,$$

*where*

$$\hat{g}(\mathbf{q}) := \hat{Q}_-(\mathbf{q})\hat{C}(\phi(\mathbf{q}))\hat{Q}_+(\mathbf{q})\hat{g}$$
$$\hat{g}(\mathbf{q}, \tilde{\mathbf{q}}) := \hat{Q}_-(\mathbf{q})\hat{C}(\phi(\mathbf{q}))\hat{Q}_+(\mathbf{q})\hat{g}\hat{\tilde{Q}}_-(\tilde{\mathbf{q}})\hat{C}(\phi(\tilde{\mathbf{q}}))\hat{\tilde{Q}}_+(\tilde{\mathbf{q}}).$$

# Triangular boundary operators $\hat{Q}(\mathbf{q})$, $\hat{\hat{Q}}(\tilde{\mathbf{q}})$ of Toeplitz type

## Example

Let

$$q_j = e^{i\alpha} q^{-j}, \quad j \in \mathbb{Z}$$

where

$$q = e^{2\pi i \tau}, \quad \Im(\tau) > 0$$

and $\alpha = \alpha(q)$ is a real valued function of $q$. Then

$$Q(\mathbf{q})_{mn} = e^{im\alpha} q^{-mn} = e^{im\alpha} q^{-\frac{1}{2}m^2} q^{\frac{1}{2}(m-n)^2} e^{-\frac{1}{2}n^2}$$

$$Q(\mathbf{q}) = Q_0(q) \left( \sum_{m=-\infty}^{\infty} q^{\frac{m^2}{2}} a^{im\alpha} \Lambda^m \right) Q_0(q),$$

where

$$Q_0(q) = \mathrm{diag}(q^{-\frac{1}{2}m^2})_{m\in\mathbb{Z}}$$

# Triangular boundary operators $\hat{Q}(\mathbf{q})$, $\hat{\hat{Q}}(\tilde{\mathbf{q}})$ of Toeplitz type (cont'd)

## Example (cont'd)

The infinite product formula for Jacobi theta functions implies

$$\sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} e^{i\alpha m} z^n = \nu(q) \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}} e^{i\alpha} z)(1 + q^{n-\frac{1}{2}} e^{-i\alpha} z^{-1})$$

where

$$\nu(q) = \prod_{n=1}^{\infty} (1 - q^n).$$

Expressing the factors in the infinite product as

$$1 + q^{n-\frac{1}{2}} e^{i\alpha} z = \exp\left( - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{i\alpha k} q^{k(n-\frac{1}{2})} z^k \right)$$

$$1 + q^{n-\frac{1}{2}} e^{-i\alpha} z = \exp\left( - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{-i\alpha k} q^{k(n-\frac{1}{2})} z^{-k} \right)$$

# Triangular boundary operators $\hat{Q}(\mathbf{q})$, $\hat{\tilde{Q}}(\tilde{\mathbf{q}})$ of Toeplitz type (cont'd)

## Example (cont'd)

Replacing the complex parameter $z$ by the infinite shift matrix $\Lambda$, we obtain the factorization

$$Q(q) = \nu(q) Q_0(q) Q_-(\alpha, q) Q_+(\alpha, q) Q_0(q)$$

where

$$Q_\pm(\alpha, q) = \prod_{n=1}^{\infty} \gamma_\pm(m, \alpha, q)$$

are lower/upper triangular infinite Toeplitz matrices, and

$$\gamma_\pm(n, \alpha, q) := \exp\left( -\sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{i\alpha k} q^{k(n-\frac{1}{2})} \Lambda^{\pm k} \right).$$

# Triangular boundary operators $\hat{Q}(\mathbf{q})$, $\hat{\hat{Q}}(\tilde{\mathbf{q}})$ of Toeplitz type (cont'd)

## Example (cont'd)

The fermionic representation of this infinite matrix is therefore given by

$$\hat{Q} = \nu(q)\hat{C}(\phi(q))\hat{Q}_{-}(\alpha, q)\hat{Q}_{+}(\alpha, q)\hat{C}(\phi(q))$$

where

$$\hat{Q}_{\pm}(\alpha, q) = \prod_{n=1}^{\infty} \hat{\gamma}_{\pm}(n, \alpha, q) = \exp\left(-\sum_{k=1}^{\infty} \frac{(-1)^k e^{i\alpha k} q^{\frac{k}{2}}}{k(1-q^k)} J_{\pm k}\right)$$

$$\hat{\gamma}_{\pm}(n, \alpha, q) := \exp\left(-\sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{i\alpha k} q^{k(n-\frac{1}{2})} J_{\pm k}\right)$$

$$\phi(q) := \{\phi_j(q)\}, \quad \phi_j(q) = -i\pi\tau j^2, \quad j \in \mathbb{Z}.$$

# Triangular boundary operators $\hat{Q}(\mathbf{q})$, $\hat{\tilde{Q}}(\tilde{\mathbf{q}})$ of Toeplitz type (cont'd)

## Example (cont'd)

The formula for the $\tau$ function therefore becomes

$$\tau_g(N, \mathbf{t}) = r_0(N, \phi(q)) \langle N | \hat{Q}_+(\alpha, q) \hat{C}(\mathbf{T}) \hat{g}(\alpha, q) | N \rangle,$$

where

$$\hat{g}(\alpha, q) := \hat{Q}_+^{-1}(\alpha, q) \hat{Q}_-^{-1}(\alpha, q) \hat{C}^{-1}(\phi(q)) \hat{g},$$

$$T_j(q, \mathbf{t}) := \sum_{k=1}^{\infty} t_j e^{ik\alpha} q^{-jk}.$$

# Triangular boundary operators $\hat{Q}(\mathbf{q})$, $\hat{\tilde{Q}}(\tilde{\mathbf{q}})$ of Toeplitz type (cont'd)

## Example (cont'd)

Similarly, we introduce a second pair $(\alpha(\tilde{q}), \tilde{q} = e^{2\pi i\tilde{\tau}})$ and define

$$\hat{\tilde{Q}}_{\pm}(\tilde{\alpha}, \tilde{q}) := \hat{Q}_{\pm}^{-1}(\tilde{\alpha}, \tilde{q}). \tag{2.1}$$

Then the 2-Toda $\tau$ function becomes $\tau_g(N, \mathbf{t}, \tilde{\mathbf{t}}) =$

$$r_0(N, \phi(q) - \tilde{\phi}(\tilde{q}))\langle N|\hat{Q}_+(\alpha, q)\hat{C}(\mathbf{T})\hat{g}(\alpha, \tilde{\alpha}, q, \tilde{q})\hat{C}(\tilde{\mathbf{T}})\hat{\tilde{Q}}_-(\tilde{\alpha}, \tilde{q})|N\rangle,$$

where $\hat{g}(\alpha, \tilde{\alpha}, q, \tilde{q}) =$

$$:= \hat{Q}_+^{-1}(\alpha, q)\hat{Q}_-^{-1}(\alpha, q)\hat{C}^{-1}(\phi(q))\hat{g}\,\hat{C}^{-1}(\phi(\tilde{q}))\hat{\tilde{Q}}_+^{-1}(\tilde{\alpha}, \tilde{q})\hat{\tilde{Q}}_-^{-1}((\tilde{\alpha}, \tilde{q}),$$

$$\tilde{T}_j := \sum_{k=1}^{\infty} \tilde{t}_j e^{ik\tilde{\alpha}} \tilde{q}^{-jk}, \quad \phi_j(\tilde{q}) = -i\pi\tilde{\tau} j^2, \quad j \in \mathbb{Z}.$$

## Triangular boundary operators $\hat{Q}(\mathbf{q})$, $\hat{\tilde{Q}}(\tilde{\mathbf{q}})$ of Toeplitz type (cont'd)

In particular, choosing $\hat{g}$ so that

$$\hat{g}(\alpha, \tilde{\alpha}, q, \tilde{q}) = \mathbf{I},$$

setting

$$\tilde{\alpha} = \alpha = \pi, \quad q = \tilde{q}, \quad t_i = \tilde{t}_i$$

and replacing $t_i$ by $\frac{1}{2}t_i$, we obtain the $q$-deformed partition function for plane partitions that was studied by Okounkov and Pandharipande, and by Nakatsu and Takahashi.

Other choices for the $q_j$'s give other "convolution flow" representations of various $\tau$ functions (cf. e.g. Wiegmann, Bettelheim, et al).

# Background and related work

## Fermionic approach to $\tau$ functions


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## Convolution symmetries, Matrix Models, $\tau$ functions


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## Applications of convolution flows


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