Compatible metrics and integrable systems

Oleg I. Mokhov

Department of Geometry and Topology,
Faculty of Mechanics and Mathematics,
Moscow State University

XIIIth International Conference
on Geometry, Integrability and Quantization
June 3-8, 2011, Varna, Bulgaria

- Classification (an integrable description) of compatible Dubrovin–Novikov brackets (flat pencils of metrics or quasi-Frobenius manifolds)
- Classification (an integrable description) of compatible nonlocal Poisson brackets of hydrodynamic type (Mokhov–Ferapontov and general Ferapontov type)
- Classification of multi-dimensional Dubrovin–Novikov brackets of hydrodynamic type
- Riemann invariants for nonlocally bi-Hamiltonian systems of hydrodynamic type
- Integrable classes of compatible metrics

Compatible and almost compatible metrics. Definitions

Definition

Two Riemannian or pseudo-Riemannian contravariant metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are called compatible if for any linear combination of these metrics

$$g^{ij}(u) = \lambda_1 g_1^{ij}(u) + \lambda_2 g_2^{ij}(u),$$

where λ_1 and λ_2 are arbitrary constants, the coefficients of the corresponding Levi–Civita connections and the components of the corresponding Riemann curvature tensors are related by the same linear formula:

$$\Gamma_k^{ij}(u) = \lambda_1 \Gamma_{1,k}^{ij}(u) + \lambda_2 \Gamma_{2,k}^{ij}(u),$$

$$R_{kl}^{ij}(u) = \lambda_1 R_{1,kl}^{ij}(u) + \lambda_2 R_{2,kl}^{ij}(u).$$

Compatible and almost compatible metrics. Definitions

Definition

Two Riemannian or pseudo-Riemannian contravariant metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are called almost compatible if for any linear combination of these metrics

$$g^{ij}(u) = \lambda_1 g_1^{ij}(u) + \lambda_2 g_2^{ij}(u),$$

where λ_1 and λ_2 are arbitrary constants, the coefficients of the corresponding Levi–Civita connections are related by the same linear formula:

$$\Gamma_k^{ij}(u) = \lambda_1 \Gamma_{1,k}^{ij}(u) + \lambda_2 \Gamma_{2,k}^{ij}(u).$$

Any two almost compatible metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ form *a pencil of almost compatible metrics*.

Compatible metrics of constant Riemannian curvature

Consider two flat metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$.
In this case, the condition

$$R^{ij}_{kl}(u) = \lambda_1 R^{ij}_{1,kl}(u) + \lambda_2 R^{ij}_{2,kl}(u)$$

means exactly that any of the metrics of the pencil

$$g^{ij}(u) = \lambda_1 g_1^{ij}(u) + \lambda_2 g_2^{ij}(u),$$

where λ_1 and λ_2 are arbitrary constants, is also flat.
Thus, any two compatible flat metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ form *a pencil of compatible flat metrics*.
Generally speaking, it is not true for almost compatible flat metrics: for example, the flat two-component metrics $g_1^{ij}(u) = exp(u^1 u^2)\delta^{ij}$, $1 \leq i, j \leq 2$, and $g_2^{ij} = \delta^{ij}$, $1 \leq i, j \leq 2$, are almost compatible but they are not compatible and do not form a pencil of almost compatible flat metrics.

Compatible metrics of constant Riemannian curvature

Consider two metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ of constant Riemannian curvature K_1 and K_2, respectively, that is,

$$R^{ij}_{1,kl}(u) = K_1(\delta^i_l \delta^j_k - \delta^i_k \delta^j_l), \quad R^{ij}_{2,kl}(u) = K_2(\delta^i_l \delta^j_k - \delta^i_k \delta^j_l).$$

Here, K_1 and K_2 are arbitrary constants.
In this case, the condition

$$R^{ij}_{kl}(u) = \lambda_1 R^{ij}_{1,kl}(u) + \lambda_2 R^{ij}_{2,kl}(u)$$

means exactly that any of the metrics of the pencil

$$g^{ij}(u) = \lambda_1 g_1^{ij}(u) + \lambda_2 g_2^{ij}(u),$$

where λ_1 and λ_2 are arbitrary constants, is a metric of constant Riemannian curvature $\lambda_1 K_1 + \lambda_2 K_2$.
Thus, any two compatible metrics of constant Riemannian curvature form *a pencil of compatible metrics of constant Riemannian curvature*.

Motivation. Compatible Poisson brackets of hydrodynamic type

A Hamiltonian operator given by an arbitrary matrix homogeneous first-order ordinary differential operator, that is, a Hamiltonian operator of the form

$$P^{ij}[u(x)] = g^{ij}(u(x))\,\frac{d}{dx} + b^{ij}_k(u(x))\,u^k_x,$$

is called *a local Hamiltonian operator of hydrodynamic type* or *Dubrovin–Novikov Hamiltonian operator*.
The operator is called *nondegenerate* if $\det(g^{ij}(u)) \not\equiv 0$.
If $\det(g^{ij}(u)) \not\equiv 0$, then operator is Hamiltonian if and only if 1) $g^{ij}(u)$ is an arbitrary contravariant flat pseudo-Riemannian metric (a metric of zero Riemannian curvature), 2) $b^{ij}_k(u) = -g^{is}(u)\Gamma^j_{sk}(u)$, where $\Gamma^j_{sk}(u)$ is the Levi-Civita connection generated by the metric $g^{ij}(u)$ (the Dubrovin–Novikov theorem).

Motivation. Compatible Poisson brackets of hydrodynamic type

For any nondegenerate local Hamiltonian operator of hydroodynamic type there always exist local coordinates $v^1, ..., v^N$ (flat coordinates of the metric $g^{ij}(u)$) in which all the coefficients of the operator are constant:

$$\widetilde{g}^{ij}(v) = \eta^{ij} = \text{ const}, \ \ \widetilde{\Gamma}^i_{jk}(v) = 0, \ \ \widetilde{b}^{ij}_k(v) = 0,$$

that is the corresponding Poisson bracket has the form

$$\{I, J\} = \int \frac{\delta I}{\delta v^i(x)} \eta^{ij} \frac{d}{dx} \frac{\delta J}{\delta v^j(x)} dx,$$

where (η^{ij}) is a nondegenerate symmetric constant matrix:

$$\eta^{ij} = \eta^{ji}, \ \ \eta^{ij} = \text{const}, \ \ \det(\eta^{ij}) \neq 0.$$

Two nondegenerate Dubrovin–Novikov Hamiltonian operators $P_1^{ij}[u(x)]$ and $P_2^{ij}[u(x)]$ given by flat metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are compatible if and only if 1) any linear combination of these metrics

$$g^{ij}(u) = \lambda_1 g_1^{ij}(u) + \lambda_2 g_2^{ij}(u),$$

where λ_1 and λ_2 are arbitrary constants, is a flat metric, 2) the coefficients of the corresponding Levi-Civita connections are related by the same linear formula:

$$\Gamma^{ij}_k(u) = \lambda_1 \Gamma^{ij}_{1,k}(u) + \lambda_2 \Gamma^{ij}_{2,k}(u).$$

These conditions on flat metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ define a pencil of compatible flat metrics (Dubrovin's flat pencil of metrics or quasi-Frobenius manifold). So the problem of description of compatible nondegenerate Dubrovin–Novikov brackets is exactly the problem of description of pencils of compatible flat metrics.

Compatible and almost compatible metrics and the Nijenhuis tensor

Consider two arbitrary contravariant Riemannian or pseudo-Riemannian metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$.
Introduce the affinor

$$v_j^i(u) = g_1^{is}(u) g_{2,sj}(u),$$

where $g_{2,sj}(u)$ is the covariant metric inverse to the metric $g_2^{ij}(u)$: $g_2^{is}(u) g_{2,sj}(u) = \delta_j^i$.
Consider the Nijenhuis tensor of this affinor

$$N_{ij}^k(u) = v_i^s(u) \frac{\partial v_j^k}{\partial u^s} - v_j^s(u) \frac{\partial v_i^k}{\partial u^s} + v_s^k(u) \frac{\partial v_i^s}{\partial u^j} - v_s^k(u) \frac{\partial v_j^s}{\partial u^i}.$$

Theorem

Any two metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are almost compatible if and only if the corresponding Nijenhuis tensor $N_{ij}^k(u)$ vanishes.

Compatible and almost compatible metrics and the Nijenhuis tensor

Definition

Two Riemannian or pseudo-Riemannian metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are called a nonsingular (semisimple) pair of metrics if the eigenvalues of this pair of metrics, that is, the roots of the equation

$$\det(g_1^{ij}(u) - \lambda g_2^{ij}(u)) = 0,$$

are distinct. A pencil of metrics is called nonsingular if it is formed by a nonsingular pair of metrics.

Theorem

If a pair of metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ is nonsingular, then the metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are compatible if and only if the Nijenhuis tensor of the affinor $v_j^i(u) = g_1^{is}(u)g_{2,sj}(u)$ vanishes. Thus, a nonsingular pair of metrics is compatible if and only if the metrics are almost compatible.

Compatible and almost compatible metrics and the Nijenhuis tensor

Assume that a pair of metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ is nonsingular and the corresponding Nijenhuis tensor vanishes.

The eigenvalues of the pair of metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ coincide with the eigenvalues of the affinor $v_j^i(u) = g_1^{is}(u) g_{2,sj}(u)$.

If all eigenvalues of an affinor are distinct, then by the Nijenhuis theorem the vanishing of the Nijenhuis tensor of this affinor implies that there exist local coordinates such that, in these coordinates, the affinor reduces to a diagonal form in the corresponding neighbourhood.

So we can consider that the affinor $v_j^i(u)$ is diagonal in the local coordinates $u^1, ..., u^N$, that is,

$$v_j^i(u) = \lambda^i(u) \delta_j^i.$$

Compatible and almost compatible metrics and the Nijenhuis tensor

The eigenvalues $\lambda^i(u),\ i = 1, ..., N$, are distinct: $\lambda^i \neq \lambda^j$ if $i \neq j$.

Lemma

If the affinor $v^i_j(u) = g^{is}_1(u) g_{2,sj}(u)$ is diagonal in local coordinates and all its eigenvalues are distinct, then, in these coordinates, the metrics $g^{ij}_1(u)$ and $g^{ij}_2(u)$ are also diagonal.

We have $g^{ij}_1(u) = \lambda^i(u) g^{ij}_2(u)$. It follows from symmetry of the metrics $g^{ij}_1(u)$ and $g^{ij}_2(u)$ that for any indices i and j

$$(\lambda^i(u) - \lambda^j(u)) g^{ij}_2(u) = 0,$$

that is, $g^{ij}_2(u) = g^{ij}_1(u) = 0$ if $i \neq j$.

Compatible and almost compatible metrics and the Nijenhuis tensor

Lemma

Let an affinor $w^i_j(u)$ be diagonal in local coordinates $u = (u^1, ..., u^N)$: $w^i_j(u) = \mu^i(u)\delta^i_j$.

1) *If all the eigenvalues $\mu^i(u),\ i = 1, ..., N$, of the diagonal affinor are distinct, that is, $\mu^i(u) \neq \mu^j(u)$ for $i \neq j$, then the Nijenhuis tensor of this affinor vanishes if and only if the ith eigenvalue $\mu^i(u)$ depends only on the coordinate u^i.*
2) *If all the eigenvalues coincide, then the Nijenhuis tensor vanishes.*
3) *In the general case of an arbitrary diagonal affinor $w^i_j(u) = \mu^i(u)\delta^i_j$, the Nijenhuis tensor vanishes if and only if $\partial\mu^i/\partial u^j = 0$ for all indices i and j such that $\mu^i(u) \neq \mu^j(u)$.*

Compatible and almost compatible metrics and the Nijenhuis tensor

For any diagonal affinor $w^i_j(u) = \mu^i(u)\delta^i_j$, the Nijenhuis tensor $N^k_{ij}(u)$ has the form

$$N^k_{ij}(u) = (\mu^i - \mu^k)\frac{\partial \mu^j}{\partial u^i}\delta^{kj} - (\mu^j - \mu^k)\frac{\partial \mu^i}{\partial u^j}\delta^{ki}$$

(no summation over indices).
Thus, the Nijenhuis tensor vanishes if and only if for any indices i and j

$$(\mu^i(u) - \mu^j(u))\frac{\partial \mu^i}{\partial u^j} = 0,$$

where is no summation over indices.

Compatible and almost compatible metrics and the Nijenhuis tensor

Theorem

A nonsingular pair of metrics is compatible if and only if there exist local coordinates $u = (u^1, ..., u^N)$ *such that* $g_2^{ij}(u) = g^i(u)\delta^{ij}$ *and* $g_1^{ij}(u) = f^i(u^i)g^i(u)\delta^{ij}$, *where* $f^i(u^i)$, $i = 1, ..., N$, *are arbitrary (generally speaking, complex) functions of single variables (the functions* $f^i(u^i)$ *are not equal identically to zero and, for nonsingular pairs of metrics, all these functions must be distinct and they can not be equal to one another if they are constants but, nevertheless, in this special case, the metrics will be also compatible).*

Integrability of the class of compatible flat metrics

Consider the problem on nonsingular pairs of compatible flat metrics. It is sufficient to classify all pairs of flat metrics of the following special diagonal form $g_2^{ij}(u) = g^i(u)\delta^{ij}$ and $g_1^{ij}(u) = f^i(u^i)g^i(u)\delta^{ij}$, where $f^i(u^i)$, $i = 1, ..., N$, are arbitrary (possibly, complex) functions of single variables.

The problem of description of diagonal flat metrics, that is, flat metrics $g_2^{ij}(u) = g^i(u)\delta^{ij}$, is a classical problem of differential geometry. This problem is equivalent to the problem of description of curvilinear orthogonal coordinate systems in an N-dimensional pseudo-Euclidean space and it was studied in detail and mainly solved in the beginning of the 20th century. Locally, such coordinate systems are determined by $N(N-1)/2$ arbitrary functions of two variables. In 1998 Zakharov showed that the Lamé equations describing curvilinear orthogonal coordinate systems can be integrated by the inverse scattering method.

Integrability of the class of compatible flat metrics

The condition that the metric $g_1^{ij}(u) = f^i(u^i) g^i(u) \delta^{ij}$ is also flat exactly gives $N(N-1)/2$ additional equations linear with respect to the functions $f^i(u^i)$.
Introduce the standard classical notation

$$g^i(u) = \frac{1}{(H_i(u))^2}, \quad d\,s^2 = \sum_{i=1}^{N} (H_i(u))^2 (du^i)^2, \tag{1}$$

$$\beta_{ik}(u) = \frac{1}{H_i(u)} \frac{\partial H_k}{\partial u^i}, \quad i \neq k, \tag{2}$$

where $H_i(u)$ are the *Lamé coefficients* and $\beta_{ik}(u)$ are the *rotation coefficients* of the diagonal metric.

Integrability of the class of compatible flat metrics

Theorem (Lamé)

The class of flat diagonal metrics is described by the following nonlinear system (the Lamé equations):

$$\frac{\partial \beta_{ij}}{\partial u^k} = \beta_{ik}\beta_{kj}, \quad i \neq j, \ \ i \neq k, \ \ j \neq k,$$

$$\frac{\partial \beta_{ij}}{\partial u^i} + \frac{\partial \beta_{ji}}{\partial u^j} + \sum_{s\neq i,\ s\neq j} \beta_{si}\beta_{sj} = 0, \quad i \neq j.$$

For the diagonal metric $g_1^{ij}(u) = f^i(u^i) g^i(u) \delta^{ij}$, we have

$$\widetilde{H}_i(u) = \frac{H_i(u)}{\sqrt{f^i(u^i)}}, \quad \widetilde{\beta}_{ik}(u) = \frac{1}{\widetilde{H}_i(u)} \frac{\partial \widetilde{H}_k}{\partial u^i} =$$

$$\frac{\sqrt{f^i(u^i)}}{\sqrt{f^k(u^k)}} \left(\frac{1}{H_i(u)} \frac{\partial H_k}{\partial u^i} \right) = \frac{\sqrt{f^i(u^i)}}{\sqrt{f^k(u^k)}} \beta_{ik}(u), \quad i \neq k.$$

Integrability of the class of compatible flat metrics

Theorem

Nonsingular pairs of compatible flat metrics are described by the following nonlinear reduction of the Lamé equations:

$$\frac{\partial \beta_{ij}}{\partial u^k} = \beta_{ik}\beta_{kj}, \quad i \neq j, \ \ i \neq k, \ \ j \neq k,$$

$$\frac{\partial \beta_{ij}}{\partial u^i} + \frac{\partial \beta_{ji}}{\partial u^j} + \sum_{s\neq i,\ s\neq j} \beta_{si}\beta_{sj} = 0, \quad i \neq j,$$

$$f^i(u^i)\frac{\partial \beta_{ij}}{\partial u^i} + \frac{1}{2}(f^i(u^i))'\beta_{ij} + f^j(u^j)\frac{\partial \beta_{ji}}{\partial u^j} + \\ + \frac{1}{2}(f^j(u^j))'\beta_{ji} + \sum_{s\neq i,\ s\neq j} f^s(u^s)\beta_{si}\beta_{sj} = 0, \quad i \neq j,$$

where $f^i(u^i)$ are nonzero arbitrary functions (the eigenvalues).

Integrability of the class of compatible flat metrics

The two-dimensional case $N = 2$ is trivial.
The Lamé equations:

$$\frac{\partial \beta_{12}}{\partial u^1} + \frac{\partial \beta_{21}}{\partial u^2} = 0.$$

Hence there exist locally a function $F(u^1, u^2)$ such that

$$\beta_{12}(u) = \frac{\partial F}{\partial u^2}, \quad \beta_{21}(u) = -\frac{\partial F}{\partial u^1},$$

$$\frac{\partial H_1}{\partial u^2} = -\frac{\partial F}{\partial u^1} H_2(u), \quad \frac{\partial H_2}{\partial u^1} = \frac{\partial F}{\partial u^2} H_1(u).$$

Integrability of the class of compatible flat metrics

Theorem

The two-dimensional metrics $g_1^{ij}(u) = (f^i(u^i)/(H_i(u))^2)\delta^{ij}$ *and* $g_2^{ij}(u) = (1/(H_i(u))^2)\delta^{ij}$ *form a pencil of flat compatible metrics if and only if the Lamé coefficients* $H_i(u),\ i = 1, 2,$ *are solutions of the linear system*

$$\frac{\partial H_1}{\partial u^2} = -\frac{\partial F}{\partial u^1} H_2(u), \quad \frac{\partial H_2}{\partial u^1} = \frac{\partial F}{\partial u^2} H_1(u),$$

where the function $F(u)$ *is a solution of the following linear equation:*

$$2\frac{\partial^2 F}{\partial u^1 \partial u^2}(f^1(u^1) - f^2(u^2)) + \frac{\partial F}{\partial u^2}\frac{df^1(u^1)}{du^1} - \frac{\partial F}{\partial u^1}\frac{df^2(u^2)}{du^2} = 0.$$

Integrability of the class of compatible flat metrics

Recall the Zakharov method for integrating the Lamé equations (1998).
We must choose a matrix function $F_{ij}(s, s', u)$ and solve the linear integral equation

$$K_{ij}(s, s', u) = F_{ij}(s, s', u) + \int_s^\infty \sum_l K_{il}(s, q, u) F_{lj}(q, s', u) dq.$$

Then we obtain a one-parameter family of solutions of the Lamé equations by the formula

$$\beta_{ij}(s, u) = K_{ji}(s, s, u).$$

Integrability of the class of compatible flat metrics

If $F_{ij}(s, s', u) = f_{ij}(s - u^i, s' - u^j)$, where $f_{ij}(x, y)$ is an arbitrary matrix function of two variables, then the formula

$$\beta_{ij}(s, u) = K_{ji}(s, s, u)$$

produces solutions of the Darboux equations.
To satisfy the Lamé equations, Zakharov proposed to impose on the "dressing matrix function" $F_{ij}(s, s', u)$ a certain additional linear differential relation

$$\frac{\partial F_{ij}(s, s', u)}{\partial s'} + \frac{\partial F_{ji}(s', s, u)}{\partial s} = 0.$$

If $F_{ij}(s - u^i, s' - u^j)$ satisfy the Zakharov differential relation, then the rotation coefficients $\beta_{ij}(u)$ satisfy the Lamé equations.

Integrability of the class of compatible flat metrics

Lemma

If both the function $F_{ij}(s - u^i, s' - u^j)$ and the function

$$\widetilde{F}_{ij}(s - u^i, s' - u^j) = \frac{\sqrt{f^j(u^j - s')}}{\sqrt{f^i(u^i - s)}} F_{ij}(s - u^i, s' - u^j)$$

satisfy the Zakharov differential relation, then the corresponding rotation coefficients $\beta_{ij}(u)$ satisfy the equations describing all nonsingular pairs of compatible flat metrics.

Integrability of the class of compatible flat metrics

To resolve the Zakharov differential relations for the matrix function $F_{ij}(s - u^i, s' - u^j)$, we can introduce $N(N-1)/2$ arbitrary functions of two variables $\Phi_{ij}(x, y)$, $i < j$, and put for $i < j$

$$F_{ij}(s - u^i, s' - u^j) = \frac{\partial \Phi_{ij}(s - u^i, s' - u^j)}{\partial s},$$
$$F_{ji}(s - u^i, s' - u^j) = -\frac{\partial \Phi_{ij}(s' - u^i, s - u^j)}{\partial s},$$

and

$$F_{ii}(s - u^i, s' - u^i) = \frac{\partial \Phi_{ii}(s - u^i, s' - u^i)}{\partial s},$$

where $\Phi_{ii}(x, y)$, $i = 1, ..., N$, are arbitrary skew-symmetric functions of two variables:

$$\Phi_{ii}(x, y) = -\Phi_{ii}(y, x).$$

Integrability of the class of compatible flat metrics

For the function

$$\widetilde{F}_{ij}(s-u^i, s'-u^j) = \frac{\sqrt{f^j(u^j-s')}}{\sqrt{f^i(u^i-s)}} F_{ij}(s-u^i, s'-u^j),$$

the Zakharov differential relation exactly gives $N(N-1)/2$ linear partial differential equations of the second order for $N(N-1)/2$ functions $\Phi_{ij}(s-u^i, s'-u^j),\ i<j$, of two variables:

$$2\frac{\partial^2 \Phi_{ij}(s-u^i, s'-u^j)}{\partial u^i \partial u^j}\left(f^i(u^i-s) - f^j(u^j-s')\right) +$$
$$\frac{\partial \Phi_{ij}(s-u^i, s'-u^j)}{\partial u^j}\frac{df^i(u^i-s)}{du^i} -$$
$$\frac{\partial \Phi_{ij}(s-u^i, s'-u^j)}{\partial u^i}\frac{df^j(u^j-s')}{du^j} = 0, \quad i<j.$$

It is very interesting that these equations coincide with the single equation for the two-component case.

Integrability of the class of compatible flat metrics

For N functions $\Phi_{ii}(s - u^i, s' - u^i)$, we have also N linear partial differential equations of the second order from the Zakharov differential relation:

$$2\frac{\partial^2 \Phi_{ii}(s - u^i, s' - u^i)}{\partial s \partial s'} \left(f^i(u^i - s) - f^i(u^i - s') \right) -$$
$$\frac{\partial \Phi_{ii}(s - u^i, s' - u^i)}{\partial s} \frac{d f^i(u^i - s')}{d s'} +$$
$$\frac{\partial \Phi_{ii}(s - u^i, s' - u^i)}{\partial s'} \frac{d f^i(u^i - s)}{d s} = 0.$$

Any solution of these linear partial differential equations generates a one-parameter family of solutions of the system describing all nonsingular pairs of compatible flat metrics. Thus, our problem is linearized.

Compatible metrics and compatible Poisson brackets of hydrodynamic type

Consider two arbitrary nonlocal Poisson brackets of hydrodynamic type (of the Ferapontov type)

$$\{I, J\}_1 = \int \frac{\delta I}{\delta u^i(x)} \left(g_1^{ij}(u(x)) \frac{d}{dx} + b^{ij}_{1,k}(u(x))\, u^k_x + \sum_{\alpha=1}^{L_1} \varepsilon_{1,\alpha} (w_1^\alpha)^i_k(u(x)) u^k_x \left(\frac{d}{dx}\right)^{-1} (w_1^\alpha)^j_s(u(x)) u^s_x \right) \frac{\delta J}{\delta u^j(x)} dx$$

and

$$\{I, J\}_2 = \int \frac{\delta I}{\delta u^i(x)} \left(g_2^{ij}(u(x)) \frac{d}{dx} + b^{ij}_{2,k}(u(x))\, u^k_x + \sum_{\alpha=1}^{L_2} \varepsilon_{2,\alpha} (w_2^\alpha)^i_k(u(x)) u^k_x \left(\frac{d}{dx}\right)^{-1} (w_2^\alpha)^j_s(u(x)) u^s_x \right) \frac{\delta J}{\delta u^j(x)} dx.$$

Compatible metrics and compatible Poisson brackets of hydrodynamic type

Theorem

If the pair of metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ is nonsingular, then the Poisson brackets $\{I, J\}_1$ and $\{I, J\}_2$ are compatible if and only if the metrics are compatible and both the metrics $g_1^{ij}(u)$, $g_2^{ij}(u)$ and the affinors $(w_1^\alpha)_j^i(u)$, $(w_2^\alpha)_j^i(u)$ can be samultaneously diagonalized in a domain of local coordinates.

If the pair of metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ is nonsingular, then the Poisson brackets $\{I, J\}_1$ and $\{I, J\}_2$ are compatible if and only if there exist local coordinates such that in these coordinates we have

$$g_2^{ij}(u) = g^i(u)\delta^{ij}, \quad g_1^{ij}(u) = f^i(u^i)g^i(u)\delta^{ij},$$

$$(w_2^\alpha)_j^i(u) = (w_2^\alpha)^i(u)\delta_j^i, \quad (w_1^\alpha)_j^i(u) = (w_1^\alpha)^i(u)\delta_j^i.$$

Equations for nonsingular pairs of compatible nonlocal Poisson brackets of hydrodynamic type

Consider an arbitrary nonsingular pair of compatible nonlocal Poisson brackets of hydrodynamic type, that is, we assume that the Poisson brackets are compatible and the pair of metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ is nonsingular.

Theorem

General nonsingular pairs of compatible nonlocal Poisson brackets of hydrodynamic type are described by the following consistent integrable nonlinear systems:

$$\frac{\partial H^{\alpha}_{2,j}}{\partial u^i} = \beta_{ij} H^{\alpha}_{2,i}, \quad i \neq j,$$

$$\frac{\partial \beta_{ij}}{\partial u^k} = \beta_{ik}\beta_{kj}, \quad i \neq j, \ \ i \neq k, \ \ j \neq k,$$

$$\epsilon^i_2 \frac{\partial \beta_{ij}}{\partial u^i} + \epsilon^j_2 \frac{\partial \beta_{ji}}{\partial u^j} + \sum_{s \neq i, \ s \neq j} \epsilon^s_2 \beta_{si}\beta_{sj} + \sum_{\alpha=1}^{L_2} \varepsilon_{2,\alpha} H^{\alpha}_{2,i} H^{\alpha}_{2,j} = 0, \quad i \neq j.$$

$$\epsilon^i_2 f^i(u^i) \frac{\partial \beta_{ij}}{\partial u^i} + \frac{1}{2}\epsilon^i_2 (f^i)' \beta_{ij} + \epsilon^j_2 f^j(u^j) \frac{\partial \beta_{ji}}{\partial u^j} + \frac{1}{2}\epsilon^j_2 (f^j)' \beta_{ji} +$$

$$\sum_{s \neq i, \ s \neq j} \epsilon^s_2 f^s(u^s) \beta_{si}\beta_{sj} + \sum_{\alpha=1}^{L_1} \varepsilon_{1,\alpha} H^{\alpha}_{1,i} H^{\alpha}_{1,j} = 0, \quad i \neq j,$$

$$\frac{\partial H^{\alpha}_{1,j}}{\partial u^i} = \beta_{ij} H^{\alpha}_{1,i}, \quad i \neq j.$$

We introduce here the standard classical notation

$$g^i(u) = \frac{\epsilon_2^i}{(H_i(u))^2}, \quad d\,s^2 = \sum_{i=1}^{N} \epsilon_2^i (H_i(u))^2 (du^i)^2, \tag{3}$$

$$\beta_{ik}(u) = \frac{1}{H_i(u)} \frac{\partial H_k}{\partial u^i}, \quad i \neq k, \tag{4}$$

where $H_i(u)$ are *the Lamé coefficients* and $\beta_{ik}(u)$ are *the rotation coefficients*, $\epsilon_2^i = \pm 1$, $i = 1, ..., N$, and introduce the functions $H^{\alpha}_{2,i}(u)$, $1 \leq i \leq N$, $1 \leq \alpha \leq L_2$, such that

$$(w_2^{\alpha})^i(u) = \frac{H^{\alpha}_{2,i}(u)}{H_i(u)},$$

and the functions $H^{\alpha}_{1,i}(u)$, $1 \leq i \leq N$, $1 \leq \alpha \leq L_1$, such that

$$(w_1^{\alpha})^i(u) = \frac{H^{\alpha}_{1,i}(u)}{H_i(u)}.$$

Lax pair for the general nonsingular pair of compatible nonlocal Poisson brackets of hydrodynamic type

The Lax pair with a spectral parameter for the system describing all nonsingular pairs of compatible nonlocal Poisson brackets of hydrodynamic type can be derived from the linear problem for the system describing all submanifolds with flat normal bundle and holonomic net of curvature lines. The equations describing all submanifolds with flat normal bundle and holonomic net of curvature lines are the conditions of consistency for the following linear system:

Lax pair for the general nonsingular pair of compatible nonlocal Poisson brackets of hydrodynamic type

$$\frac{\partial \varphi_i}{\partial u^k} = \frac{\sqrt{\epsilon_2^i}}{\sqrt{\epsilon_2^k}} \beta_{ik} \varphi_k, \quad i \neq k,$$

$$\frac{\partial \varphi_i}{\partial u^i} = -\sum_{k \neq i} \frac{\sqrt{\epsilon_2^k}}{\sqrt{\epsilon_2^i}} \beta_{ki} \varphi_k + \sum_{\alpha=1}^{L_2} \frac{\sqrt{\varepsilon_{2,\alpha}}}{\sqrt{\epsilon_2^i}} H^{\alpha}_{2,i} \psi^{\alpha},$$

$$\frac{\partial \psi^{\alpha}}{\partial u^i} = -\frac{\sqrt{\varepsilon_{2,\alpha}}}{\sqrt{\epsilon_2^i}} H^{\alpha}_{2,i} \varphi_i.$$

Lax pair for the general nonsingular pair of compatible nonlocal Poisson brackets of hydrodynamic type

The condition that the bracket $\{I, J\}_1 + \lambda\{I, J\}_2$ is a Poisson bracket for any λ is equivalent to the system corresponding to the nonlocal Poisson bracket of hydrodynamic type with the metric $(\lambda + f^i(u^i))g^i(u)\delta^{ij}$ and the affinors $(w_1^\beta)^i_j(u)$, $1 \leq \beta \leq L_1$, and $\sqrt{\lambda}(w_2^\alpha)^i_j(u)$, $1 \leq \alpha \leq L_2$. In this case, the linear problem becomes the Lax pair with the spectral parameter λ for the general nonsingular pair of arbitrary compatible nonlocal Poisson brackets of hydrodynamic type:

$$\frac{\partial \varphi_i}{\partial u^k} = \frac{\sqrt{\epsilon_2^i(\lambda + f^i)}}{\sqrt{\epsilon_2^k(\lambda + f^k)}}\beta_{ik}\varphi_k, \quad i \neq k, \tag{5}$$

$$\frac{\partial \varphi_i}{\partial u^i} = -\sum_{k \neq i} \frac{\sqrt{\epsilon_2^k(\lambda + f^k)}}{\sqrt{\epsilon_2^i(\lambda + f^i)}}\beta_{ki}\varphi_k +$$

$$\sum_{\alpha=1}^{L_2} \frac{\sqrt{\varepsilon_{2,\alpha}\lambda}}{\sqrt{\epsilon_2^i(\lambda + f^i)}} H_{2,i}^{\alpha}\psi^{\alpha} + \sum_{\beta=1}^{L_1} \frac{\sqrt{\varepsilon_{1,\beta}}}{\sqrt{\epsilon_2^i(\lambda + f^i)}} H_{1,i}^{\beta}\chi^{\beta}, \tag{6}$$

$$\frac{\partial \psi^{\alpha}}{\partial u^i} = -\frac{\sqrt{\varepsilon_{2,\alpha}\lambda}}{\sqrt{\epsilon_2^i(\lambda + f^i)}} H_{2,i}^{\alpha}\varphi_i, \tag{7}$$

$$\frac{\partial \chi^{\beta}}{\partial u^i} = -\frac{\sqrt{\varepsilon_{1,\beta}}}{\sqrt{\epsilon_2^i(\lambda + f^i)}} H_{1,i}^{\beta}\varphi_i. \tag{8}$$

If $H^{\alpha}_{1,i}(u) = 0$, $1 \leq \alpha \leq L_1$, and $H^{\alpha}_{2,i}(u) = 0$, $1 \leq \alpha \leq L_2$, then we describe all compatible local Poisson brackets of hydrodynamic type (compatible Dubrovin–Novikov brackets or flat pencils of metrics). (This Lax pair was found by Ferapontov.)
If $H^{\alpha}_{1,i}(u) = \sqrt{\varepsilon_{1,1} K_1} H_i(u)$, $\alpha = 1$, $L_1 = 1$, and $H^{\alpha}_{2,i}(u) = \sqrt{\varepsilon_{2,1} K_2} H_i(u)$, $\alpha = 1$, $L_2 = 1$, then we describe all compatible nonlocal Poisson brackets of hydrodynamic type generated by metrics of constant Riemannian curvature. The Lax pair corresponding to arbitrary nonsingular pencils of metrics of constant Riemannian curvature can be also easily derived from the linear problem for the system describing all the orthogonal curvilinear coordinate systems in N-dimensional spaces of constant curvature K_2:

$$\frac{\partial \varphi_i}{\partial u^j} = \frac{\sqrt{\varepsilon^i}}{\sqrt{\varepsilon^j}} \beta_{ij} \varphi_j, \quad i \neq j,$$

$$\frac{\partial \varphi_i}{\partial u^i} = -\sum_{k \neq i} \frac{\sqrt{\varepsilon^k}}{\sqrt{\varepsilon^i}} \beta_{ki} \varphi_k + \frac{\sqrt{K_2}}{\sqrt{\varepsilon^i}} H_i \psi,$$

$$\frac{\partial \psi}{\partial u^i} = -\frac{\sqrt{K_2}}{\sqrt{\varepsilon^i}} H_i \varphi_i.$$

The condition of consistensy for the linear system gives the equations for all orthogonal curvilinear coordinate systems in N-dimensional spaces of constant Riemannian curvature K_2.

The corresponding Lax pair with a spectral parameter for nonsingular pencils of metrics of constant Riemannian curvature has the form:

$$
\begin{aligned}
\frac{\partial \varphi_i}{\partial u^j} &= \sqrt{\frac{\varepsilon^i(\lambda + f^i)}{\varepsilon^j(\lambda + f^j)}}\beta_{ij}\varphi_j, \quad i \neq j,\\
\frac{\partial \varphi_i}{\partial u^i} &= -\sum_{k\neq i}\sqrt{\frac{\varepsilon^k(\lambda + f^k)}{\varepsilon^i(\lambda + f^i)}}\beta_{ki}\varphi_k + \sqrt{\frac{\lambda K_2 + K_1}{\varepsilon^i(\lambda + f^i)}}H_i\psi,\\
\frac{\partial \psi}{\partial u^i} &= -\sqrt{\frac{\lambda K_2 + K_1}{\varepsilon^i(\lambda + f^i)}}H_i\varphi_i,
\end{aligned}
\tag{9}
$$

where λ is a spectral parameter. The condition of consistency for the linear system is equivalent to the equations for nonsingular pencils of metrics of constant Riemannian curvature.

If $H^{\alpha}_{2,i}(u) = 0$, $1 \leq i \leq N$, $1 \leq \alpha \leq L_1$, then the corresponding integrable systems describe compatible pairs of Poisson brackets of hydrodynamic type one of which is local. These systems always give integrable reductions of the classical Lamé equations. The corresponding Lax pairs with a spectral parameter have the form:

$$\frac{\partial \varphi_i}{\partial u^k} = \frac{\sqrt{\epsilon^i_2(\lambda + f^i)}}{\sqrt{\epsilon^k_2(\lambda + f^k)}} \beta_{ik} \varphi_k, \quad i \neq k,$$

$$\frac{\partial \varphi_i}{\partial u^i} = -\sum_{k \neq i} \frac{\sqrt{\epsilon^k_2(\lambda + f^k)}}{\sqrt{\epsilon^i_2(\lambda + f^i)}} \beta_{ki} \varphi_k + \sum_{\beta=1}^{L_1} \frac{\sqrt{\varepsilon_{1,\beta}}}{\sqrt{\epsilon^i_2(\lambda + f^i)}} H^{\beta}_{1,i} \chi^{\beta},$$

$$\frac{\partial \chi^{\beta}}{\partial u^i} = -\frac{\sqrt{\varepsilon_{1,\beta}}}{\sqrt{\epsilon^i_2(\lambda + f^i)}} H^{\beta}_{1,i} \varphi_i.$$

Theorem. *For an arbitrary non-singular* (*semisimple*) *non-locally bi-Hamiltonian system of hydrodynamic type, there exist local coordinates* (*Riemann invariants*) *such that all the related matrix differential-geometric objects, namely, the matrix* $V^i_j(u)$ *of this system of hydrodynamic type, the metrics* $g^{ij}_1(u)$ *and* $g^{ij}_2(u)$ *and the affinors* $(w_{1,n})^i_j(u)$ *and* $(w_{2,n})^i_j(u)$ *of the non-local bi-Hamiltonian structure of this system, are diagonal in these local coordinates.*