XIIIth International Conference

Geometry, Integrability and Quantization

June 3-8, 2011, Varna, Bulgaria

ANALYTIC DESCRIPTION OF THE VISCOUS FINGERING INTERFACE IN A HELE-SHAW CELL

Peter Djondjorov, Vassil Vassilev,

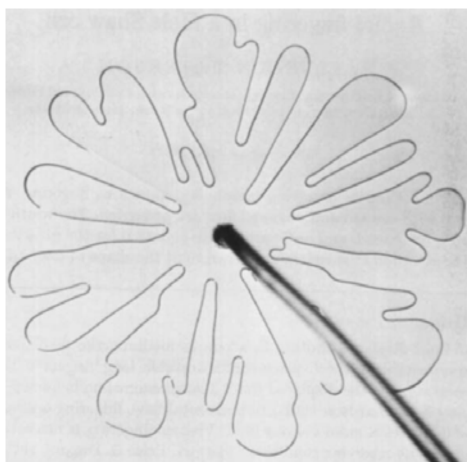
Institute of Mechanics – Bulgarian Academy of Sciences

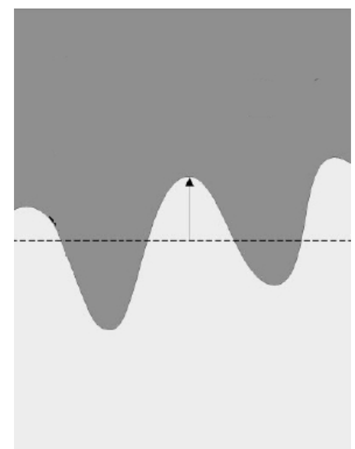
Mariana Hadzhilazova and Ivaïlo M. Mladenov

Institute of Biophysics and Biomedical Engineering – Bulgarian Academy of Sciences

Introduction

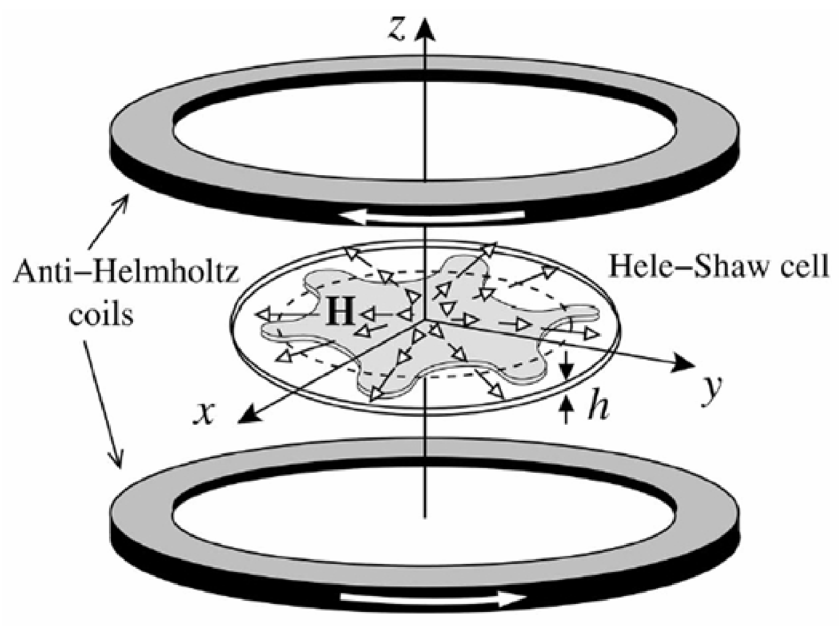
Viscous fingering (Saffman-Taylor instability) is the formation of patterns in the interface between two fluids in a Hele-Shaw cell. It occurs during injection when a less viscous fluid displaces a more viscous one. It can also occur due to gravity if a horizontal interface separates two fluids of different densities and the heavier one is above the other.

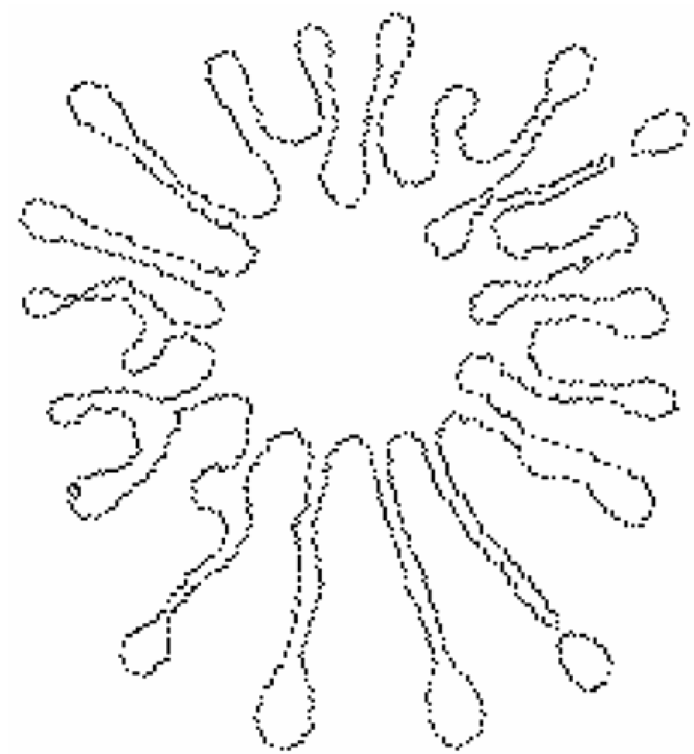




Introduction

Saffman-Taylor instability also occurs in many other frameworks, e.g. in a Hele-Shaw cell subjected to pressure, radial magnetic field or rotation.

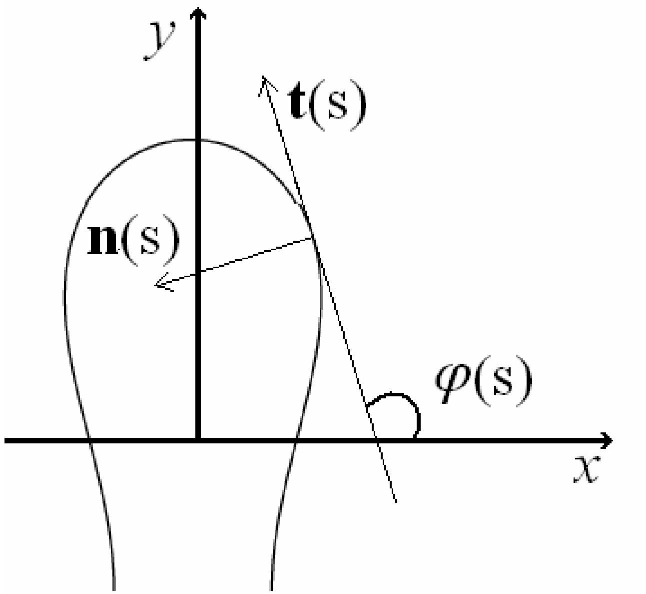




Coordinates and interface description

Let the interface be given by means of the coordinates $x(s)$, $y(s)$ in a certain Cartesian coordinate frame in the Euclidean plane with s being the interface arclength. The unit tangent vector $\mathbf{t}(s)$ and the unit normal vector $\mathbf{n}(s)$ are related to the curvature $\kappa(s)$ through the Frenet-Serret formulas

$$\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s), \qquad \mathbf{n}'(s) = -\kappa(s)\mathbf{t}(s).$$



State of the art

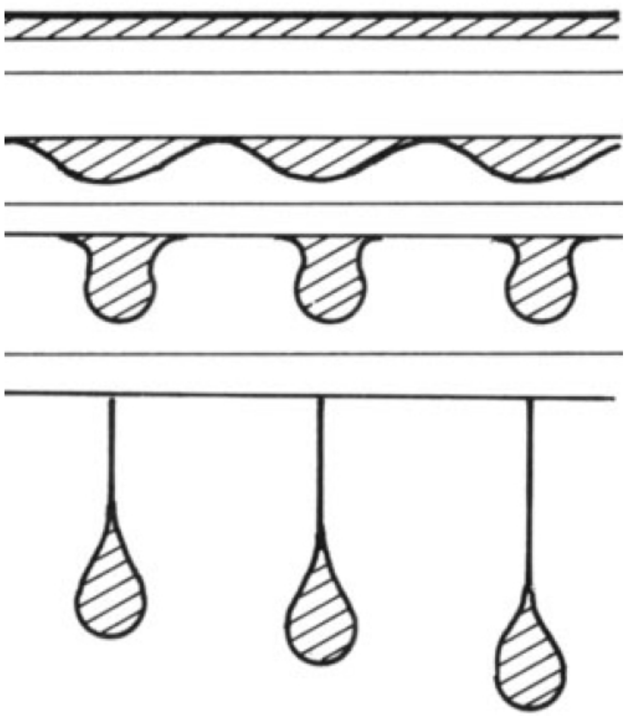
Nye *et.al.* (1984) studied gravitationally driven Saffman-Taylor fingering using oil above air in a Hele-Shaw cell. They derived the expression

$$\kappa = \frac{\mathrm{d}\varphi}{\mathrm{d}s} = -Y$$

for the curvature of the interface between the two fluids and

$$X(\psi) = -\sqrt{2} \int \frac{\psi^2 - c}{\left(1 - (\psi^2 - c)^2\right)^{1/2}} \mathrm{d}\psi$$

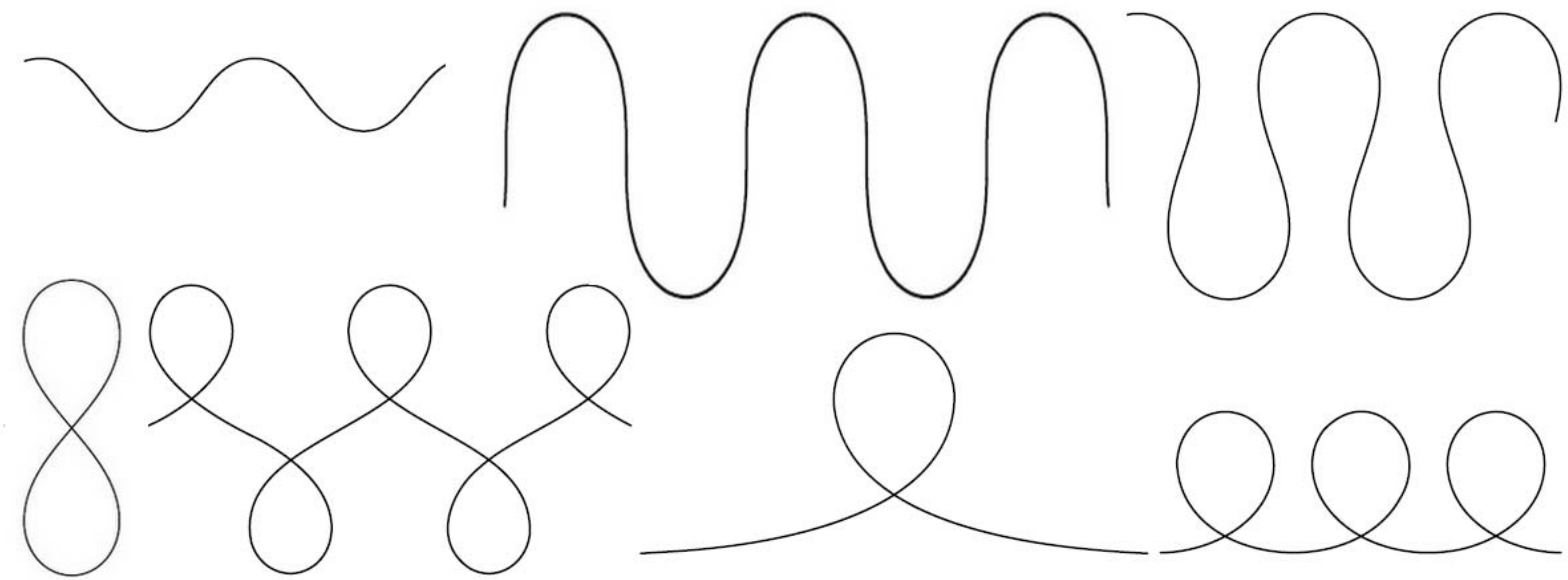
where $\psi = (c + \cos\varphi)^{1/2}$ and (X, Y) are the scalled coordinates.



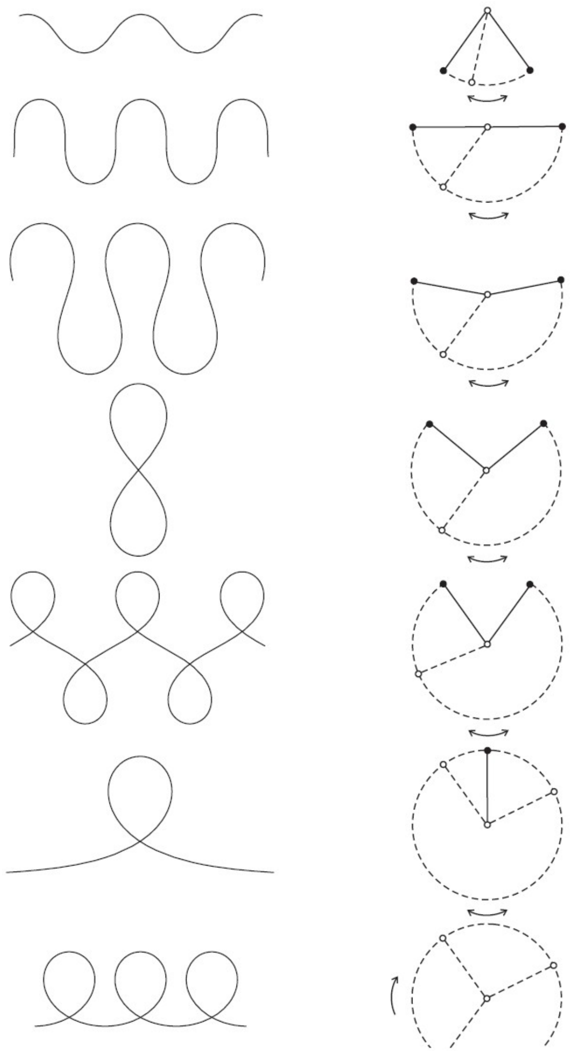
Nye J.F., H.W.Lean and A.N.Wright. Interfaces and falling drops in a Hele-Shaw cell. *Eur. J. Phys.* **5**, 73-80, 1984.

State of the art

Recall that within the approximations used by Nye *et.al.* an expression for the curvature of form $\kappa = -Y$ is obtained. It is not a surprize that such curvature gives rise to the prominent *Euler elasticae* presented below



State of the art



It is noteworthy that *Euler elasticae* is inherent in many different fields of science. Most of all, *elasticae* appears in problems more or less related to the shapes of plane curves.

However, other examples also exist. Here is an illustration of *elasticae* curves vizualizing the vibration of a pendulum.

Djondjorov P.A., M.TS. Hadzhilazova, I.M. Mladenov and V.M. Vassilev. Explicit parameterization of Euler elastica. Ninth International Conference on Geometry, Integrability and Quantization, June 8–13, 2007, Varna, Bulgaria, Ed. I.M. Mladenov, SOFTEX, Sofia 2008, pp 1–12.

State of the art

It is shown by many authors, see e.g., Leandro *et.al.* (2008) and Oliveira *et.al.* (2008), that in a rotating Hele-Shaw cell the equation, balancing the centrifugal force and the surface tension can be integrated to yield the expression

$$\kappa(r) = \Omega(r^2 - \mathring{r}^2),$$

for the curvature of the interface between the two fluids, where Ω is the dimensionless angular velocity and $\mathring{r}$ is the radius at which κ vanishes.

References

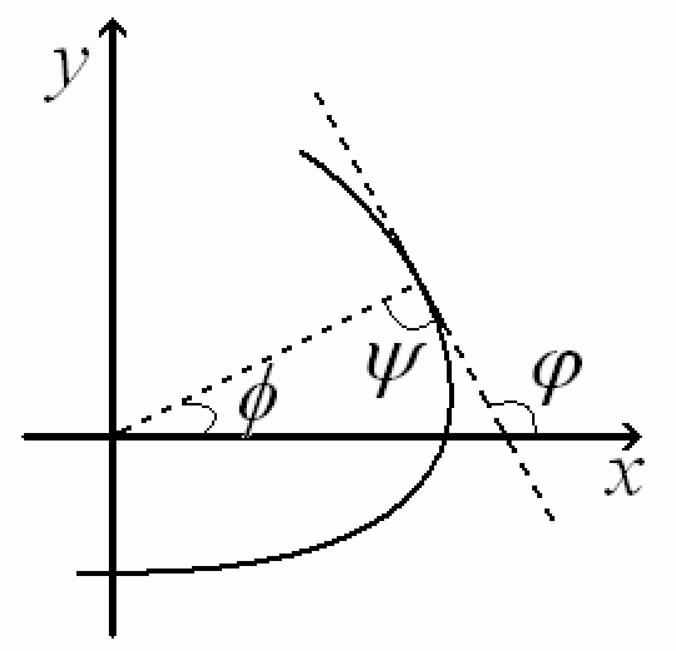
Leandro E.S.G., R.M. Oliveira, J.A. Miranda. Geometric approach to stationary shapes in rotating HeleShaw flows. *Physica D: Nonlinear Phenomena* **237**, 652-664, 2008.

Oliveira R.M., J.A. Miranda, Leandro E.S.G. Ferrofluid patterns in a radial magnetic field: Linear stability, nonlinear dynamics, and exact solutions. *Phys Rev* E **77**, 016304, 2008.

State of the art

In these papers, the embedding $\phi = \phi(r)$ of the interface is obtained in terms of the angle ψ by the following two integrals

$$\psi = \arcsin\left(\frac{1}{r}\int_{r_0}^{r} t\kappa(t)\mathrm{d}t + \frac{r_0}{r}\sin\psi_0,\right), \qquad \phi = \phi_0 + \int_{r_0}^{r}\frac{1}{t}\tan\psi(t)\mathrm{d}t,$$



where r_0 is the radius of a certain fixed point of the interface.

However, the integrals are too complicated and the authors of the aforementioned papers proceed the analysis evaluating the foregoing integrals numerically.

Here, we follow another way to determine the parametric equations for the interface in an explicit analytic form.

Differential equations

It is easy to see that Frenet–Serret formulae can be written as the following system for the coordinates of the position vector $\mathbf{r}(s) = (x(s), y(s))$ of the interface

$$x'' + \kappa(r)y = 0, \qquad y'' - \kappa(r)x = 0, \qquad \kappa(r) = \Omega(r^2 - \mathring{r}^2).$$

Such a dynamical system is studied by the authors of the present analysis within another context in

Djondjorov, P., Vassilev, V., Mladenov, I., Plane curves associated with integrable dynamical systems of the Frenet-Serret type, Trends in differential geometry, complex analysis and mathematical physics. World Scientific - Singapore, pp. 56-62, 2009.

Vassilev, V., Djondjorov, P., Mladenov, I., Integrable dynamical systems of the Frenet-Serret type, *Ibid*, pp. 234-244.

Parametric equations

In these papers we show that the coordinates of the position vector can be expressed through the curvature $\kappa(s)$ and the slope angle

$$\varphi(s) = \int \kappa(s)\mathrm{d}s$$

in the form

$$x(s) = \frac{1}{2\Omega}\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\cos\varphi(s) + \frac{1}{4\Omega}(\kappa^{2}(s) - \mu)\sin\varphi(s)$$

$$z(s) = \frac{1}{2\Omega}\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\sin\varphi(s) - \frac{1}{4\Omega}(\kappa^{2}(s) - \mu)\cos\varphi(s)$$

Hence, if the curvature $\kappa(s)$ and the slope angle $\varphi(s)$ are known functions of the arclength, the foregoing expressions for $x(s)$ and $y(s)$ are parametric equations of the interface.

Differential equation for the curvature

In polar coordinates the foregoing dynamical system trahsforms to an independent differential equation for $r(s)$ and another equation for the polar angle. Substituting the expression $r^2 = \mathring{r}^2 + \kappa/\Omega$ in the equation for $r(s)$, one easily obtains an equation for the curvature of form

$$\kappa'' + \kappa^3 - \mu\kappa - 4\Omega = 0.$$

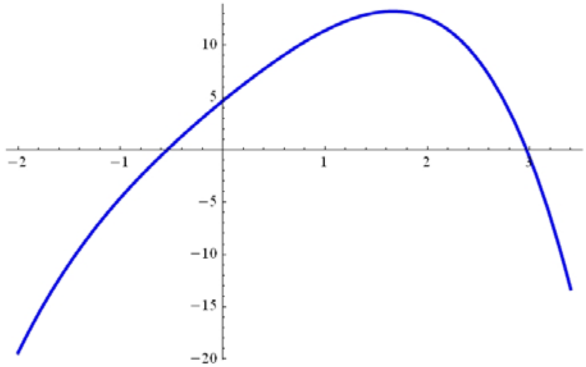
It possesses an apparent first integral of form

$$(\kappa')^2 = P(\kappa(s)), \qquad P(\kappa) = 2E - \frac{1}{4}\kappa^4 + \frac{1}{2}\mu\kappa^2 + 4\Omega\kappa,$$

where E is an integration constant.

We now proceed with determination of the solutions to this equation for the curvature. Depending on the values of the coefficients μ, Ω and E, two cases for the intrinsic equation of the interface and the corresponding slope angle $\varphi(s)$ are to be considered.

Case 1. The polynomial $P(\kappa)$ has two real roots $\alpha < \beta$ and two complex conjugate roots γ and δ. In this case there exist both periodic and nonperiodic solutions for the curvature $\kappa(s)$.



Periodic solutions exist in the cases when $(3\alpha+\beta)(\alpha+3\beta) \neq 0$ and $\eta = (\gamma-\delta)/(2i) \neq 0$. They are of the form

$$\kappa_1(s) = \frac{(A\beta + B\alpha) - (A\beta - B\alpha)\,\mathrm{cn}(us,k)}{(A+B) - (A-B)\,\mathrm{cn}(us,k)},$$

$$\varphi_1(s) = \frac{A\beta - B\alpha}{A-B}s + \frac{(A+B)(\alpha-\beta)}{2u(A-B)}\Pi\left(-\frac{(A-B)^2}{4AB}, \mathrm{am}(us,k), k\right)$$

$$+\frac{\alpha-\beta}{2u\sqrt{k^2+\frac{(A-B)^2}{4AB}}}\arctan\left(\sqrt{k^2+\frac{(A-B)^2}{4AB}}\frac{\mathrm{sn}(us,k)}{\mathrm{dn}(us,k)}\right)$$

where

$$A=\sqrt{4\eta^2+(3\alpha+\beta)^2},\quad B=\sqrt{4\eta^2+(\alpha+3\beta)^2},\quad u=\frac{1}{4}\sqrt{AB}$$

$$k=\frac{1}{\sqrt{2}}\sqrt{1-\frac{4\eta^2+(3\alpha+\beta)(\alpha+3\beta)}{\sqrt{\left[4\eta^2+(3\alpha+\beta)(\alpha+3\beta)\right]^2+16\eta^2(\beta-\alpha)^2}}}.$$

Nonperiodic solutions are obtained in the cases in which

$$(3\alpha+\beta)(\alpha+3\beta)=\eta=0.$$

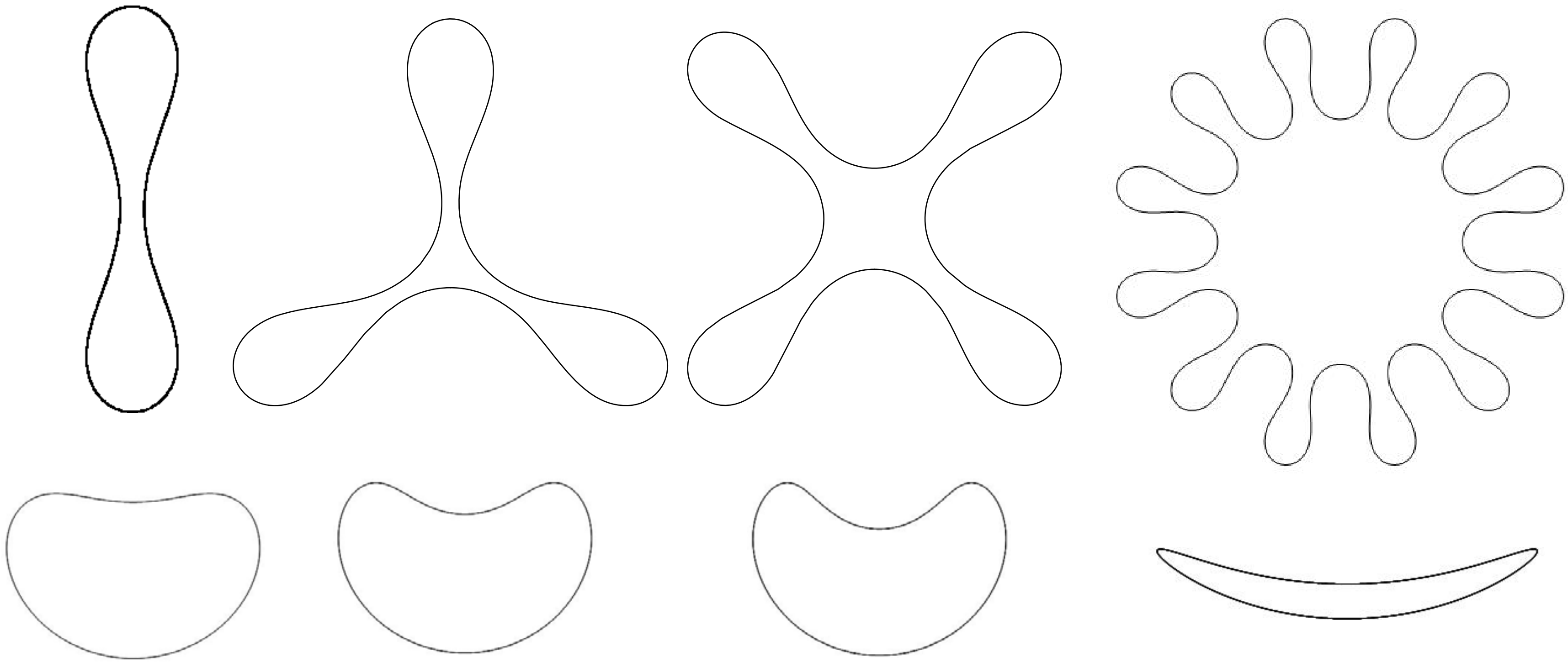
They are of the form

$$\kappa_2(s)=\zeta-\frac{4\zeta}{1+\zeta^2s^2},\qquad \varphi_2(s)=\zeta s-4\arctan(\zeta s)$$

where $\zeta=\alpha$ if $3\alpha+\beta=0$ and $\zeta=\beta$ if $\alpha+3\beta=0$.

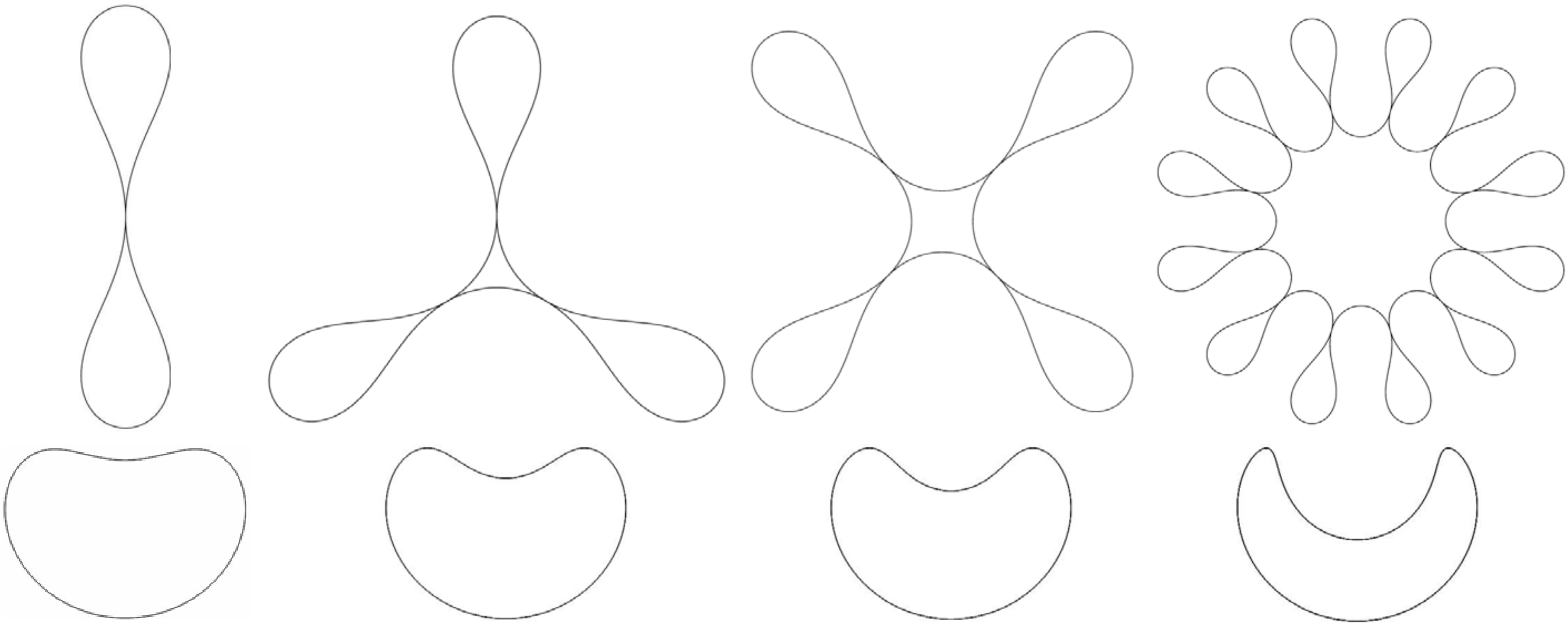
Equilibrium shapes of the fluid interface

Examples of simple curves that are appropriate for the observed interface shapes in the fixed Cartesian frame (above) and in the moving co-frame associated with the curve (below).

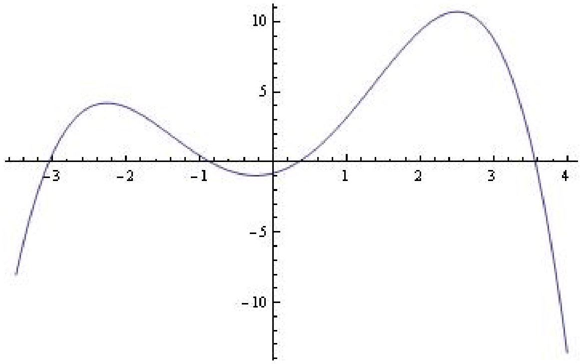


Equilibrium shapes with points of contact

In a rotating Hele-Shaw cell, there exist values of the angular velocity at which the interface exhibits shapes with points of contact that can be considered as the onset of drop formation and separation. Shapes in the fixed Cartesian frame (above) and in the moving co-frame associated with the curve (below).



Case 2. The polynomial $P(\xi)$ has four real roots $\alpha < \beta < \gamma < \delta$. Then, two periodic solutions exist:



$$\kappa_3(s) = \delta - \frac{(\delta-\alpha)(\delta-\beta)}{(\delta-\beta)+(\beta-\alpha)\,\mathrm{sn}^2(us,k)}$$

$$\varphi_3(s) = \delta s - \frac{\delta-\alpha}{u}\Pi\left(\frac{\beta-\alpha}{\beta-\delta}, \mathrm{am}(us,k), k\right)$$

and

$$\kappa_4(s) = \beta + \frac{(\gamma-\beta)(\delta-\beta)}{(\delta-\beta)-(\delta-\gamma)\,\mathrm{sn}^2(us,k)}$$

$$\varphi_4(s) = \beta s - \frac{\beta-\gamma}{u}\Pi\left(\frac{\delta-\gamma}{\delta-\beta}, \mathrm{am}(us,k), k\right)$$

where

$$u = \frac{1}{4}\sqrt{(\gamma-\alpha)(\delta-\beta)}, \qquad k = \sqrt{\frac{(\beta-\alpha)(\delta-\gamma)}{(\gamma-\alpha)(\delta-\beta)}}.$$

Equilibrium shapes of the fluid interface

Curvatures belonging to **Case 2** always give rise to self-intersecting curves. Here is an example of 4-fold symmetric curves in the fixed Cartesian frame (above) and in the moving co-frame associated with the curve (below). The curves in the most right column clarify the actual size and positions of the two curves.