Rotation and Orientation: *Fundamentals*

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VARNA, 2011

What is Rotation ?

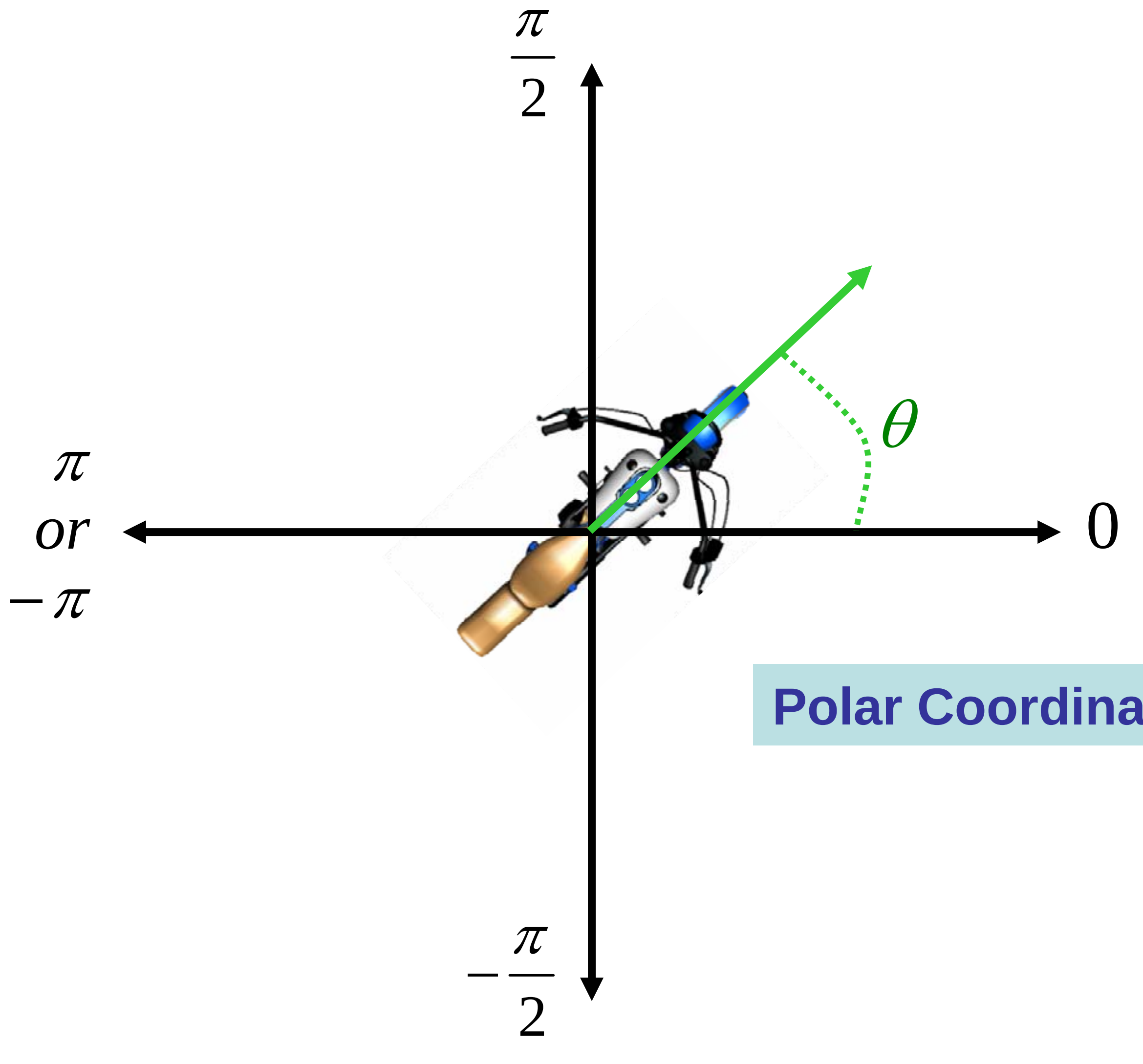
- Not intuitive
  - Formal definitions are also confusing

- Many different ways to describe
  - Rotation (direction cosine) matrix
  - Euler angles
  - Axis-angle
  - Rotation vector
  - Helical angles
  - Unit quaternions

Orientation vs. Rotation

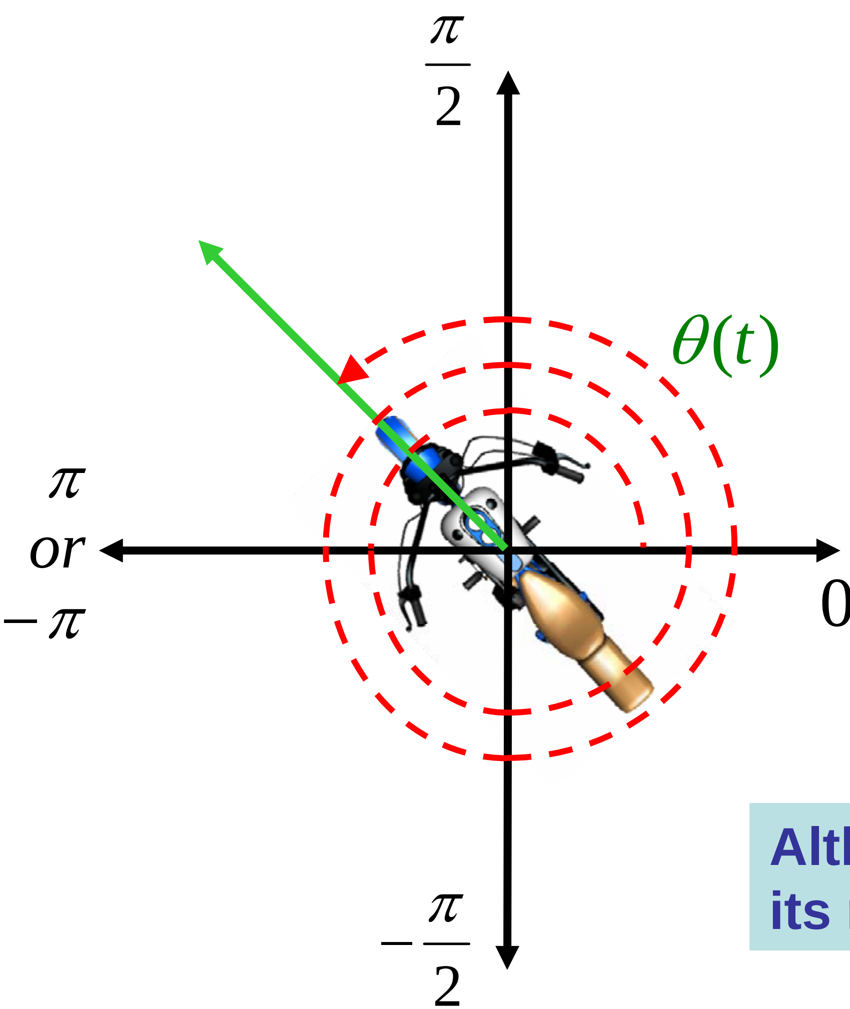
- ***Rotation***
  - Circular movement
- ***Orientation***
  - The state of being oriented
  - Given a coordinate system, the orientation of an object can be represented as a rotation from a reference pose
- Analogy
  - (point : vector) is similar to (orientation : rotation)
  - Both represent a sort of (state : movement)

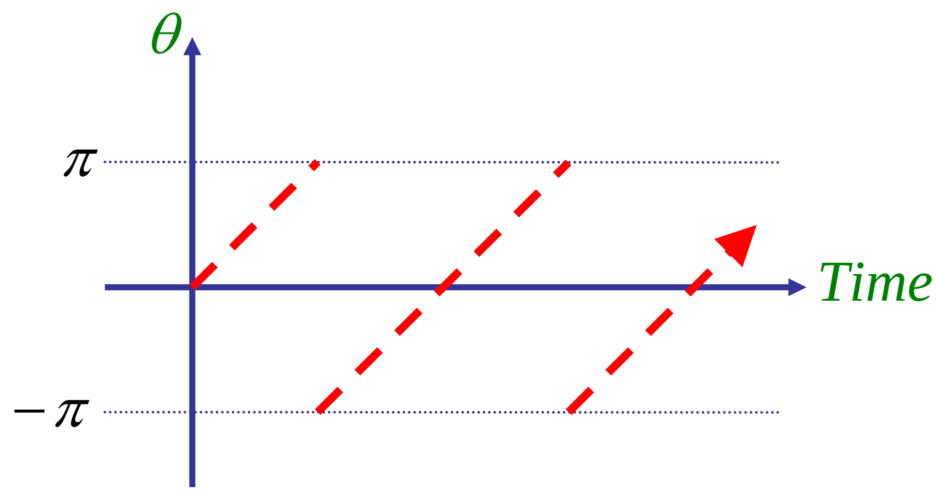
2D Orientation



Polar Coordinates

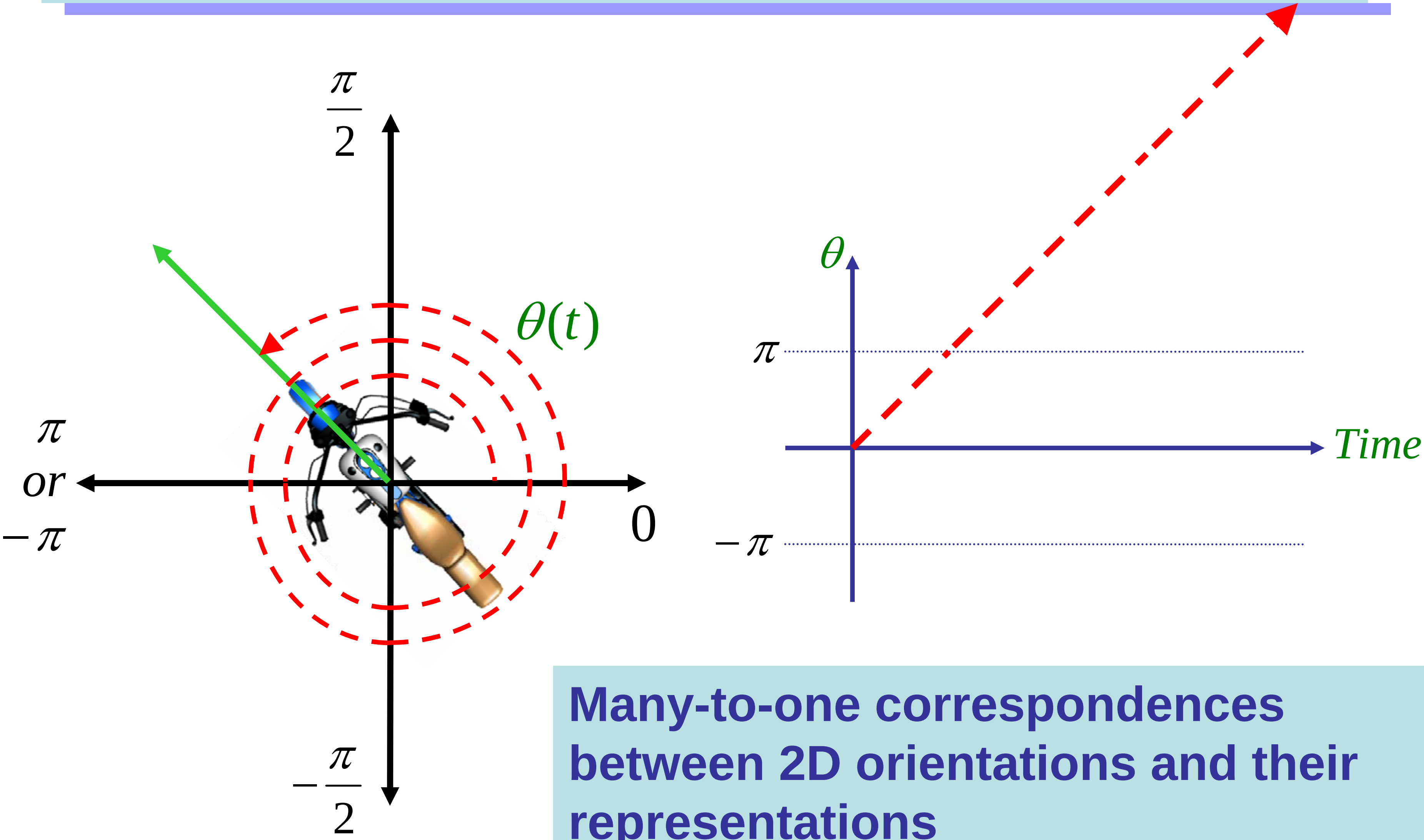
2D Orientation





Although the motion is continuous,
its representation could be discontinuous

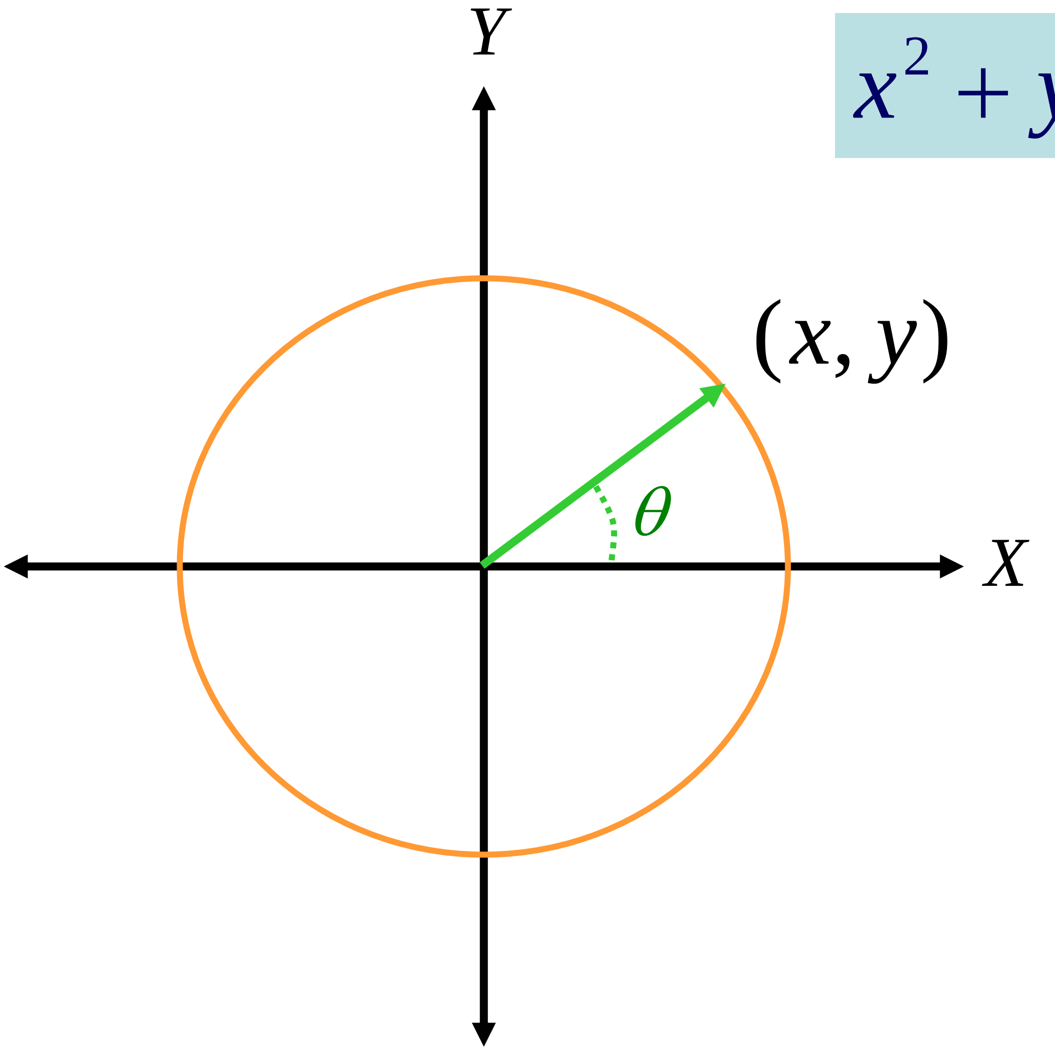
2D Orientation



Many-to-one correspondences between 2D orientations and their representations

Extra Parameter

$$x^2 + y^2 = 1$$

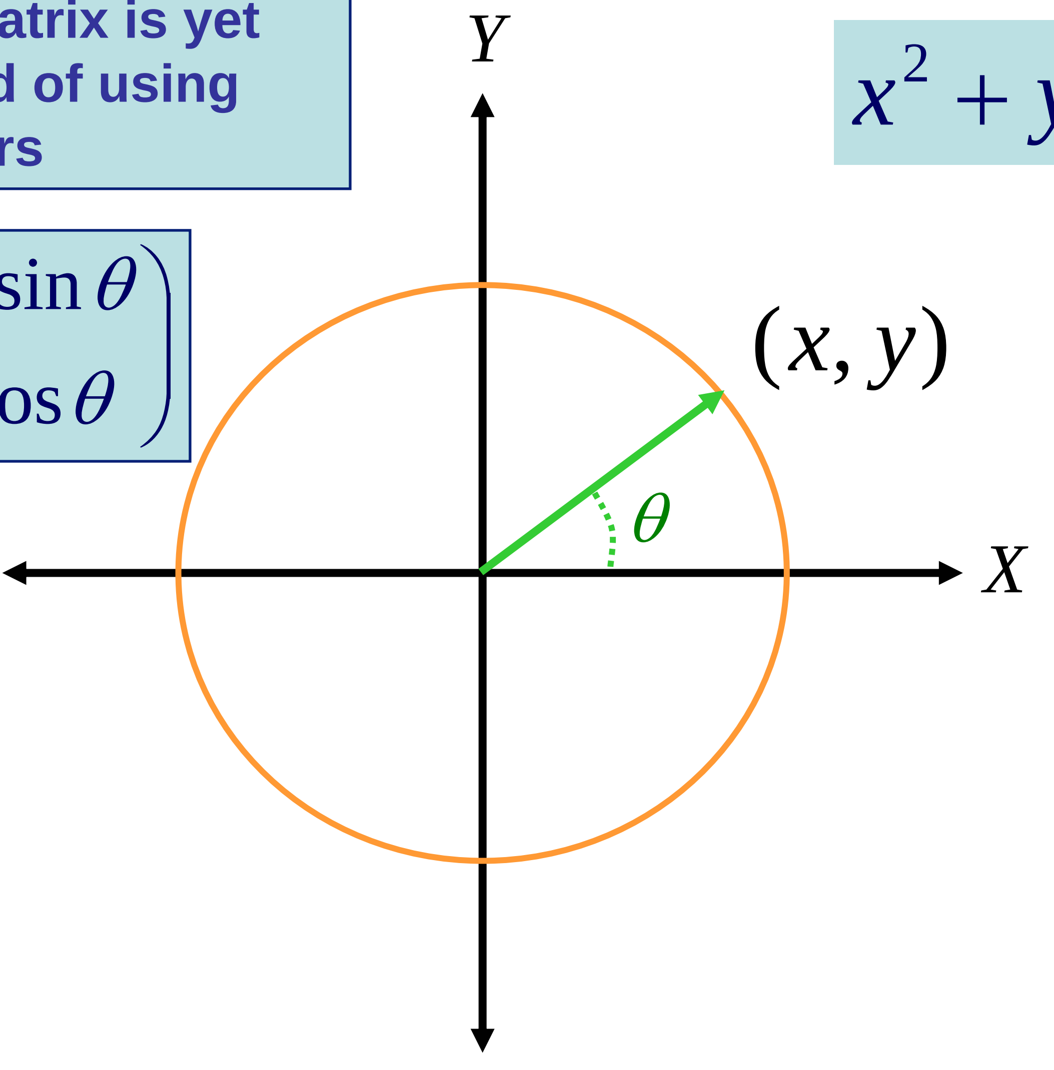


Extra Parameter

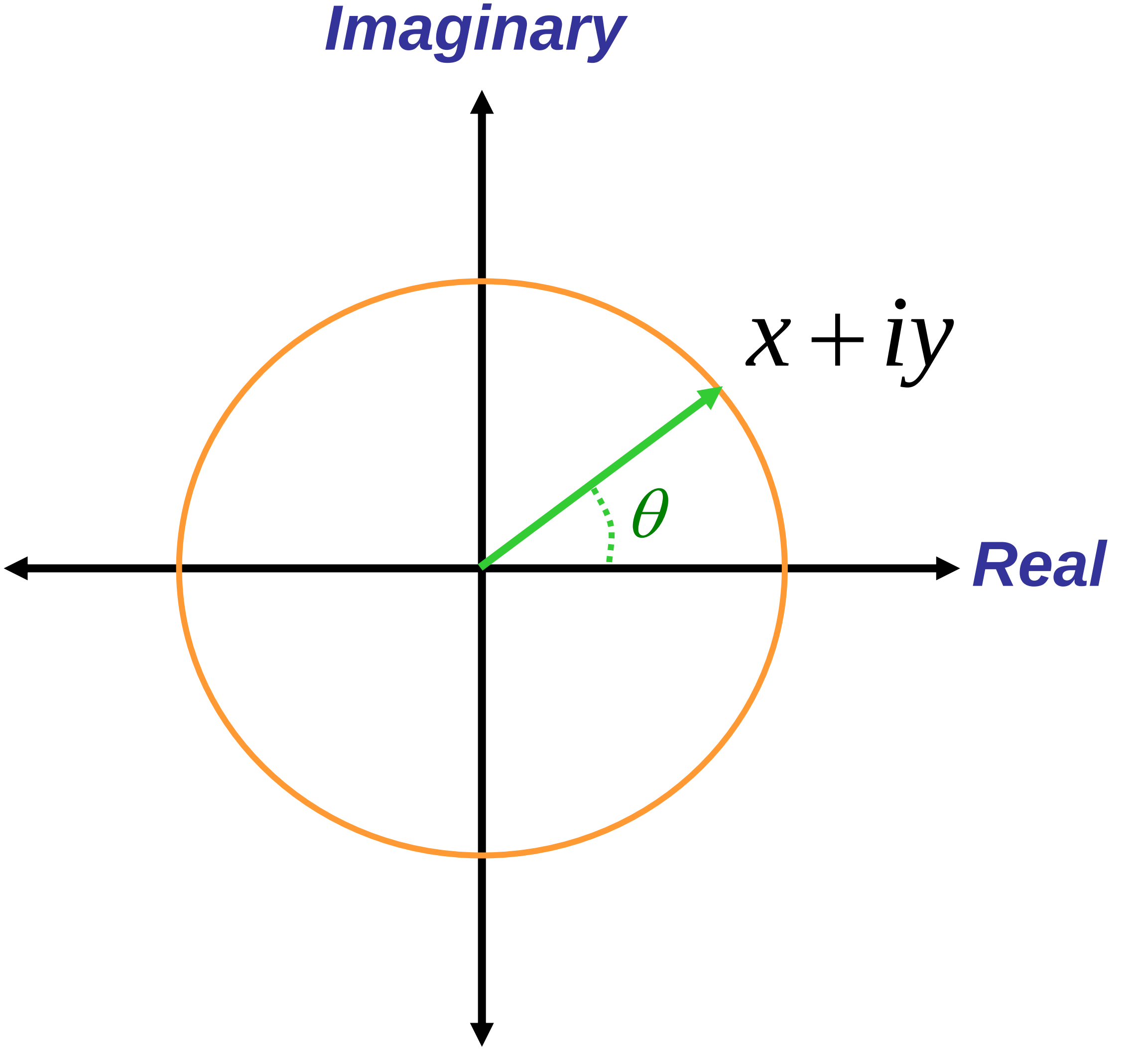
2x2 Rotation matrix is yet another method of using extra parameters

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$x^2 + y^2 = 1$$

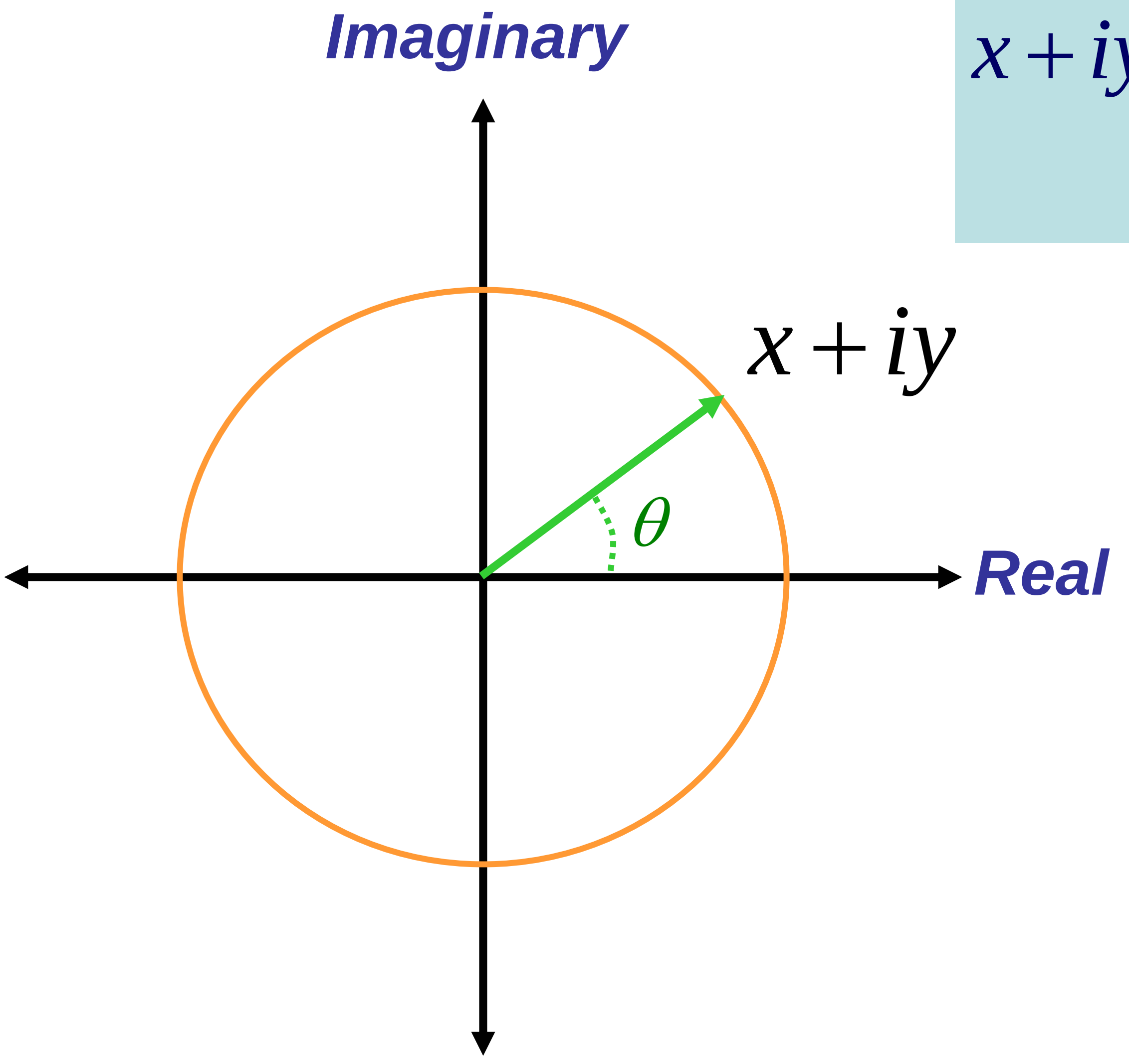


Complex Number

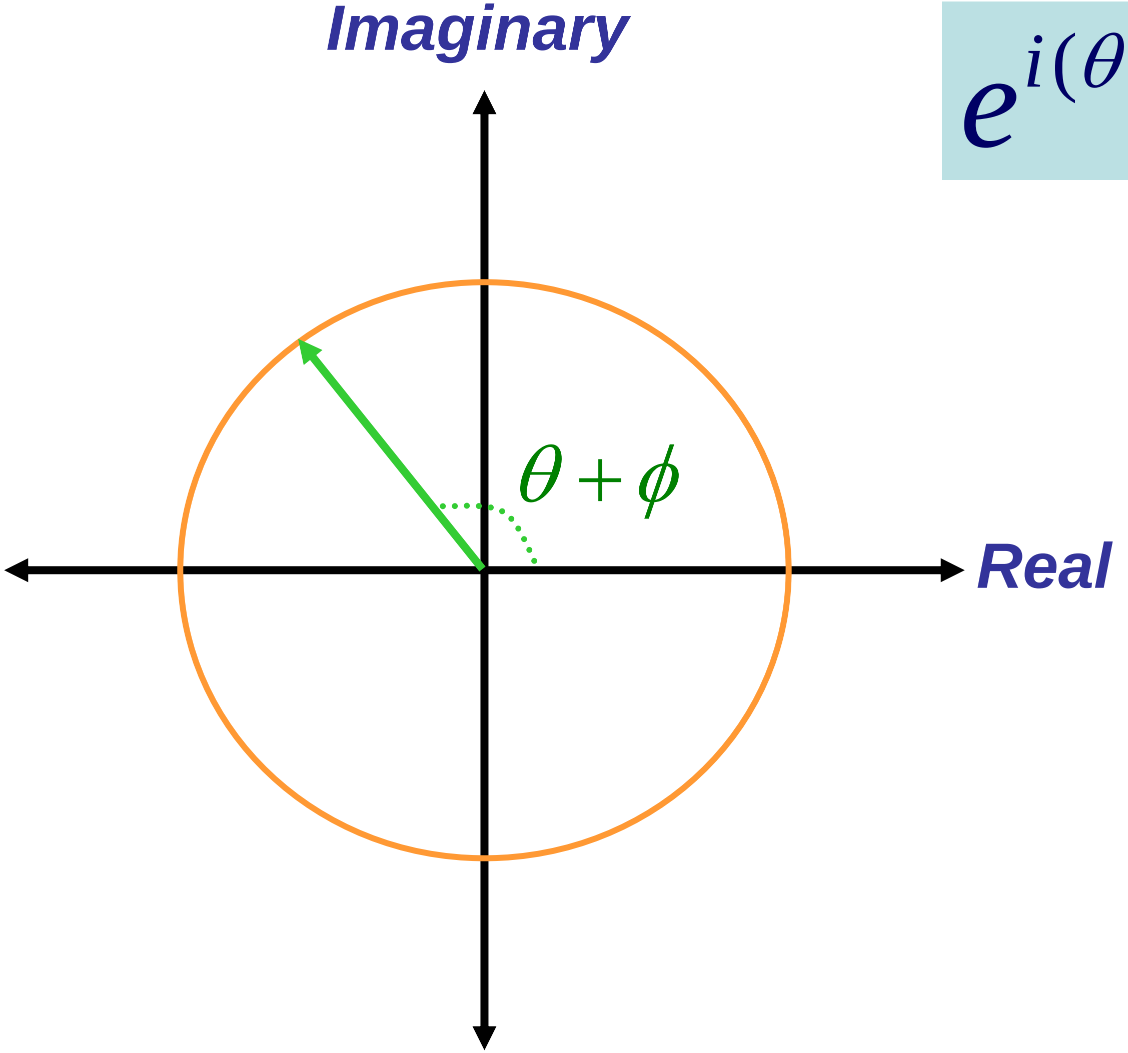


Complex Exponentiation

$$x + iy = \cos\theta + i\sin\theta$$
$$= e^{i\theta}$$



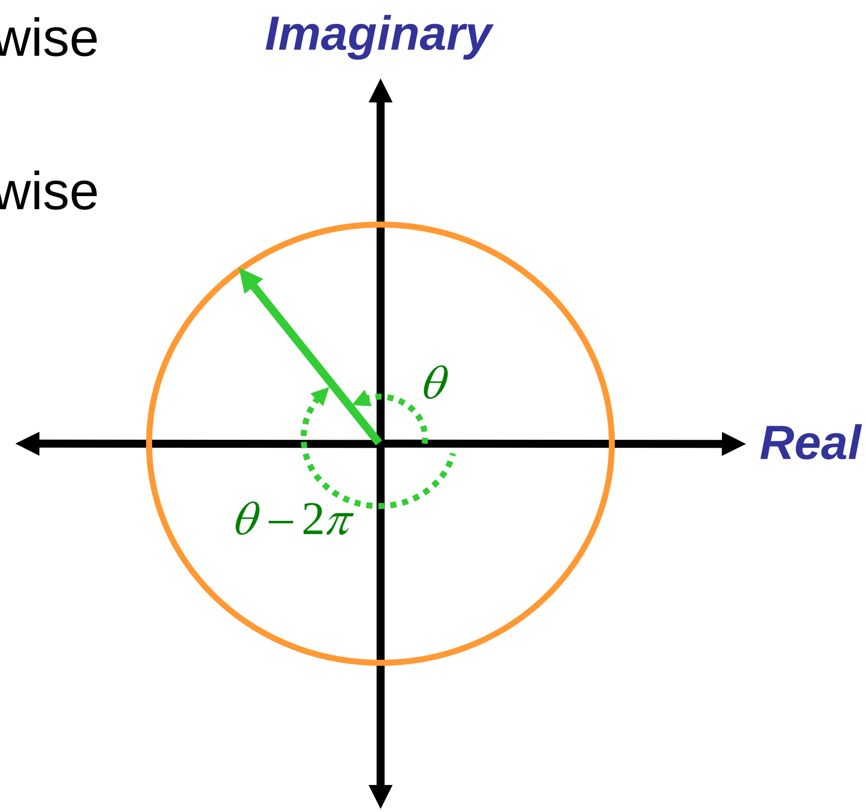
Rotation Composition



$$e^{i(\theta+\phi)} = e^{i\theta} e^{i\phi}$$

2D Rotation

- Complex numbers are good for representing ***2D orientations***, but inadequate for ***2D rotations***
- A complex number cannot distinguish different rotational movements that result in the same final orientation
  - Turn 120 degree counter-clockwise
  - Turn -240 degree clockwise
  - Turn 480 degree counter-clockwise

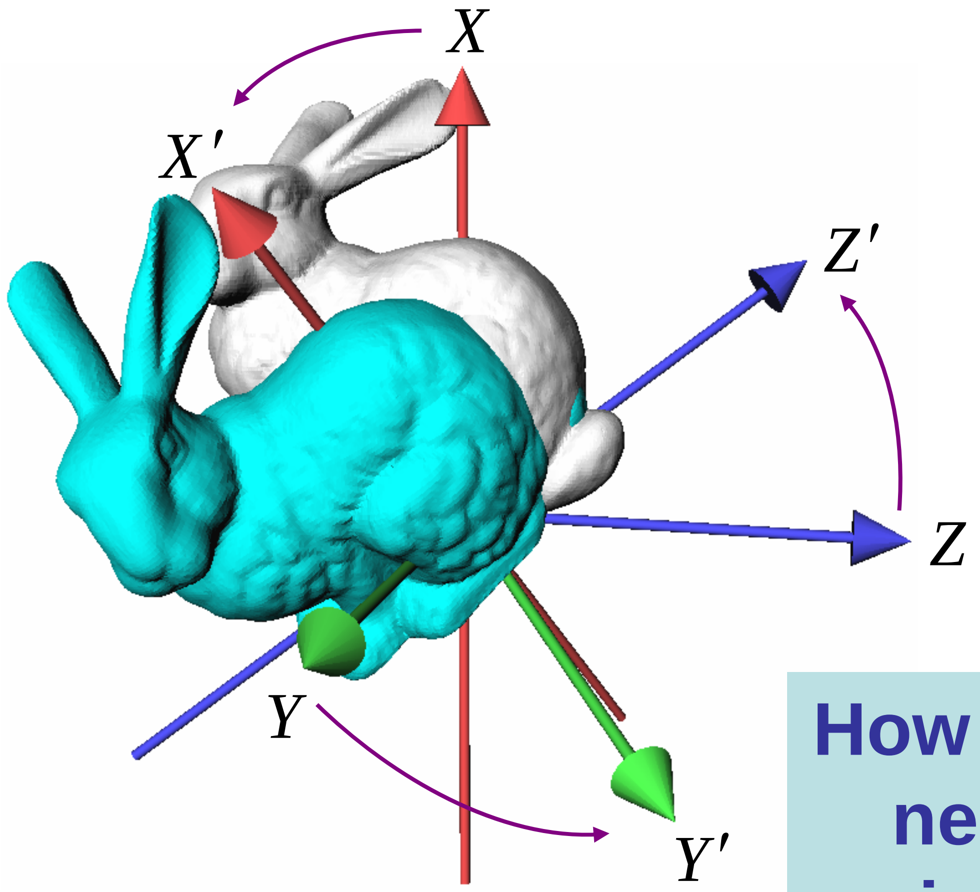


2D Rotation and Orientation

- ***2D Rotation***
  - The consequence of any ***2D rotational*** movement can be uniquely represented by a turning angle
  - A turning angle is ***independent*** of the choice of the reference orientation
- ***2D Orientation***
  - The non-singular parameterization of ***2D orientations*** requires extra parameters
    - Eg) Complex numbers, 2x2 rotation matrices
  - The parameterization is ***dependent*** on the choice of the reference orientation

3D Rotation

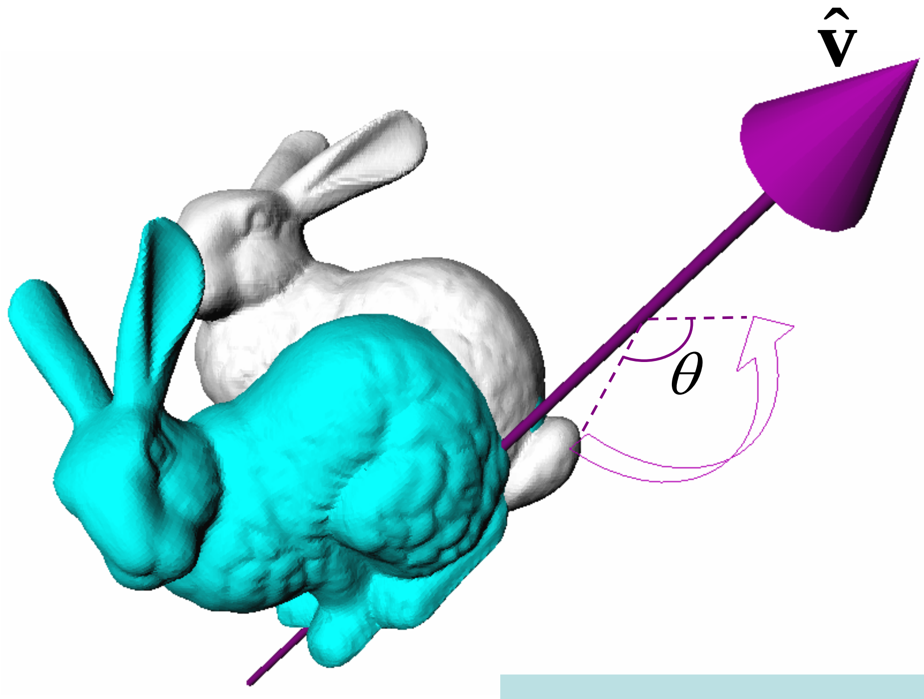
- Given two arbitrary orientations of a rigid object,



How many rotations do we need to transform one orientation to the other ?

3D Rotation

- Given two arbitrary orientations of a rigid object,



we can always find a fixed axis of rotation and a rotation angle about the axis

Euler's Rotation Theorem

The general displacement of a rigid body with one point fixed is a rotation about some axis

Leonhard Euler (1707-1783)

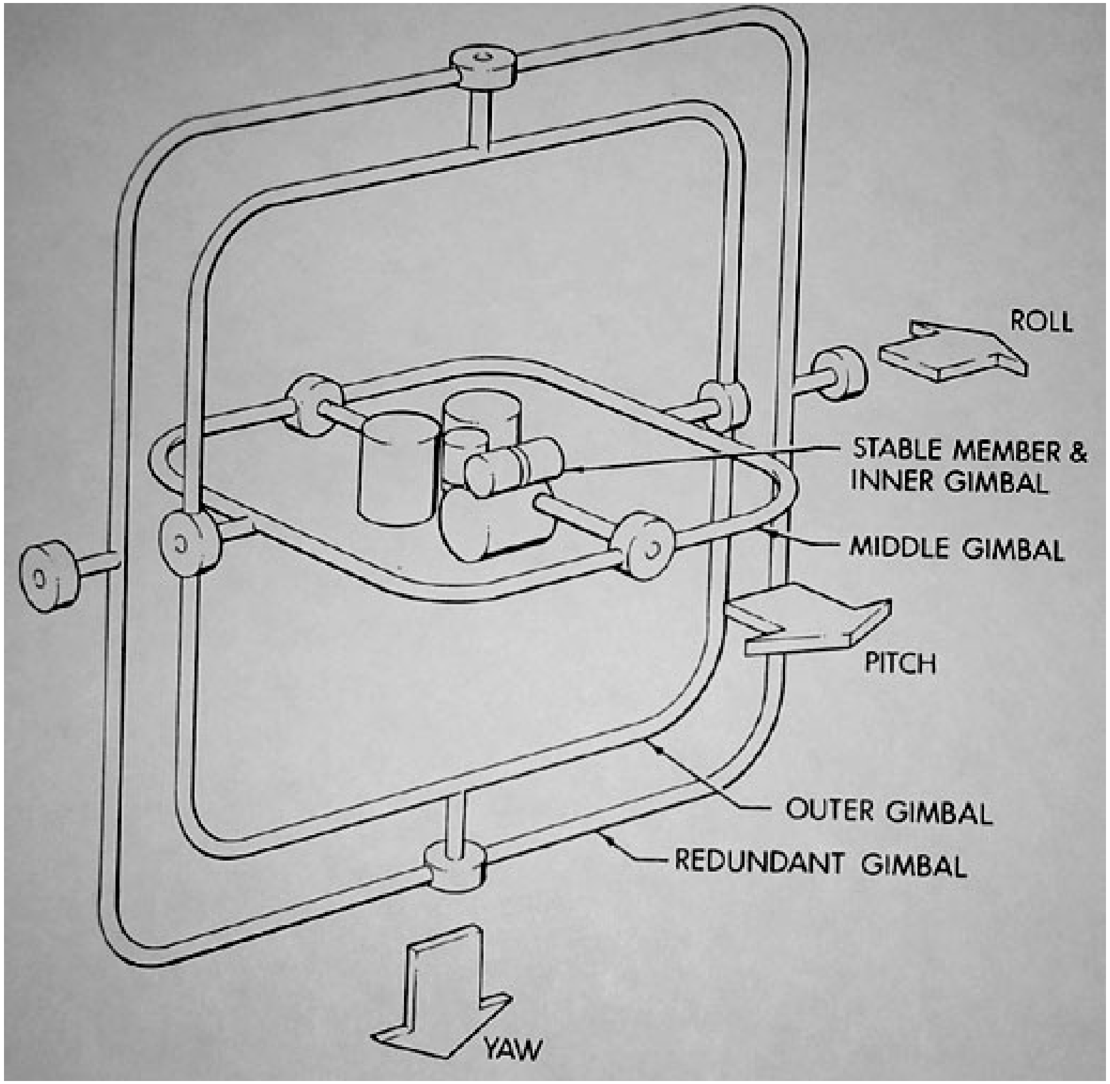
In other words,

- Arbitrary 3D rotation equals to one rotation around an axis
- Any 3D rotation leaves one vector unchanged

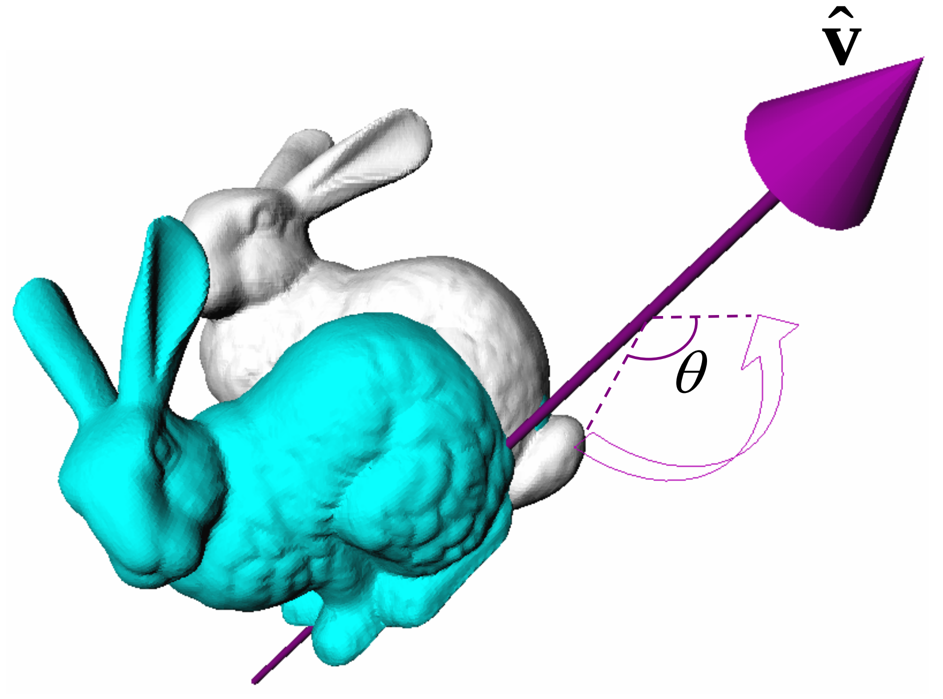
Euler Angles

- Rotation about three orthogonal axes
  - 12 combinations
    - XYZ, XYX, XZY, XZX
    - YZX, YZY, YXZ, YXY
    - ZXY, ZXZ, ZYX, ZYZ

- ***Gimble lock***
  - Coincidence of inner most and outmost gimbles' rotation axes
  - Loss of degree of freedom



Rotation Vector



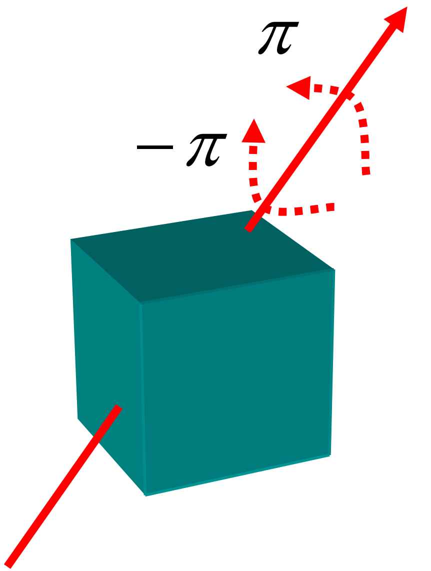
$\hat{\mathbf{v}}$: unit vector

θ : scalar angle

- Rotation vector (3 parameters) $\mathbf{v} = \theta\,\hat{\mathbf{v}} = (x, y, z)$
- Axis-Angle (2+1 parameters) $(\theta, \hat{\mathbf{v}})$

3D Orientation

- Unhappy with three parameters
  - Euler angles
    - Discontinuity (or many-to-one correspondences)
    - Gimble lock
  - Rotation vector (a.k.a Axis/Angle)
    - Discontinuity (or many-to-one correspondences)



Using an Extra Parameter

- ***Euler parameters***

$$e_0 = \cos\left(\frac{\theta}{2}\right)$$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \hat{\mathbf{v}} \sin\left(\frac{\theta}{2}\right)$$

θ : rotation angle

$\hat{\mathbf{v}}$: rotation axis

Quaternions

- William Rowan Hamilton (1805-1865)
  - Algebraic couples (complex number) 1833

$x + iy$ **where** $i^2 = -1$

Quaternions

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$$x + iy \quad \textbf{where} \quad i^2 = -1$$

  - Quaternions 1843

$$w + ix + jy + kz \quad \textbf{where} \quad \begin{aligned} & i^2 = j^2 = k^2 = ijk = -1 \\ & ij = k, \quad jk = i, \quad ki = j \\ & ji = -k, kj = -i, ik = -j \end{aligned}$$

Quaternions

William Thomson

"… though beautifully ingenious, have been an unmixed evil to those who have touched them in any way."

Arthur Cayley

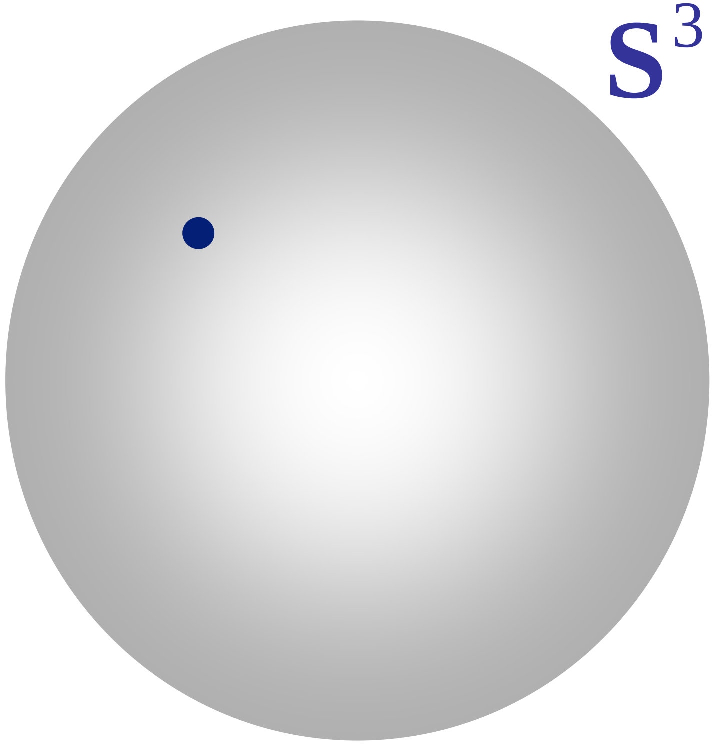
"… which contained everything but had to be unfolded into another form before it could be understood."

Unit Quaternions

- Unit quaternions represent 3D rotations

$$
\begin{aligned}
\mathbf{q} &= w + ix + jy + kz \\
&= (w, x, y, z) \\
&= (w, \mathbf{v})
\end{aligned}
$$

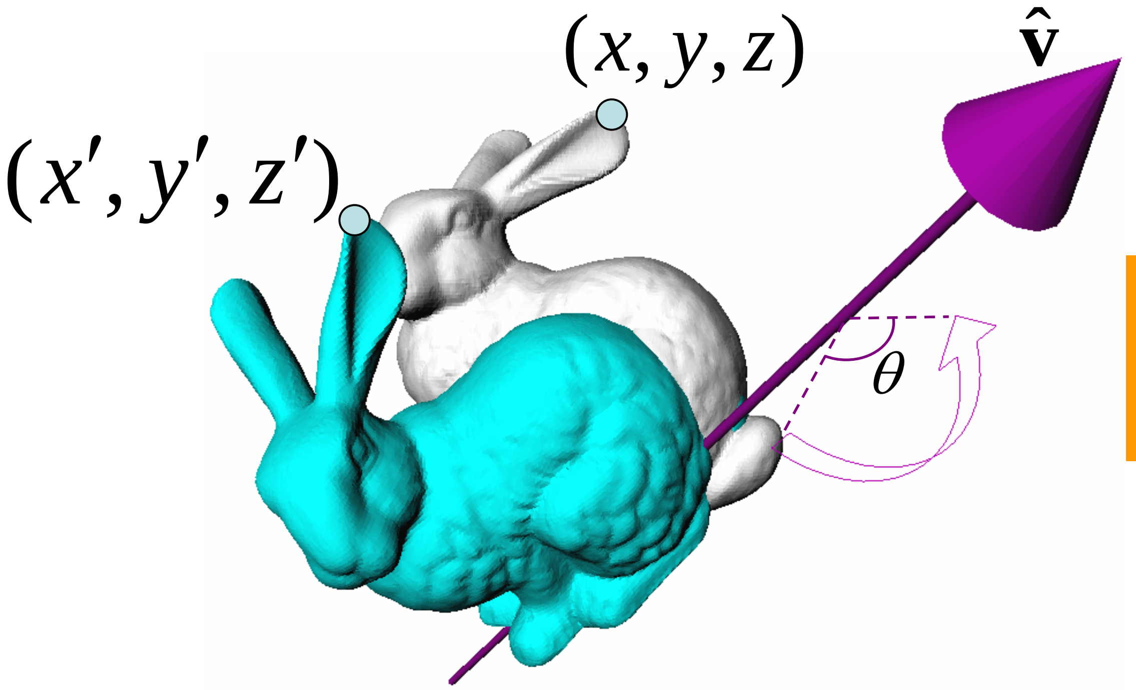
$\mathbf{S}^3$



$$w^2 + x^2 + y^2 + z^2 = 1$$

Rotation about an Arbitrary Axis

- Rotation about axis $\hat{\mathbf{v}}$ by angle θ



$$\mathbf{q} = \left(\cos\frac{\theta}{2}, \hat{\mathbf{v}} \sin\frac{\theta}{2} \right)$$

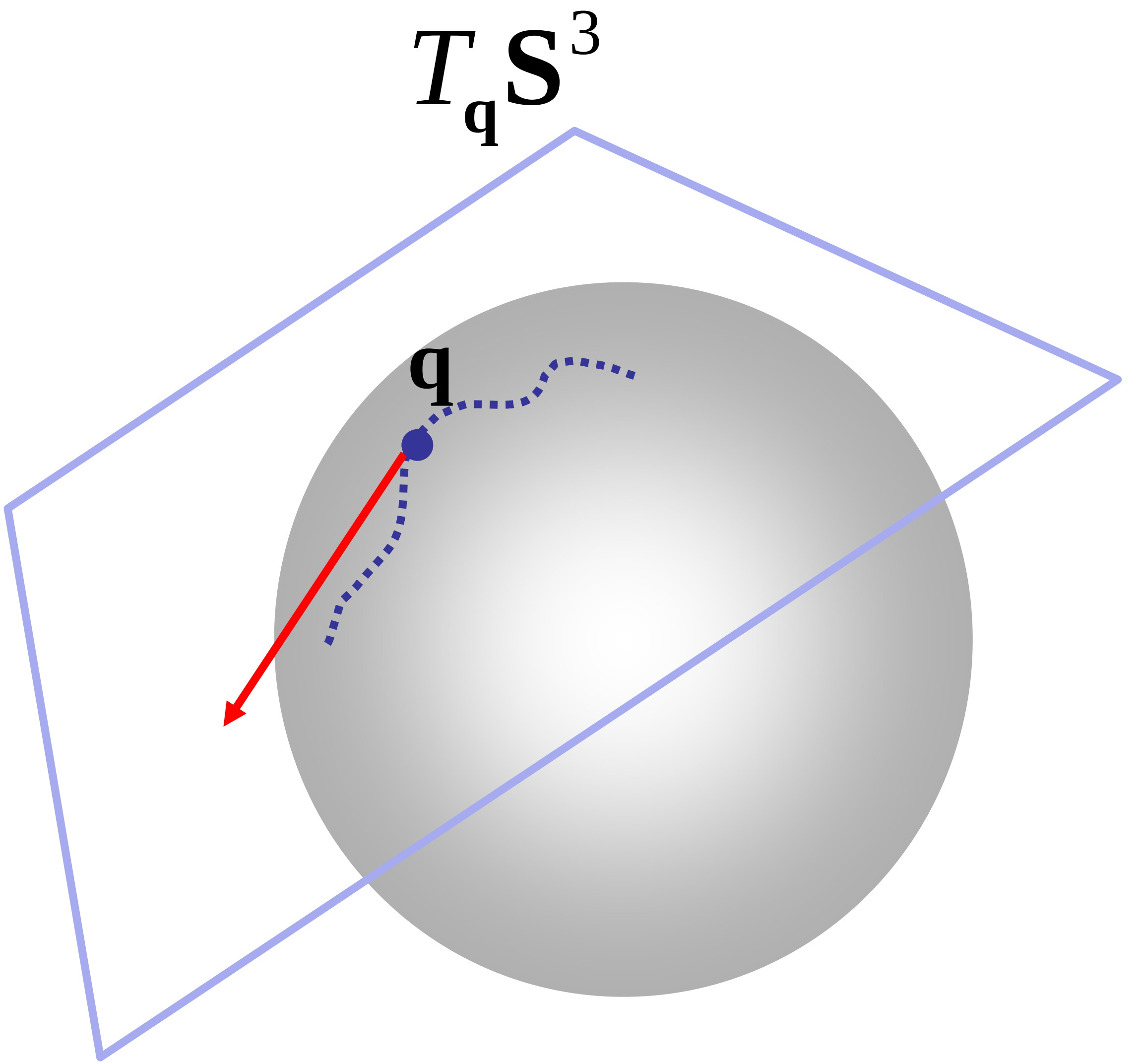
$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}$ **where** $\mathbf{p} = (0, x, y, z)$

Purely Imaginary Quaternion

Unit Quaternion Algebra

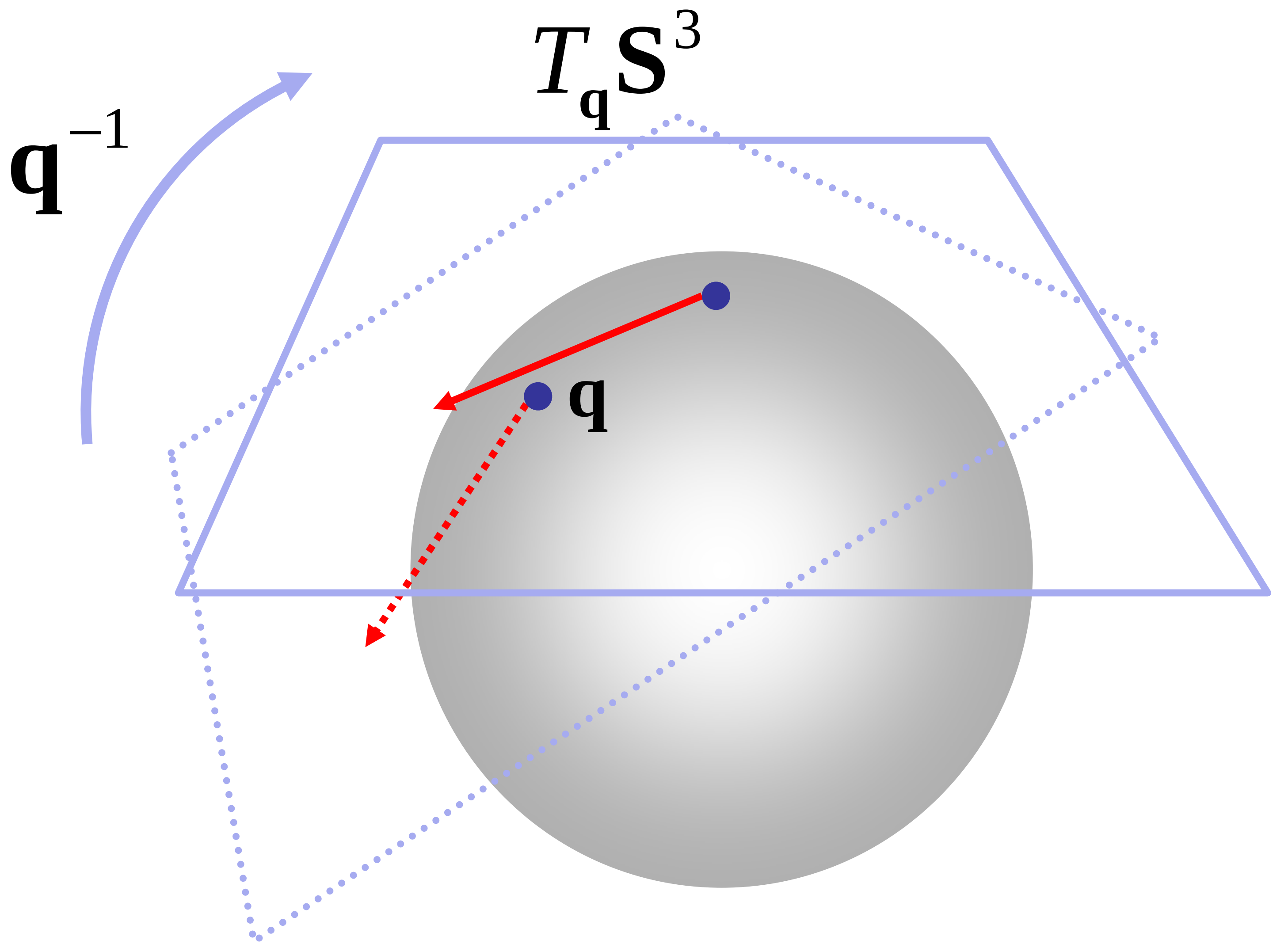
- Identity $\mathbf{q} = (1,0,0,0)$
- Multiplication
$$\begin{aligned}\mathbf{q}_1\mathbf{q}_2 &= (w_1, \mathbf{v}_1)(w_2, \mathbf{v}_2) \\ &= (w_1 w_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, w_1\mathbf{v}_2 + w_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2)\end{aligned}$$
- Inverse
$$\begin{aligned}\mathbf{q}^{-1} &= (w,-x,-y,-z)/(w^2 + x^2 + y^2 + z^2) \\ &= (-w, x, y, z)/(w^2 + x^2 + y^2 + z^2)\end{aligned}$$
- Unit quaternion space is
  - closed under multiplication and inverse,
  - but not closed under addition and subtraction

Tangent Vector
(Infinitesimal Rotation)

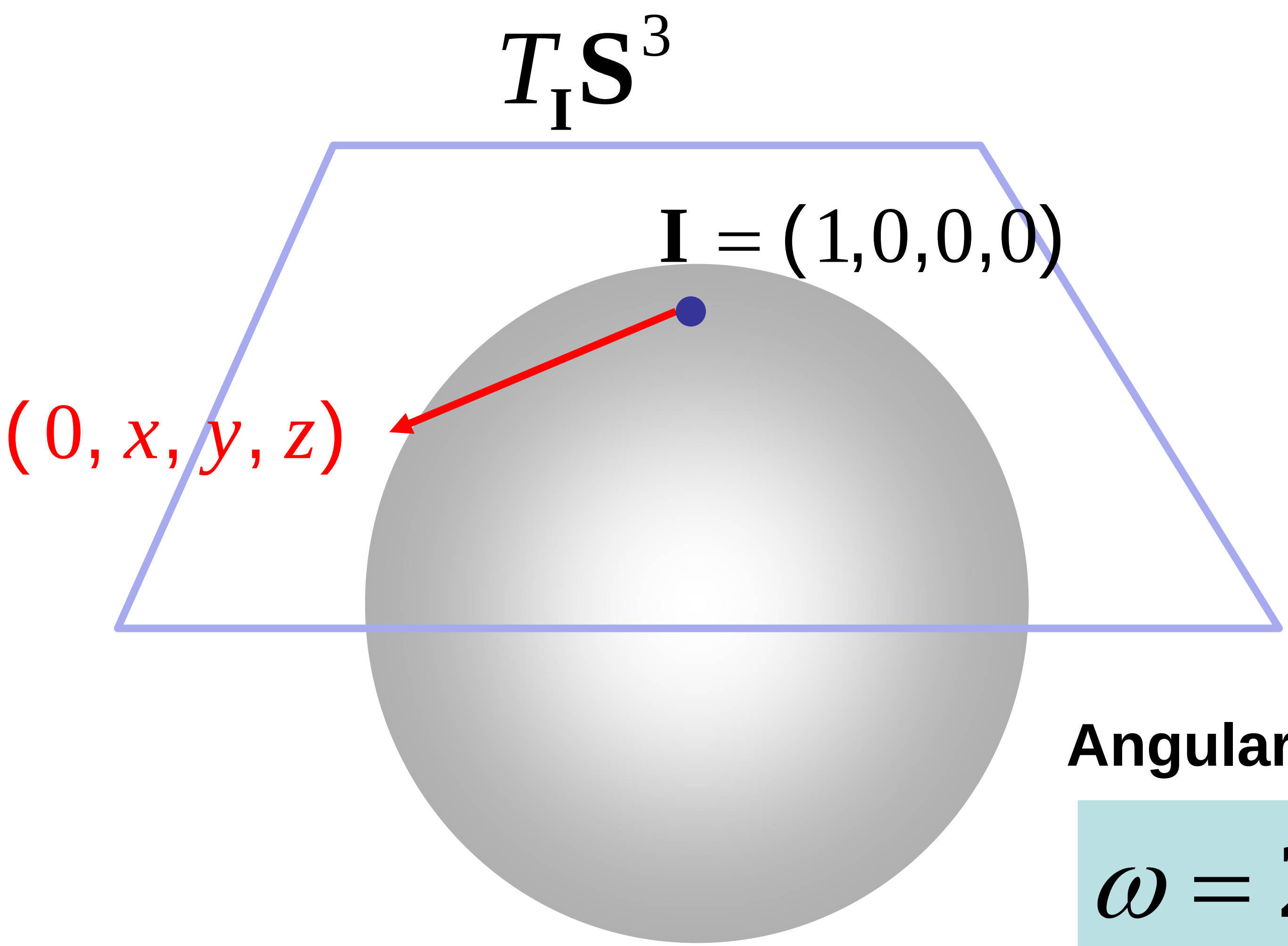


Tangent Vector

(Infinitesimal Rotation)



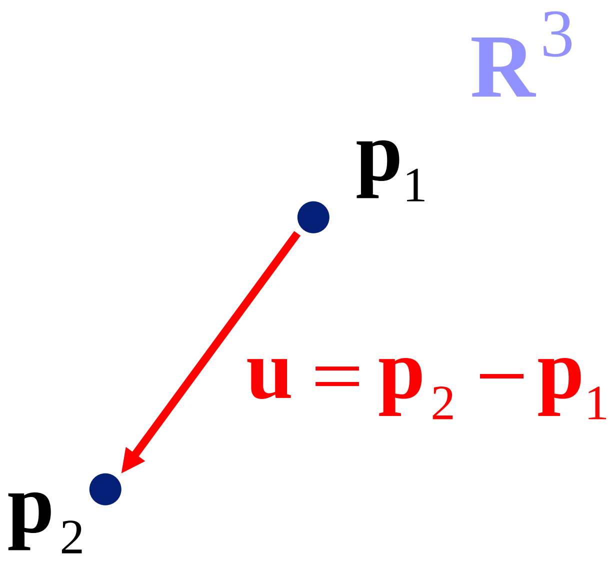
Tangent Vector
(Infinitesimal Rotation)

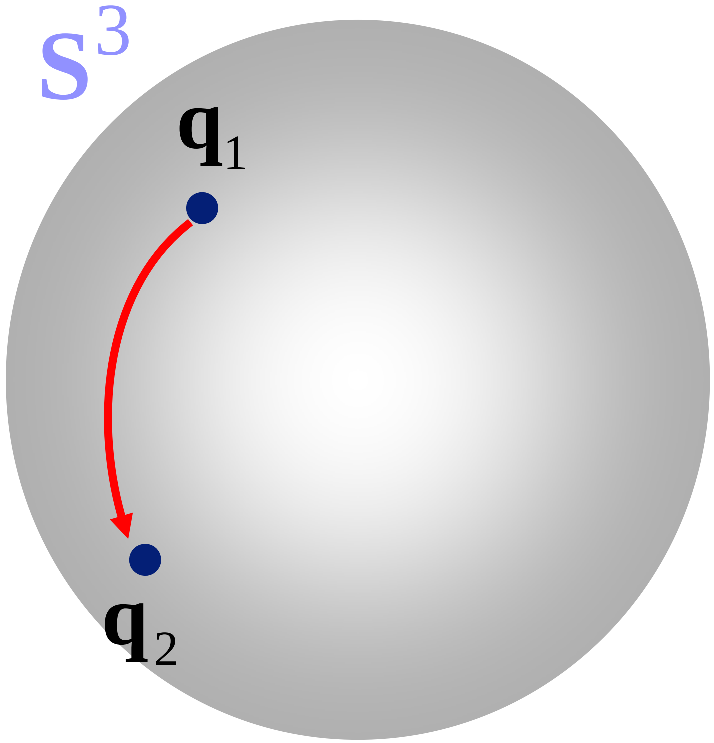


Angular Velocity

$$\omega = 2\mathbf{q}^{-1}\dot{\mathbf{q}}$$

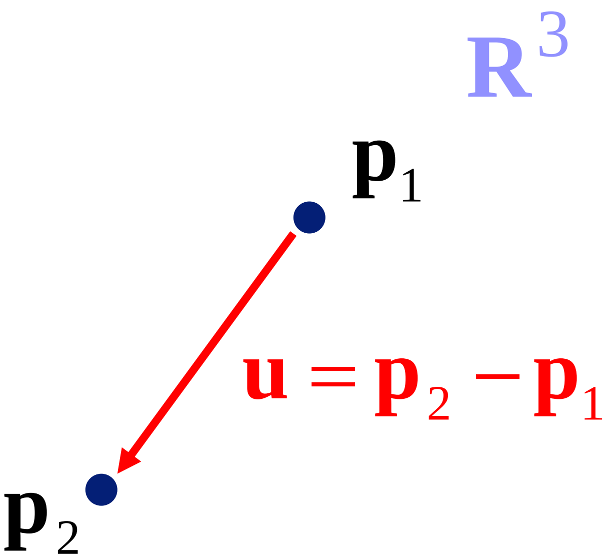
Rotation Vector

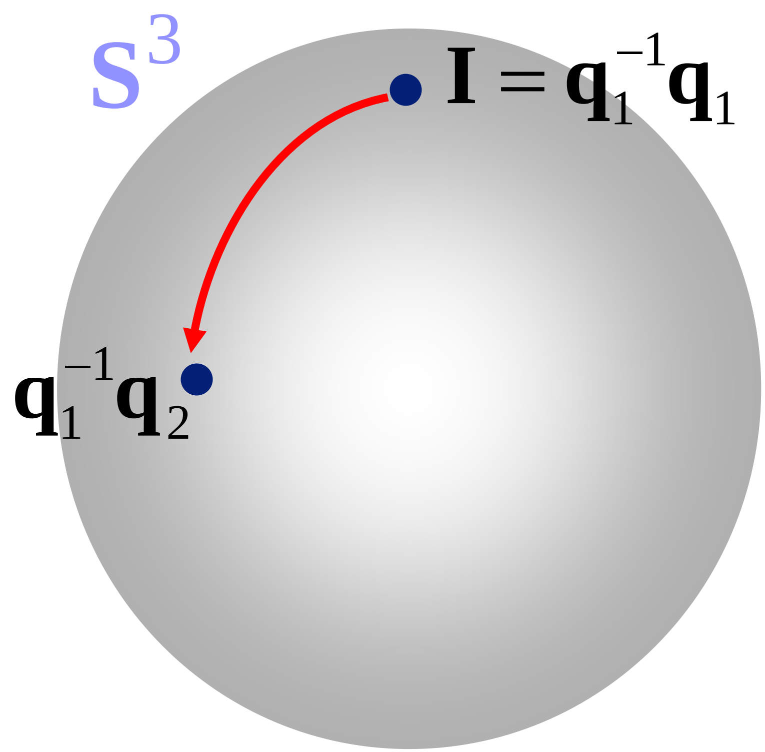




$$\begin{aligned} \mathbf{p}_2 &= \mathbf{p}_1 + \mathbf{u} \\ &= \mathbf{p}_1 + (\mathbf{p}_2 - \mathbf{p}_1) \end{aligned}$$

Rotation Vector





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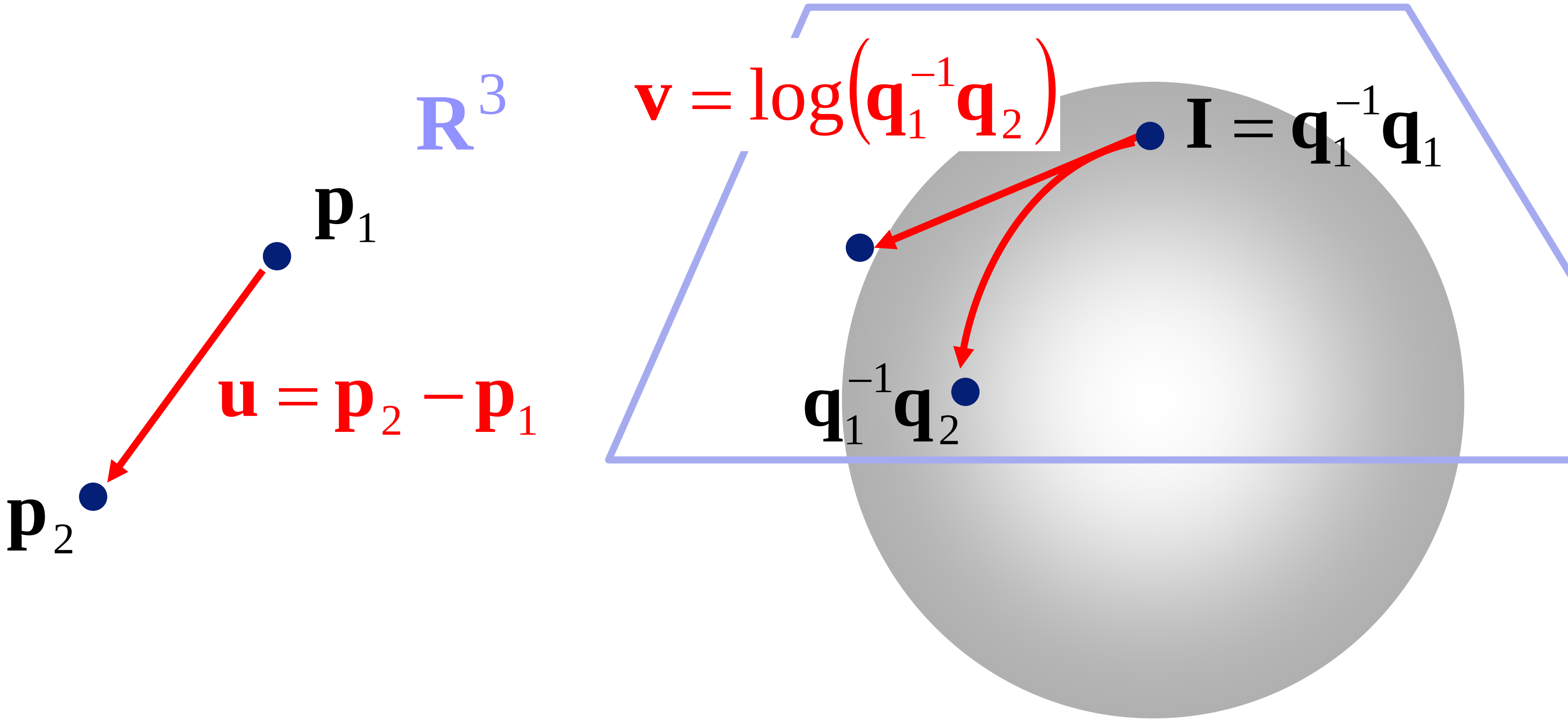

$$\begin{aligned}\mathbf{p}_2 &= \mathbf{p}_1 + \mathbf{u} \\ &= \mathbf{p}_1 + (\mathbf{p}_2 - \mathbf{p}_1)\end{aligned}$$

$$\begin{aligned}\mathbf{q}_2 &= \mathbf{q}_1 \exp(\mathbf{v}) \\ &= \mathbf{q}_1 \exp\left(\log\left(\mathbf{q}_1^{-1}\mathbf{q}_2\right)\right)\end{aligned}$$

Rotation Vector

- ***Finite rotation***
  - Eg) Angular displacement
  - Be careful when you add two rotation vectors

$$e^u e^v \neq e^{u+v}$$

- ***Infinitesimal rotation***
  - Eg) Instantaneous angular velocity
  - Addition of angular velocity vectors are meaningful

Coordinate-Invariant Operations

A. (orientation) $\cdot$ (orientation) $\longrightarrow$ (UNDEFINED)
B. exp(rotation) $\cdot$ exp(rotation) $\longrightarrow$ exp(rotation)
C. (orientation) $\cdot$ exp(rotation) $\longrightarrow$ (orientation)
exp(rotation) $\cdot$ (orientation) $\longrightarrow$ (orientation)
D. $(\text{orientation})^{-1}$ $\cdot$ (orientation) $\longrightarrow$ exp(rotation)
(orientation) $\cdot$ $(\text{orientation})^{-1}$ $\longrightarrow$ exp(rotation)
E. log(exp(rotation)) $\longrightarrow$ (rotation)
F. log(orientation) $\longrightarrow$ (UNDEFINED)
G. (scalar) $\cdot$ (rotation) $\longrightarrow$ (rotation)
$\text{exp(rotation)}^{(\text{scalar})}$ $\longrightarrow$ exp(rotation)
H. $(\text{orientation})^{(\text{scalar})}$ $\longrightarrow$ (orientation) if scalar=1
$\longrightarrow$ exp(rotation) if scalar=0
$\longrightarrow$ (UNDEFINED) otherwise
I. (rotation) $\pm$ (rotation) $\longrightarrow$ (rotation)
J. $\sum$ (scalar) $\cdot$ (rotation) $\longrightarrow$ (rotation)
K. affine_combi(orientations) $\longrightarrow$ (ILL-DEFINED)

Analogy

- (point : vector) is similar to (orientation : rotation)

a. (point) + (point) $\to$ (UNDEFINED)
b. (vector) $\pm$ (vector) $\to$ (vector)
c. (point) $\pm$ (vector) $\to$ (point)

d. (point) - (point) $\to$ (vector)

g. (scalar) $\cdot$ (vector) $\to$ (vector)

h. (scalar) $\cdot$ (point) $\to$ (point) if scalar=1
$\to$ (vector) if scalar=0
$\to$ (UNDEFINED) otherwise

j. $\sum$ (scalar) $\cdot$ (vector) $\to$ (vector)
k. $\sum$ (scalar) $\cdot$ (point) $\to$ (point) if $\sum$ scalar=1
$\to$ (vector) if $\sum$ scalar=0
$\to$ (UNDEFINED) otherwise

A. (orientation) $\cdot$ (orientation) $\to$ (UNDEFINED)
B. exp(rotation) $\cdot$ exp(rotation) $\to$ exp(rotation)
C. (orientation) $\cdot$ exp(rotation) $\to$ (orientation)
exp(rotation) $\cdot$ (orientation) $\to$ (orientation)
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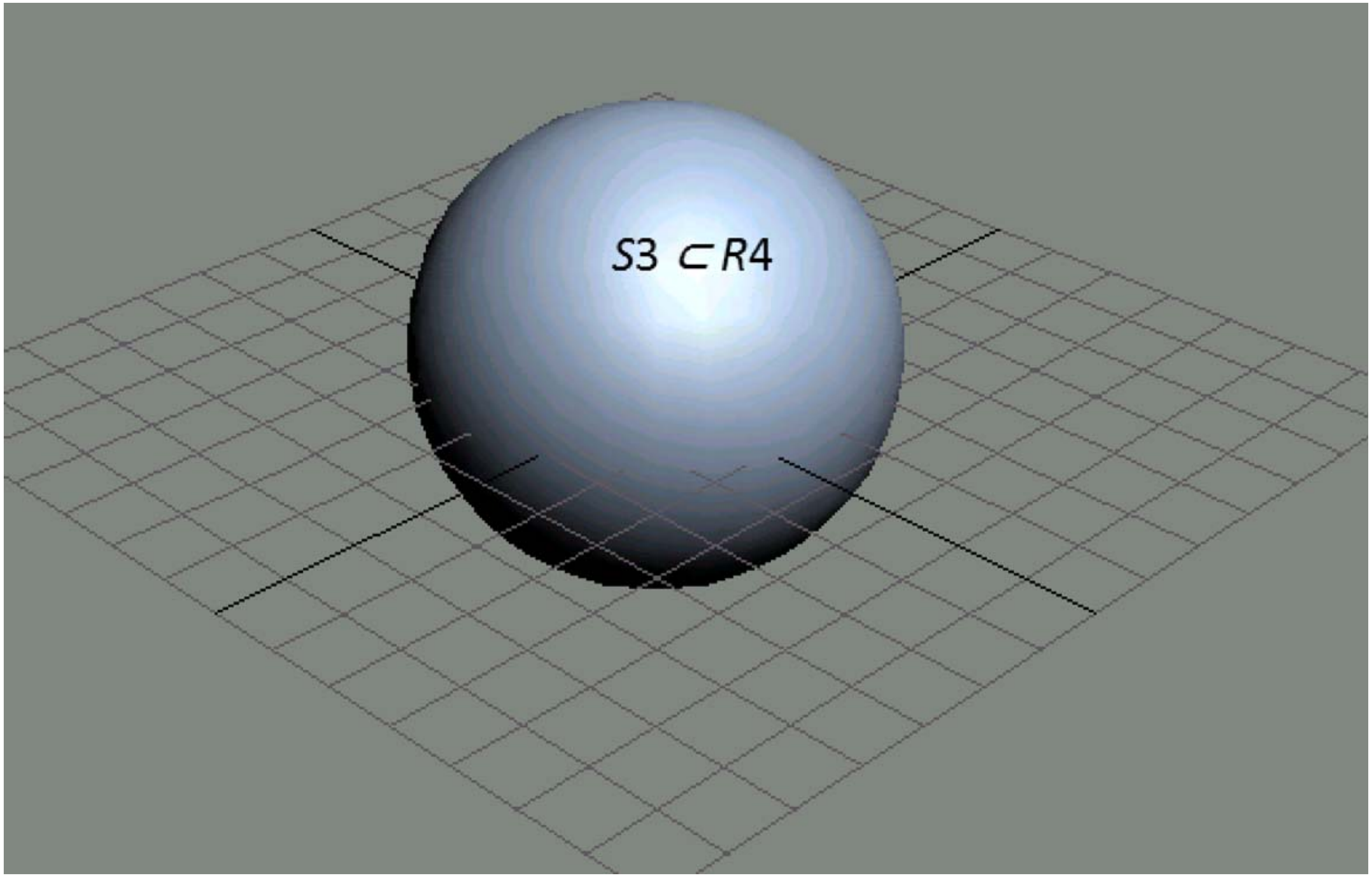
K. affine_combi(orientations) $\to$ (ILL-DEFINED)

Rotation Conversions

- In theory, conversion between any representations is possible
- In practice, conversion is not simple because of different conventions
- Quaternion to Matrix

$$R = \begin{pmatrix} q_0^2 + q_x^2 - q_y^2 - q_z^2 & 2q_x q_y - 2q_0 q_z & 2q_x q_z + 2q_0 q_y & 0 \\ 2q_x q_y + 2q_0 q_z & q_0^2 - q_x^2 + q_y^2 - q_z^2 & 2q_y q_z - 2q_0 q_x & 0 \\ 2q_x q_z - 2q_0 q_y & 2q_y q_z + 2q_0 q_x & q_0^2 - q_x^2 - q_y^2 + q_z^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Method for Mapping the Four-Dimensional Space onto the Oriented Three-Dimensional Space



For further presentation, we recall the notion of three-dimensional sphere $S3 \subset R4$. Such a sphere defined as a subspace of the four-dimensional vector space $R4$ is determined by the well-known expression

$$S^3 = \left\{ x(x_1, x_2, x_3, x_4) \in R^4 : \ |x|^2 = \sum_{k=1}^{4} (x_k)^2 = 1 \right\}.$$

The sphere ***S*3** has the structure of a three-dimensional analytic connected closed oriented manifold, just as the three-dimensional rotation group ***SO*(3)**. Therefore, such a sphere ***S*3** can in a standard way be embedded in a four-dimensio nal arithmetic space **R4** equipped with the structure of quaternion algebra.
In this case, the four-dimensional vector **$x = (x_1, x_2, x_3, x_4)^T$** whose coordinates are

$$x_1 = \lambda_0,\ x_2 = \lambda_1,\ x_3 = \lambda_2,\ x_4 = \lambda_3,$$

respectively, can be represented in the well-known algebraic form (2.2) of the classi-cal Hamiltonian quaternion Λ. The sphere projection $S^3 \to RP^3$ associates each s uch **a** quaternion $\Lambda \in S^3 \subset R^4$ with a pair of quaternions $(\Lambda, -\Lambda)$, which corres-pond to two identified opposite points on the surface of the three-dimensional sphere ***S*3.**

If the four real parameters $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in R^1$ of the classical Hamiltonian quaternions $\Lambda \in R^4$ are used, the mapping of the sphere $S^3 \subset R^4$ onto the space ***SO*(3)** of all possible configurations of a rigid body with a single immovable (fixed) point is two-sheeted.

METHOD OF LOCAL THREE-DIMENSIONAL PARAMETRIZATION

Consider the stereographic projection of the above-introduced three-dimensional sphere **S3 ⊂ R4** onto the oriented three-dimensional vector subspace **R3** (the hyp-erplane **Γ3 ⊂ R3**) in more detail. For the standard sphere **S3** of unit radius **|r| = 1**, we have the well-known relation (2.6). Inturn, the sphere **S3** itself as a subspace of the space **R4** has the structure of an analytic connected oriented manifold, which is a submanifold of the space **R4**. In the case of stereographic projection (mapping) **S3 → R3**, any point on **S3** opposite to the hyperplane **Γ3 ⊂ R3** can be t he center of the projection. Note that, in addition, the mapping considered here is also a conformal mapping. Indeed, the stereographic projection of the sphere **S3** canbe considered as part of the conformal mapping of the finite four-dimensional **R4 → R4** (into itself), because the stereographic projection can be continued to such a mapping.

An exception is the projection center **α**, which corresponds to the single point at infinity in **R4**. Under the stereographic projection, the point at infinity of the hyperp-lane **Γ3 ⊂ R3** is associated with a single point of the sphere **S3**, i.e., the pole poi-nt **α**. Because of the above property and the fact that the mapping itself is confor mal, we use the method of the stereographic projection **S3 ⊂ R3**.

The mapping considered here associates the four co-ordinates **(x1, x2, x3, x4)** of a global vector **x** ∈ **R4** with the three coordinates **(y1, y2, y3)** of a local vector **y** ∈ **R3**. Usually, the operation of such projection can be written symbolic-ally as the chain **S3\{α} → R3**. We prescribe the center of the stereographic projection α, n amely, the pole of the three-dimensional sphere **S3**, for which we take the chosen

- Then the straight line passing through the given pole **α(0, 0, 0, 1)** and an arbitrary point **x** ∈ **S3** on the surface of the sphere **S3** intersects the oriented vector subspa-ce **R3** at some point, which we denote by **ϕ(x)**.
- Just themapping taking such **a** point **x**∈**R4** to the oriented subspace **R3** (**x→ϕ(x)**∈**R3**) homeomorphism between the sphere **S3** (with a single punctured point α) and the space **R3.** In this case, there exists a stereographic projection of the four-dimensional vector **x** ∈ **M3 ⊂ R4** onto the oriented subspace ***R*3.**
- Therefore, the point of intersection of the straight line drawn from the pole $\alpha \in$ ***M*3** through an arbitrary point $x \in$ ***R*4** on the surface of the sphere ***S*3** corresponding to the vector ***x*(*x*1*, x*2*, x*3*, x*4)** with the oriented space ***R*3** gives a single point of intersection ***ϕ*(*x*)** on the hyperplane **Γ3** $\subset$ ***R*3**, i.e., the desired three-dimensional vector $y \in$ ***R*3.** Here we present the three coordinates of this point in the form

$$\varphi(x) = \left\{ \frac{x_1}{1 - x_4}, \frac{x_2}{1 - x_4}, \frac{x_3}{1 - x_4} \right\}$$

For the subsequent calculations, we introduce a rectangular 3 × 4 matrix ***V*** of the projective transformation satisfying the identities

$$VV^{T} = E \qquad\qquad V\alpha = 0$$

where ***E*** is the unit 3 × 3 matrix and ***α*=(0, 0, 0, 1)*T*** is a 4 × 1 column vector.

Under the mapping considered here, i.e., under the stereographic projection,

the intersection point ***ϕ*(*x*) ∈ *R*3** coincides with the desired three-dimensional vector of local parameters ***y* ∈ *R*3**. Then, changing the notation ***ϕ*(*x*) ⇔ *y*** and using identities (3.1) and (3.2), we have the coupling equation for the two vecto-rs ***x* ∈ *R*4** and ***y* ∈ *R*3** introduced above:

$$y = \frac{Vx}{1 - \alpha^{T} x}$$

where ***x* ∈ *M*3 ⊂ *R*4** and *V* is the rectangular 3 × 4 matrix of projection written as two matrices: *V* = *E*3×3|03×1. Thus, Eq. (3.3) obtained above is the point of intersection ***ϕ*(*x*) ≡ *y* ∈ *R*3** of the straight line connecting the point *α* of the center (pole) of the stereographic projection and an arbitrary **point *x* ∈ *M*3 ⊂ *R*4** on the sphere ***S*3** itself with the oriented subspace ***R*3.** Note that Eq. (3.3) relating three- and four-dimensional vectors is defined

for all ***x* ∈ *M*3 ⊂ *R*4** except ***x* ∈ *α***. The latter can readily be proved, because the point ***α*** of the projection center (pole) does not belong to the set ***M*3**. Then, prescribing the four linear coordinates ***x*1, *x*2, *x*3, *x*4** of a point ***x* ∈ *M*3 ⊂ *R*4** and using (3.3), one can readily obt-ain the three desired local parameters, i.e., the coordinates ***y*1, *y*2, *y*3** of the point of intersection ***ϕ*(*x*) ⇔ *y* (*y* ∈ *R*3)** We illustrate this by an example of the above mapping

Explaining slide