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Balanced metrics, TYZ expansion and quantization of Kähler manifolds

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joint with

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Balanced metrics

(M, L) polarized manifold (M compact complex manifold, L very ample holomorphic line bundle over M).

Let g be a Kähler metric on M such that $\omega \in c_1(L)$ and h hermitian metric on L such that $Ric(h) = \omega$.

Kempf's distortion function $T_g \in C^\infty(M, \mathbb{R}^+)$

$$T_g(x) = \sum_{j=0}^{N} h(s_j(x), s_j(x)),\ x \in M$$

where $\{s_0, \dots, s_N\}$, $N + 1 = \dim H^0(L)$, is an o.b. with respect to

$$\langle s, t\rangle_h = \int_M h(s,t)\frac{\omega^n}{n!}, s, t \in H^0(L)$$

Definition (Donaldson): a polarized metric $g \in c_1(L)$ is said to be balanced if $T_g = cost = \frac{N+1}{V(M)}$, $V(M) = \int_M \frac{\omega^n}{n!}$.

Main results on balanced metrics

Theorem (Zhang, 1996): $\exists\ g$ balanced, $g \in c_1(L) \Leftrightarrow (M, L)$ Chow polystable.

Theorem (Donaldson, 2001): *Let* $g_{cscK} \in c_1(L)$ *and* $\frac{Aut(M,L)}{\mathbb{C}^*}$ *discrete. Then, for all* $m >> 1$, $\exists!$ *balanced metric* $g_m \in c_1(L^m)$ *such that* $\frac{g_m}{m} \xrightarrow{C^\infty} g_{cscK}$. *Moreover, if* $g_m \in c_1(L^m)$ *is a sequence of balanced metrics such that* $\frac{g_m}{m} \xrightarrow{C^\infty} g_\infty$ *then* g_∞ *is cscK.*

Corollary: *Let* $g_{cscK} \in c_1(L)$ *and* $\frac{Aut(M,L)}{\mathbb{C}^*}$ *discrete. Then* (M, L) *is asymptotically Chow stable.*

Corollary: If $\frac{\text{Aut}(M,L)}{\mathbb{C}^*}$ is discrete and it exists $g_{cscK} \in c_1(L)$ then g_{cscK} is unique in $c_1(L)$.

What happens without the assumption on Aut(M, L)

Theorem (C. Arezzo – L. , 2004): *Let g and $\tilde{g}$ be two balanced metrics in $c_1(L)$. Then there exists $F \in$ Aut(M, L) such that $F^*\tilde{g} = g$.*

Theorem (A. Della Vedova – F. Zuddas, 2011): *Let $M = Bl_{p_1,\ldots,p_4}\mathbb{C}P^2$ (four points in the same line except one). Then there exists a polarization L of M and $g_{cscK} \in c_1(L)$ such that (M, L^m) is not Chow polystable for $m >> 1$.*

Theorem (Chen –Tian, 2008): If $\tilde{g}_{cscK} \sim g_{cscK} \Rightarrow \exists\ F \in \mathrm{Aut}(M)$ such that $F^*\tilde{g}_{cscK} = g_{cscK}$.

Some problems on balanced metrics

$$\mathcal{B}(L) = \{g_B \ \text{ balanced} \mid g_B \in c_1(L^{m_0}), \text{for some } \ m_0\}$$

$$\mathcal{B}_c(L) = \mathcal{B}(L)/\sim$$

$$\mathcal{B}_{g_B} = \{m g_B \in \mathcal{B}(L) \mid m \in \mathbb{N}\}, \qquad g_B \in \mathcal{B}(L)$$

Problem: study $\#\mathcal{B}_c(L)$ and $\#\mathcal{B}_{g_B}$.

Some problems on balanced metrics

$$\overset{\Longrightarrow ?}{}$$

$$\# \mathcal{B}_{g_B} = \infty \quad \Longrightarrow \# \mathcal{B}_c(L) = \infty \quad \Longleftarrow (M, L) \ \text{asynt.Chow pol.}$$

$$\Uparrow \quad \Downarrow ? \qquad\qquad\qquad\qquad \Uparrow \quad \not\Downarrow$$

$$\{m g_B \quad \text{balanced} \quad \forall m >> 1 \quad \Leftrightarrow \quad \exists \ \text{CGR} \ *\text{-product on} \ (M, \omega_B)\}$$

$$\Uparrow \quad \Downarrow ?$$

$$\{L \quad \text{polarization of} \ (M, g_{hom} = g_B), \pi_1(M) = 1\}$$

A conjecture

Conjecture: *Let (M, L) be a polarized manifold. If there exists $g_B \in \mathcal{B}(L)$ such that $\#\mathcal{B}_{g_B} = \infty$ then (M, g_B) is homogeneous and $\pi_1(M) = 1$.*

Some results

Theorem 1: *Let* (M, L) *be a polarized manifold,* $\dim M = 1$*. If there exists* $g_B \in \mathcal{B}(L)$ *such that* $\#\mathcal{B}_{g_B} = \infty$ *then* $M = \mathbb{C}P^1$*.*

Theorem 2: *Let* M *be a toric manifold,* $\dim M \leq 4$*. If* $g_{KE} \in c_1(L)$*,* $L = K^*$*. Then* $\#\mathcal{B}_c(L) = \infty$*. Moreover, there exists* $g_B \in \mathcal{B}(L)$ *such that* $\#\mathcal{B}_{g_B} = \infty$ *iff* M *is either the projective space or the product of projective spaces.*

Theorem 3: *Let* g_{cscK} *be a cscK on a manifold* M *and let* $\tilde{g}_{cscK}$ *be a cscK on* $\tilde{M} = Bl_{p_1,\ldots,p_k}M$ *obtained by Arezzo-Pacard construction. Assume that there exists a polarization* L *of* $\tilde{g}_{cscK}$*. Then* $\#\mathcal{B}_{g_B} < \infty$ *for all* $g_B \in \mathcal{B}(L)$*.*

Balanced and projectively induced metrics

(M, L) polarized manifold, $g \in c_1(L)$, $m \in \mathbb{N}^+$, $Ric(h_m) = m\omega$,

$\{s_0, \dots, s_{d_m}\}$, $d_m + 1 = \dim H^0(L^m)$, o.b. for

$$\langle s, t\rangle_h = \int_M h_m(s,t)\frac{\omega^n}{n!}, s, t \in H^0(L^m).$$

$\varphi_m : M \to \mathbb{C}P^{d_m} : x \mapsto [s_0(x) : \dots : s_{d_m}(x)]$ *coherent states map*

$$\boxed{\varphi_m^*\omega_{FS} = m\omega + \tfrac{i}{2}\partial\bar\partial \log T_{mg}(x)}$$

$T_{mg}(x) = \sum_{j=0}^{d_m} h_m(s_j(x), s_j(x))$.

Therefore: $mg \in c_1(L^m)$ is balanced $\Leftrightarrow$ mg is projectively induced by φ_m.

Approximation of polarized metrics

Theorem (G. Tian, 1990): *Let* (M, L) *be a polarized manifold and* $g \in c_1(L)$*. Then*

$$\frac{\varphi_m^* g_{FS}}{m} \xrightarrow{C^2} g.$$

TYZ (Tian–Yau–Zelditch) expansion

Theorem (S. Zelditch, 1998): *Let* (M, L) *be a polarized manifold and* $g \in c_1(L)$*. Then*

$$T_{mg}(x) \sim \sum_{j=0}^{\infty} a_j(x) m^{n-j}, a_0(x) = 1,$$

namely, for all r *and* k *there exists* $C_{k,r}$ *such that*

$$||T_{mg}(x) - \sum_{j=0}^{k} a_j(x) m^{n-j}||_{C^r} \leq C_{k,r} m^{n-k-1}.$$

Corollary: Let (M, L) be polarized manifold and $g \in c_1(L)$. Then $\frac{\varphi_m^* g_{FS}}{m} \xrightarrow{C^\infty} g$.

Theorem *(Z. Lu, 2000):* Each $a_j(x)$ is a polynomial of the curvature (of the metric g) and of its covariant derivatives. Moreover,

$$\begin{cases} a_1(x) = \frac{1}{2}\rho \\ a_2(x) = \frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|Ric|^2 + 3\rho^2) \\ a_3(x) = \frac{1}{8}\Delta\Delta\rho + \frac{1}{24}\,\mathrm{div}\,\mathrm{div}(R, Ric) - \frac{1}{6}\,\mathrm{div}\,\mathrm{div}(\rho Ric) + \\ + \frac{1}{48}\Delta(|R|^2 - 4|Ric|^2 + 8\rho^2) + \frac{1}{48}\rho(\rho^2 - 4|Ric|^2 + |R|^2) + \\ + \frac{1}{24}(\sigma_3(Ric) - Ric(R, R) - R(Ric, Ric)) \end{cases}$$

Lemma 1: *Let* (M, L) *be a polarized manifold and* $g \in c_1(L)$. *Let* $\mathcal{B}_g = \{mg \text{ is balanced} \mid m \in \mathbb{N}\}$. *If* $\#\mathcal{B}_g = \infty$ *then the coefficients* $a_j(x)$ *of* $T_{mg}(x) \sim \sum_{j=0}^{\infty} a_j(x) m^{n-j}$ *are constants for all* $j = 0, 1, \ldots$.

proof: *Let* $\{m_s\}_{s=1,2,\ldots}$ *be an unbounded sequence such that* $T_{m_s g}(x) = T_{m_s}$. *We know that* $a_0 = 1$. *Assume that* $a_j(x) = a_j$, *for* $j = 0, \ldots, k-1$. *Then,*

$$|T_{s,k,n} - a_k(x) m_s^{n-k}| \leq C_k m_s^{n-k-1}, \quad T_{s,k,n} = T_{m_s} - \sum_{j=0}^{k-1} a_j m_s^{n-j}$$

for some constants C_k.

Then $|m_s^{k-n} T_{s,k,n} - a_k(x)| \leq C_k m_s^{-1}$ *and if* $s \to \infty$ *then* $m_s^{k-n} T_{s,k,n} \to a_k(x)$ *and hence* a_k *is costant.* $\square$

The proof of Theorem 1

Theorem 1*:* Let (M, L) be a polarized manifold, $\dim M = 1$. If there exists $g_B \in \mathcal{B}(L)$ such that $\#\mathcal{B}_{g_B} = \infty$ then $M = \mathbb{C}P^1$.

proof:

If $\#\mathcal{B}_{g_B} = \infty \stackrel{Lemma 1}{\Longrightarrow} g_B$ *cscK* $\Rightarrow M = \mathbb{C}P^1$ *and* $g_B = m_0 g_{FS}$. $\square$

Lemma 2*:* Let (M, L) be a polarized manifold and $g = g_{cscK} \in c_1(L)$. Assume that one of the following conditions is satisfied:

1. mg is not proj. induced $\forall m$;

2. there exists $j_0 \geq 2$ such that $a_{j_0} \neq cost$ $(T_{mg}(x) \sim \sum_{j=0}^{\infty} a_j(x) m^{n-j})$

Then $\#\mathcal{B}_{g_B} < \infty$ for all $g_B \in \mathcal{B}(L)$.

proof: *Let $g_B \in \mathcal{B}(L)$ (g_B balanced and $g_B \in c_1(L^{m_0})$ for some m_0).*

If $\#\mathcal{B}_{g_B} = \infty \overset{\text{Lemma 1}}{\Longrightarrow} a_j^B$ $(T_{mg_B}(x) \sim \sum_{j=0}^{\infty} a_j^B(x) m^{n-j})$ are constants for all $j = 0, 1, \ldots$.

In particular $a_1^B = \rho_B/2$ is constant and hence (by Chen–Tian theorem) there exists $F \in$ Aut(M) such that $F^* g_B = m_0 g$.

This implies that $m_0 g$ is proj. induced and that all the a_j's are constants for all $j = 0, 1, \ldots.$ in contrast with 1. and 2. $\Box$

Remark: *There exist polarized metrics $g_{cscK} \in c_1(L)$ such that all the coefficients of TYZ are costants but mg is not projectively induced for all m (e.g. hyperbolic metrics, flat metrics on abelian varieties).*

The proof of Theorem 2

Theorem 2: Let M be a toric manifold, $\dim M \leq 4$. If $g_{KE} \in c_1(L)$, $L = K^*$. Then $\#\mathcal{B}_c(L) = \infty$. Moreover, there exists $g_B \in \mathcal{B}(L)$ such that $\#\mathcal{B}_{g_B} = \infty$ iff M is either the projective space or the product of projective spaces.

idea of the proof:

$\#\mathcal{B}_c(L) = \infty$ follows by the fact that symmetric toric manifolds $(M, L = K^*)$ are asympt. Chow polystable.

Hard part: mg_{KE} is proj. induced for some m iff M is either the projective space or the product of projective spaces. Conclusion follows by Lemma 2. $\square$

The proof of Theorem 3

Theorem 3: Let g_{cscK} be a cscK on a manifold M and let $\tilde{g}_{cscK}$ be a cscK on $\tilde{M} = Bl_{p_1,\ldots,p_k}M$ obtained by Arezzo-Pacard construction. Assume that there exists a polarization L of $\tilde{g}_{cscK}$. Then $\#\mathcal{B}_{g_B} < \infty$ for all g_B in $\mathcal{B}(L)$.

idea of the proof: *One can prove that the coefficient a_2 of TYZ is not constant so the conclusion follows again by Lemma 2.* $\square$

Some open problems on TYZ

1. Classify the Kähler manifolds where the coefficients of TYZ are all constants.

2. Classify the Kähler manifolds where $a_k = 0$*, for* $k > n$*.*

Teorema *(L., 2005): There exists an open set* $U \subset M$ *such that:*

$$a_k(x) = C_k(1) + \sum_{\substack{r+j=k \\ r \geq 0 \;\; j \geq 1}} C_r(\tilde{a}_j(x,y))_{|y=x}$$

$$\mathcal{L}_m(f(x)) = \int_U f(y) e^{-mD(x,y)} \frac{\omega^n}{n!}(y) \sim \frac{1}{m^n} \sum_{r \geq 0} m^{-r} C_r(f)(x),$$

$$T_{mg}(x,\bar{y}) \sim \textstyle\sum_{j \geq 0} a_j(x,\bar{y}) m^{n-j} \quad \Rightarrow \quad |T_{m\omega}(x,\bar{y})|^2 \sim m^{2n}(1 + \sum_{j=1}^{+\infty} \tilde{a}_j(x,y) m^{-j})$$